Fixed Point Theory, 24(2023), No. 2, 627-640
DOI: 10.24193/fpt-ro.2023.2.11
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

AN ITERATIVE ALGORITHM WITH APPLICATIONS FOR SOLVING VARIATIONAL INCLUSION PROBLEMS AND NONEXPANSIVE MAPPINGS

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Abstract. In this paper, a viscosity splitting iterative algorithm with perturbation is introduced for solving an inclusion problem with two accretive operators and a fixed point problem of an infinite family of nonexpansive mappings. Strong convergence of the iterative algorithm is obtained in a Banach space. A convex minimization problem is also considered in Hilbert spaces as an applications.
Key Words and Phrases: Banach space, nonexpansive mapping, splitting method, variational inequality, variational inclusion, fixed point.
2020 Mathematics Subject Classification: 47H05, 47H09, 47H10, 65K15, 90C25.

1. Introduction and preliminaries

Let E^* denote the dual space of a Banach space E, and let $\langle \cdot, \cdot \rangle$ be the duality pairing between space E and its dual. One always uses $\mathfrak{J}_q$ $(q > 1)$ to denote the generalized duality mapping onto 2^{E^*}, which is defined by, $\forall x \in E$,

$$\mathfrak{J}_q(x) := \{y \in E^* : \langle x, y \rangle = \|x\|^q, \|y\| = \|x\|^{q-1}\}.$$

If $q = 2$, then one calls $\mathfrak{J}_2$ the normalized duality mapping, and it is denoted by $\mathfrak{J}$ in this paper. One knows that $\mathfrak{J}_q(x)$ is always nonempty, which is due to the Hahn-Banach Theorem and, for all $x \neq 0$, $\mathfrak{J}_q(x) = \|x\|^{q-2}\mathfrak{J}(x)$. The single-valued generalized duality mapping will be denoted by $\mathfrak{j}_q$ and the single-valued normalized duality mapping is denoted by $\mathfrak{j}$ next.

This research work was funded by Institutional Fund Projects under grant no. (IFPIP: 1487-130-1443). The authors gratefully acknowledge technical and financial support provided by the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia.

Let ϵ be a real number in $[0,2]$. The convexity modulus Banach space E is the function $\delta_E(\epsilon):(0,2]\to[0,1]$, defined by

$$\delta_E(\epsilon)=\inf\left\{\frac{2-\|x+y\|}{2}:\|x\|=\|y\|=1,\|x-y\|\geq\epsilon\right\},$$

One says that space E is p-uniformly convex with $p>1$ iff there exists a real number associated with p, $c_p>0$, such that, for any $\epsilon\in(0,2]$, $\delta_E(\epsilon)\geq c_p\epsilon^p$.

Let N be a mapping on E. The fixed point set of N is denoted by $\mathrm{Fix}(N)$ in this paper. One recalls that N is said to be contractive iff, $\forall x,y\in E$, $\|Nx-Ny\|\leq\kappa\|x-y\|$, where κ is a real number in $(0,1)$. One further recalls that N is said to be nonexpansive iff, $\forall x,y\in E$, $\|Nx-Ny\|\leq\|x-y\|$. There are numerous applications of nonexpansive mappings in various research fields; see, e.g., [4, 5, 16, 20, 13]. Normal Mann iteration is a powerful scheme to investigate fixed points of nonexpansive mappings and their extensions. But, normal Mann iteration is weakly convergent even in Hilbert spaces; see [2]. Another popular scheme is the viscosity scheme, which is based on the Halpern iterative scheme. It generates a sequence $\{x_n\}$ in the following manner:

$$x_1\in E,\ x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)Nx_n,\ \forall n\geq1,$$

where f is a contractive mapping, $\{\alpha_n\}$ is a real sequence in $(0,1)$, and x_1 is an initial point chosen arbitrarily.

Let $N_1,N_2,\cdots,N_i,\cdots$ be nonexpansive mappings on space E. Let Id denote the identity mapping on E, and let $\{\zeta_n\}$ be a sequence in $(0,\zeta]$, where ζ is some real number in $(0,1)$. Consider

$$\begin{aligned}
&V_{n,n}=(1-\zeta_n)Id+\zeta_nN_nV_{n,n+1},\\
&V_{n,n-1}=(1-\gamma_{n-1})Id+\gamma_{n-1}N_{n-1}V_{n,n},\\
&\cdots\\
&V_{n,k}=(1-\zeta_k)Id+\zeta_kN_kV_{n,k+1},\\
&\cdots\\
&V_{n,2}=(1-\zeta_2)Id+\zeta_2N_2V_{n,3},\\
&W_n=V_{n,1}=(1-\zeta_1)Id+\zeta_1N_1V_{n,2},
\end{aligned}\tag{1.1}$$

where $V_{n,n+1}=Id$.

In strictly convex Banach spaces, from [19], for every $x\in C$, the limit $\lim_{n\to\infty}V_{n,k}x$ exists in strictly convex Banach spaces. Define a mapping W on E by

$$Wx=\lim_{n\to\infty}V_{n,1}x=\lim_{n\to\infty}W_nx,\ \forall x\in E.$$

W is called the W-*mapping* defined by $N_1,N_2,\cdots$. From [19], one has

$$\mathrm{Fix}(W)=\cap_{i=1}^{\infty}\mathrm{Fix}(N_i).$$

Recall that the smoothness modulus of E, $\rho_E:[0,\infty)\to[0,\infty)$, is defined by

$$\rho_E(t)=\sup\left\{\frac{\|x+y\|-\|x-y\|-2}{2}:x\in B_E,\|y\|\leq t\right\}.$$

E is said to be uniformly smooth iff $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$. E is said to be q-uniformly smooth iff there exists a fixed constant $c > 0$ such that $\frac{\rho_E(t)}{t^q} \leq c$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth; see [21] and the references therein.

It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $qp = q + p$. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p, where $p > 1$. Let $Proj_D^E : E \to D$, where E be a subset of space E, be a mapping. It is said to be [3, 10] (1) sunny iff, for each $y \in C$ and $\xi \in (0, 1)$, $Proj_D^E\big((1 - \xi)Proj_D^E y + \xi y\big) = Proj_D^E y$; (2) retraction iff $(Proj_D^E)^2 = Proj_D^E$; (3) sunny nonexpansive retractction iff $Proj_D^E$ is a sunny, nonexpansive, and retraction operator. D is said to be a nonexpansive retract of E iff there exists a nonexpansive retraction from E onto D. In a q-smooth Banach space E, $Proj_E^E$ is sunny and nonexpansive iff $\langle x - Proj_E^E x, \mathfrak{J}_q(y - Proj_E^E x)\rangle \leq 0$, $\forall x \in E$, $y \in E$. In the setting of Hilbert spaces, the sunny nonexpansive retraction coincides with the nearest point projection. Let $x \in E$ and $x_0 \in D$. $x_0 = Proj_D^E x$ iff $\langle x - x_0, \mathfrak{J}_q(y - x_0)\rangle \leq 0$ for all $y \in D$, where $Proj_C^E$ is a sunny nonexpansive retraction from E onto D in a q-smooth Banach space E.

Recall that an operator $P : E \to 2^E$ with the domain, denoted by

$$Dom(P) = \{z \in E : Pz \neq \emptyset\}$$

and the range, denoted by $Ran(P) = \cup\{Pz : z \in Dom(P)\}$, is said to be accretive iff, for $r > 0$,

$$\|y - x\| \leq \|r(\bar{y} - \bar{x}) + (y - x)\|, \quad \forall \bar{x} \in Px, \bar{y} \in Py,$$

for all $x, y \in Dom(P)$. There is a celebrated equivalent definition by Kato [11] as follows

$$\langle \bar{x} - \bar{y}, \mathfrak{j}_q(x - y)\rangle \geq 0, \quad \forall \bar{x} \in Px, \bar{y} \in Py.$$

Furthermore, nn accretive operator P is said to be m-accretive iff $Ran(Id + rP)$ is precisely E for any positive real number r. It is known that an operator P is m-accretive iff A is maximally monotone in the setting of Hilbert spaces. For m-accretive operator P, one defines a single-valued operator $Res_r^P : Ran(Id + rP) \to Dom(P)$ by $Res_r^P = (Id + rP)^{-1}$, which is nonexpansive. Indeed, it is also firmly nonexpansive. And it is called the resolvent operator of P.

Let Q be a single-valued operator on E. Recall that Q is said to be η-inverse strongly accretive iff there exist some $\mathfrak{j}_q(x - y) \in \mathfrak{J}_q(x - y)$ and some $\zeta > 0$ such that

$$\langle Qx - Qy, \mathfrak{j}_q(x - y)\rangle \geq \eta\|Qx - Qy\|^q, \quad \forall x, y \in C.$$

From the definition, one sees that each η-inverse strongly accretive operator is accretive.

The forward-backward splitting method, introduced by Peaceman and Rachford [15] and Douglas and Rachford [9], is powerful to investigate accretive operators. Recently, many new splitting schemes were introduced in Hilbert spaces, however, there few associated results in Banach spaces; see, e.g., [7, 6, 8, 23]. In this paper, we study a viscosity splitting scheme for common fixed points of an infinite family of nonexpansive mappings and zero points of $P+Q$, the sum of an m-accretive operator P

and an inverse-strong accretive operator Q. We present a strong convergence theorems in the framework of uniformly convex and q-uniformly smooth Banach spaces.

The following lemmas are essential to our main convergence theorem.

Lemma 1.1. [21] *The following inequality holds true in a real q-uniformly smooth Banach space $\|x+y\|^q \leq q\langle y, \mathfrak{J}_q(x)\rangle + \|x\|^q + K_q\|y\|^q$, $\forall x, y \in E$, where K_q is the smooth constant.*

Lemma 1.2. [17] *Let E be a strictly convex Banach space and q-uniformly smooth Banach space. Let N be a nonexpansive mapping. Let P be an m-accretive operator, and let Q be an η-inverse strongly accretive operator. Then*

$$\mathrm{Fix}((Id + rP)^{-1}(Id - rQ)) = (P+Q)^{-1}(0).$$

Define a mapping M by $Mx = \varsigma Nx + (1-\varsigma)Res_r^P(x - rQx)$, where r is a real number in $(0, (\frac{q\eta}{K_q})^{\frac{1}{q-1}})$ and ς is a real number in $(0,1)$. Then M is nonexpansive and

$$\mathrm{Fix}(M) = \mathrm{Fix}(N) \cap (P+Q)^{-1}(0).$$

Lemma 1.3. [1] *Let P be an m-accretive operator on a Banach space E. For $\lambda > 0$ and $\mu > 0$,*

$$Res_\mu^P\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)Res_\lambda^P x\Big) = Res_\lambda^P x, \quad \forall x \in E.$$

Lemma 1.4. [24] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be nonnegative real sequences with*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \ \forall n \geq 0,$$

where $\{t_n\}$ is a real sequence in $(0,1)$. If

$$\sum_{n=0}^{\infty} c_n < \infty, \ \sum_{n=0}^{\infty} t_n = \infty, \ \text{and} \ \limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5. [17] *Let p is real number greater than 1, and let r be some positive real number. In a uniformly convex Banach space E, the following inequality hold true*

$$\|ax + by + cz\|^p \leq a\|x\|^p + b\|y\|^p + c\|z\|^p - \frac{a^p b + b^p a}{(a+b)^p}\varphi(\|x-y\|),$$

where $x, y, z \in \{x \in E : \|x\| \leq r\}$, $a, b, c \in [0,1]$ such that $a+b+c = 1$, $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ is some strictly increasing continuous convex function.

Lemma 1.6. [12] *Let H be a real Hilbert space, and let C be a closed, convex, and nonempty subset of H. Let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with*

$$\cap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset.$$

Then $\lim_{n\to\infty} \sup_{x\in C} \|Wx - W_n x\| = 0$.

Lemma 1.7. [22] *Let E a uniformly convex Banach space. Let f be a contractive mapping, and let M be a nonexpansive mapping with fixed points. Let $Proj_{\mathrm{Fix}(M)}^{E}$ is the unique sunny nonexpansive retraction from E onto $\mathrm{Fix}(M)$. For each τ in $(0,1)$, let x_τ be the unique solution to $x_\tau = (1-\tau)Mx_\tau + \tau f(x_\tau)$. Then $\{x_\tau\}$ converges strongly to a fixed point $\bar{x} = Proj_{\mathrm{Fix}(M)}^{E} f(\bar{x})$ as $\tau \to 0$.*

Lemma 1.8. [14] *Let $q > 1$. Then the following inequality holds:*

$$(ab)q \le (q-1)b^{\frac{q}{q-1}} + a^q,$$

for arbitrary positive real numbers a and b.

2. Main results

Theorem 2.1. *Let E be a real uniformly convex and q-uniformly smooth Banach space with smooth constant K_q. Let f be a contractive mapping on E with the contractive coefficient $\kappa \in (0,1)$. Let P be an m-accretive operator, and let Q be a η-inverse strongly accretive operator. Let $N_1, N_2, \cdots N_i, \cdots$ be nonexpansive mappings with a common fixed point. Let $\{x_n\}$ be a sequence generated via the following scheme*

$$\begin{cases} x_1 \in E, \ z_n = (1-\delta_n)W_n x_n + \delta_n x_n, \\ x_{n+1} = \alpha_n f(z_n) + \beta_n z_n + \gamma_n Res_{r_n}^{P}(x_n + y_n - r_n Q x_n), \quad \forall n \ge 1, \end{cases}$$

where $\{W_n\}$ is defined by (1.1), $\{y_n\}$ is a sequence in E such that

$$\sum_{n=1}^{\infty} \|y_n\| < \infty,$$

$\{r_n\}$ is a real sequence with

$$0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < \left(\frac{q\eta}{K_q}\right)^{\frac{1}{q-1}}, \ \text{and} \ \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty,$$

$\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are four real sequences satisfying the condition:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty, \ \sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty,$$

$$\liminf_{n\to\infty} \beta_n\gamma_n > 0, \ \sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty, \ \sum_{n=1}^{\infty} |\delta_n - \delta_{n+1}| < \infty,$$

and $\alpha_n + \beta_n + \gamma_n = 1$. If

$$\cap_{i=1}^{\infty} \mathrm{Fix}(N_i) \cap (P+Q)^{-1}(0)$$

is nonempty, then the sequence $\{x_n\}$ generated above converges strongly to

$$\bar{x} = Proj_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(P+Q)^{-1}(0)}^{E} f(\bar{x}),$$

where $\bar{x} = Proj_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(P+Q)^{-1}(0)}^{E}$ is the unique sunny nonexpansive retraction of E onto $\cap_{i=1}^{\infty}\mathrm{Fix}(N_i) \cap (P+Q)^{-1}(0)$.

Proof. From the celebrated inequality, Lemma 1.1, we find that for all $x, x' \in E$

$$\begin{aligned}
&\|(Id - r_nQ)x' - (Id - r_nQ)x\|^q \\
&\leq r_n^q K_q \|Qx' - Qx\|^q + \|x' - x\|^q - r_n q \langle Qx - Qy, \mathfrak{J}_q(x' - x)\rangle \\
&\leq \|x' - x\|^q + r_n(r_n^{q-1}K_q - \eta q)\|Qx' - Qx\|^q, \forall x, x' \in E.
\end{aligned}$$

In view of the restriction on $\{r_n\}$, one sees that $Id - r_nQ$ is nonexpansive for all n, that is,

$$\|(Id - r_nQ)x' - (Id - r_nQ)x\| \leq \|x - y\|, \forall x', x \in E.$$

Fix a point in $\cap_{i=1}^{\infty} \text{Fix}(N_i) \cap (P + Q)^{-1}(0)$, say x. One easily sees that

$$x = (I + r_nP)^{-1}(I - r_nQ) = W_n x,$$

where $n \geq 1$. Observe

$$\begin{aligned}
\|x - z_n\| &= \|(1 - \delta_n)(x - W_n x_n) + \delta_n(x - x_n)\| \\
&\leq (1 - \delta_n)\|W_n x - W_n x_n\| + \delta_n\|x - x_n\| \\
&\leq \|x - x_n\|,
\end{aligned}$$

that implies

$$\begin{aligned}
&\|x - x_{n+1}\| \\
&\leq \alpha_n\|x - f(z_n)\| + \beta_n\|x - z_n\| + \gamma_n\|Res_{r_n}^P(x - r_nQx) - Res_{r_n}^P(x_n + y_n - r_nQx_n)\| \\
&\leq \alpha_n\|x - f(x)\| + \alpha_n\|f(z_n) - f(x)\| + \beta_n\|x - z_n\| \\
&\quad + \gamma_n\|Res_{r_n}^P(x - r_nQx) - Res_{r_n}^P(x_n + y_n - r_nQx_n)\| \\
&\leq \alpha_n\|x - f(x)\| + (\alpha_n\kappa + \beta_n)\|x - z_n\| + \gamma_n(\|(x - r_nQx) - (x_n - r_nQx_n)\| + \|y_n\|) \\
&\leq \alpha_n\|x - f(x)\| + (\alpha_n\kappa + \beta_n + \gamma_n)\|x - x_n\| + \gamma_n\|y_n\| \\
&\leq \alpha_n(1 - \kappa)\frac{\|x - f(x)\|}{1 - \kappa} + (1 - \alpha_n(1 - \kappa))\|x - x_n\| + \|y_n\|.
\end{aligned}$$

Using the mathematical induction, one asserts that

$$\|x - x_{n+1}\| \leq \max\left\{\|x - x_1\|, \frac{\|x - f(x)\|}{1 - \kappa}\right\} + \sum_{n=1}^{\infty}\|y_n\| < \infty.$$

This demonstrates that $\{x_n\}$ is a bounded vector sequence. Hence, $\{z_n\}$ is also a bounded vector sequence, which is due to the expansivity of each N_i. Using Lemma 1.2, one has

$$\text{Fix}(Z) = \cap_{i=1}^{\infty}\text{Fix}(N_i) \cap (P + Q)^{-1}(0),$$

where $Z = (1 - \vartheta)W + \vartheta Res_r^P(I - rQ)$ with $\vartheta \in (0, 1)$ and $r > 0$ is nonexpansive. Next, one demonstrates

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(\bar{x} - x_n)\rangle \geq 0,$$

where $\bar{x} = Proj^E_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(P+Q)^{-1}(0)} f(\bar{x})$ with $Proj^E_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(P+Q)^{-1}(0)}$ being the sunny nonexpansive retractction. Setting $\lambda_n = x_n + y_n - r_n Q x_n$, we find

$$\begin{aligned}\|\lambda_n - \lambda_{n+1}\| &\leq \|(x_n + y_{n+1} - r_{n+1}Qx_n) - (x_{n+1} + y_{n+1} - r_{n+1}Qx_{n+1})\| \\ &\quad + \|(x_n + y_{n+1} - r_{n+1}Qx_n) - (x_n + y_n - r_n Q x_n)\| \\ &\leq \|y_{n+1}\| + \|y_n\| + |r_{n+1} - r_n|\|Qx_n\| + \|x_{n+1} - x_n\|.\end{aligned}$$

In view of the resolvent equality in Lemma 1.3, we have

$$\begin{aligned}&\|Res^P_{r_{n+1}}\lambda_{n+1} - Res^P_{r_n}\lambda_n\| \\ &= \|Res^P_{r_n}\lambda_n - Res^P_{r_n}\big(\frac{r_{n+1}-r_n}{r_{n+1}}Res^P_{r_{n+1}}\lambda_{n+1} + \frac{r_n}{r_{n+1}}\lambda_{n+1}\big)\| \\ &\leq \|\lambda_n - \big((1-\frac{r_n}{r_{n+1}})Res^P_{r_{n+1}}\lambda_{n+1} + \frac{r_n}{r_{n+1}}\lambda_{n+1}\big)\| \\ &\leq \|\lambda_n - \lambda_{n+1}\| + \frac{|r_n - r_{n+1}|\|\lambda_{n+1} - Res^P_{r_n}\lambda_{n+1}\|}{r_{n+1}} \\ &\leq \|y_{n+1}\| + \|y_n\| + |r_{n+1} - r_n|\|Qx_n\| + \|x_{n+1} - x_n\| \\ &+ \frac{|r_n - r_{n+1}|\|\lambda_{n+1} - Res^P_{r_n}\lambda_{n+1}\|}{r_{n+1}}.\end{aligned} \tag{2.1}$$

Observe that

$$\begin{aligned}z_{n+1} - z_n &= (1-\delta_{n+1})(W_{n+1}x_{n+1} - W_n x_n) + (W_n x_n - x_n)(\delta_n - \delta_{n+1}) \\ &\quad + \delta_{n+1}(x_{n+1} - x_n).\end{aligned} \tag{2.2}$$

From the nonexpansivity of each N_i, one has

$$\begin{aligned}\|W_n x_n - W_{n+1}x_n\| &= \zeta_1\|N_1 V_{n,2}x_n - N_1 V_{n+1,2}x_n\| \\ &\leq \zeta_1\|V_{n,2}x_n - V_{n+1,2}x_n\| \\ &= \zeta_1\|V_{n,2}x_n - V_{n+1,2}x_n\| \\ &= \zeta_1\zeta_2\|N_2 V_{n,3}x_n - N_2 V_{n+1,3}x_n\| \\ &\leq \zeta_1\zeta_2\|V_{n,3}x_n - V_{n+1,3}x_n\| \\ &\leq \cdots \\ &\leq \zeta_1\zeta_2\cdots\zeta_n\|V_{n,n+1}x_n - V_{n+1,n+1}x_n\| \\ &\leq M\prod_{i=1}^{n}\zeta_i,\end{aligned} \tag{2.3}$$

where M is an appropriate constant. Combining (2.2) and (2.3), one arrives at

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq (1-\delta_{n+1})\|W_{n+1}x_{n+1} - W_n x_n\| + \|W_n x_n - x_n\||\delta_n - \delta_{n+1}| \\ &\quad + \delta_{n+1}\|x_{n+1} - x_n\| \\ &\leq \|x_{n+1} - x_n\| + (1-\delta_{n+1})\|W_{n+1}x_n - W_n x_n\| \\ &\quad + \|W_n x_n - x_n\||\delta_n - \delta_{n+1}| \\ &\leq \|x_{n+1} - x_n\| + (1-\delta_{n+1})M\prod_{i=1}^{n}\zeta_i + \|W_n x_n - x_n\||\delta_n - \delta_{n+1}|.\end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\le \|f(z_n)\||\alpha_{n+1}-\alpha_n| + \alpha_{n+1}\|f(z_{n+1})-f(z_n)\| \\
&\quad + \|z_n\||\beta_{n+1}-\beta_n| + \beta_{n+1}\|z_{n+1}-z_n\| \\
&\quad + \|Res_{r_n}^P\lambda_n\||\gamma_{n+1}-\gamma_n| + \gamma_{n+1}\|Res_{r_n}^P\lambda_n - Res_{r_{n+1}}^P\lambda_{n+1}\| \\
&\le \|f(z_n)\||\alpha_{n+1}-\alpha_n| + (\alpha_{n+1}\kappa+\beta_{n+1})\|z_{n+1}-z_n\| + \|z_n\||\beta_{n+1}-\beta_n| \\
&\quad + \|Res_{r_n}^P\lambda_n\||\gamma_{n+1}-\gamma_n| + \gamma_{n+1}\|Res_{r_n}^P\lambda_n - Res_{r_{n+1}}^P\lambda_{n+1}\|.
\end{aligned} \tag{2.5}$$

Substituting (2.1) and (2.4) into (2.5) yields that

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\le \|f(z_n)\||\alpha_{n+1}-\alpha_n| + (\alpha_{n+1}\kappa+\beta_{n+1})\|z_{n+1}-z_n\| + \|z_n\||\beta_{n+1}-\beta_n| \\
&\quad + \|Res_{r_n}^P\lambda_n\||\gamma_{n+1}-\gamma_n| + \gamma_{n+1}\|Res_{r_n}^P\lambda_n - Res_{r_{n+1}}^P\lambda_{n+1}\| \\
&\le \|f(z_n)\||\alpha_{n+1}-\alpha_n| + (1-\alpha_{n+1}(1-\kappa))\|x_{n+1}-x_n\| \\
&\quad + M\prod_{i=1}^n \zeta_i + \|W_nx_n - x_n\||\delta_n-\delta_{n+1}| + \|z_n\||\beta_{n+1}-\beta_n| \\
&\quad + \|Res_{r_n}^P\lambda_n\||\gamma_{n+1}-\gamma_n| + \|y_{n+1}\| + \|y_n\| + |r_{n+1}-r_n|\|Qx_n\| \\
&\quad + \frac{|r_n-r_{n+1}|\|\lambda_{n+1}-Res_{r_n}^P\lambda_{n+1}\|}{r_{n+1}}
\end{aligned} \tag{2.6}$$

From

$$\sum_{n=1}^\infty |\alpha_{n+1}-\alpha_n| < \infty,\ \sum_{n=1}^\infty |\beta_{n+1}-\beta_n| < \infty,\ \sum_{n=1}^\infty |\gamma_{n+1}-\gamma_n| < \infty,$$

$$\sum_{n=1}^\infty |\delta_{n+1}-\delta_n| < \infty,\ \sum_{n=1}^\infty |r_{n+1}-r_n| < \infty,\ \lim_{n\to\infty}\alpha_n = 0,\ \text{and}\ \sum_{n=1}^\infty \alpha_n = \infty,$$

one obtains from Lemma 1.4 that $\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\| = 0$. Observe that

$$\begin{aligned}
\|x - Res_{r_n}^P\lambda_n\|^2 &\le \|(x-r_nQx) - (x_n+y_n-r_nQx_n)\|^2 \\
&\le \|x-x_n\|^2 + \|y_n\|(\|y\| + 2\|(x-r_nQx)-(x_n-r_nQx_n)\|).
\end{aligned} \tag{2.7}$$

Setting $p=2$ in Lemma 1.5 yields

$$\begin{aligned}
\|x-x_{n+1}\|^2 \le \alpha_n\|x-f(z_n)\|^2 + \beta_n\|x-z_n\|^2 + \gamma_n\|x-Res_{r_n}^P\lambda_n\|^2 \\
- \beta_n\gamma_n\varphi(\|z_n - Res_{r_n}^P\lambda_n\|).
\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), one has

$$\begin{aligned}
&\|x-x_{n+1}\|^2 \\
&\le \alpha_n\|x-f(z_n)\|^2 + \|x-x_n\|^+\|y_n\|(\|y\| + 2\|(x-r_nQx)-(x_n-r_nQx_n)\|)2 \\
&\quad - \beta_n\gamma_n\varphi(\|z_n - Res_{r_n}^P\lambda_n\|),
\end{aligned}$$

which in turn implies

$$\beta_n\gamma_n\varphi(\|z_n - Res_{r_n}^P\lambda_n\|) \le \alpha_n\|x - f(z_n)\|^2 + (\|x - x_n\| + \|x - x_{n+1}\|)\|x_n - x_{n+1}\| + \|y_n\|(\|y\| + 2\|(x - r_nQx) - (x_n - r_nQx_n)\|),$$

In view of the condition $\liminf_{n\to\infty}\beta_n\gamma_n > 0$, one has

$$\lim_{n\to\infty}\varphi(\|z_n - Res_{r_n}^P\lambda_n\|) = 0.$$

Moreover, from the fact that φ is a strictly increasing convex continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, one also has

$$\lim_{n\to\infty}\|z_n - Res_{r_n}^P\lambda_n\| = 0. \tag{2.9}$$

Next, without loss of generality, one assumes that there exists a positive number ξ such that $r_n \ge \xi$. From the accretiveness of P, one has

$$\begin{aligned}&\|Res_\xi^P(x_n - \xi Qx_n) - Res_{r_n}^P(x_n - r_nQx_n)\|^q\\ &\le \frac{\langle x_n - Res_{r_n}^P(x_n - r_nQx_n), \mathfrak{J}_q\big(Res_\xi^P(x_n - \xi Qx_n) - Res_{r_n}^P(x_n - r_nQx_n)\big)\rangle|\xi - r_n|}{r_n}\\ &\le \|x_n - Res_{r_n}^P(x_n - r_nQx_n)\|\|Res_{r_n}^P(x_n - r_nQx_n) - R_\xi^P(x_n - \xi Qx_n)\|^{q-1},\end{aligned}$$

which demonstrates that

$$\|Res_{r_n}^P(x_n - r_nQx_n) - x_n\| \ge \|Res_{r_n}^P(x_n - r_nQx_n) - Res_\xi^P(x_n - \xi Qx_n)\|. \tag{2.10}$$

From the fact that

$$\|Res_{r_n}^P\lambda_n - x_n\| \le \|x_{n+1} - x_n\| + \alpha_n\|f(z_n) - Res_{r_n}^P\lambda_n\| + \beta_n\|z_n - Res_{r_n}^P\lambda_n\|,$$

one obtains from (2.9) that

$$\lim_{n\to\infty}\|Res_{r_n}^P\lambda_n - x_n\| = 0. \tag{2.11}$$

Observe that

$$\begin{aligned}\|Res_{r_n}^P(x_n - r_nQx_n) - x_n\| &\le \|Res_{r_n}^P\lambda_n - x_n\| + \|Res_{r_n}^P\lambda_n - Res_{r_n}^P(x_n - r_nQx_n)\|\\ &\le \|Res_{r_n}^P\lambda_n - x_n\| + \|\lambda_n - (x_n - r_nQx_n)\|\\ &\le \|Res_{r_n}^P\lambda_n - x_n\| + \|y_n\|.\end{aligned}$$

This implies from (2.11) that

$$\lim_{n\to\infty}\|Res_{r_n}^P(x_n - r_nQx_n) - x_n\| = 0, \tag{2.12}$$

On the other hand,

$$\begin{aligned}&\|x_n - Res_\xi^P(x_n - \xi Qx_n)\|\\ &\le \|Res_{r_n}^P(x_n - r_nQx_n) - x_n\| + \|Res_\xi^P(x_n - \xi Qx_n) - Res_{r_n}^P(x_n - r_nQx_n)\|,\end{aligned}$$

which together with (2.10) and (2.12) yields

$$\lim_{n\to\infty}\|Res_\xi(x_n - \xi Qx_n) - x_n\| = 0. \tag{2.13}$$

This also implies that $z_n - x_n \to 0$ as $n \to \infty$. Hence, $W_nx_n - x_n \to 0$ as $n \to \infty$. Note that $\|Wx_n - x_n\| \le \|Wx_n - W_nx_n\| + \|W_nx_n - x_n\|$. One asserts from Lemma 1.6 that

$Wx_n - x_n \to 0$ as $n \to \infty$. Fix $\varrho \in (0,1)$ and set $M = (1-\varrho)W + \varrho Res_{\xi}^{P}(I - \xi Q)$. Using Lemma 1.2 reaches the situation that M becomes a nonexpansive mapping with $\mathrm{Fix}(M) = \cap_{i=1}^{\infty}\mathrm{Fix}(N_i) \cap (P+Q)^{-1}(0)$. From the construction of M, one has $Mx_n - x_n \to 0$ as $n \to \infty$. The nonexpansivity of M and the contractivity of f guarantee that $(1-\tau)M + \tau f$ is contractive. Next, the unique fixed point of $(1-\tau)M + \tau f$ is denoted by x^{τ}. Let $\bar{x} = \lim_{t\to 0} x^{\tau}$. It follows from Lemma 1.7 that

$$\bar{x} = Proj_{\mathrm{Fix}(M)}^{E} f(\bar{x}) = Proj_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(P+Q)^{-1}(0)}^{E} f(\bar{x}),$$

where $Proj_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(P+Q)^{-1}(0)}^{E}$ is the unique sunny nonexpansive retraction of E onto $\cap_{i=1}^{\infty}\mathrm{Fix}(N_i) \cap (P+Q)^{-1}(0)$. Observe that

$$\begin{aligned}
\|x^{\tau} - x_n\|^q &= (1-\tau)\langle Mx^{\eta} - x_n, \mathfrak{J}_q(x^{\eta} - x_n)\rangle + \eta\langle f(x^{\tau}) - x_n, \mathfrak{J}_q(x^{\eta} - x_n)\rangle \\
&= (1-\tau)\Big(\langle Mx_n - x_n, \mathfrak{J}_q(x^{\eta} - x_n)\rangle + \langle Mx^{\tau} - Mx_n, \mathfrak{J}_q(x^{\tau} - x_n)\rangle\Big) \\
&\quad + \tau\Big(\langle x^{\tau} - x_n, \mathfrak{J}_q(x^{\tau} - x_n)\rangle + \langle f(x^{\tau}) - x^{\tau}, \mathfrak{J}_q(x^{\tau} - x_n)\rangle\Big) \\
&\le (1-\tau)\|Mx_n - x_n\|\|x_n - x^{\tau}\|^{q-1} + (1-\tau)\|x^{\tau} - x_n\|\|x^{\tau} - x_n\|^{q-1} \\
&\quad + \tau\|x^{\tau} - x_n\|^q + \tau\langle f(x^{\tau}) - x^{\eta}, \mathfrak{J}_q(x^{\tau} - x_n)\rangle \\
&\le \tau\langle f(x^{\tau}) - x^{\eta}, \mathfrak{J}_q(x^{\tau} - x_n)\rangle + \|x_n - x^{\tau}\|^q + \|x^{\tau} - x_n\|^{q-1}\|x_n - Mx_n\|,
\end{aligned}$$

which demonstrates

$$\langle x^{\tau} - f(x^{\eta}), \mathfrak{J}_q(x^{\tau} - x_n)\rangle \le \frac{\|x_n - Mx_n\|\|x^{\tau} - x_n\|^{q-1}}{\tau}.$$

Fix τ. By letting $n \to \infty$, one has

$$\limsup_{n\to\infty}\langle x^{\tau} - f(x^{\tau}), \mathfrak{J}_q(x^{\tau} - x_n)\rangle \le 0.$$

Hence, $\limsup_{n\to\infty}\langle \bar{x} - f(\bar{x}), \mathfrak{J}_q(\bar{x} - x_n)\rangle \le 0$. In view of the definition of W_n, we obtain that

$$\|z_n - \bar{x}\| \le (1-\delta_n)\|W_n x_n - \bar{x}\| + \delta_n\|x_n - \bar{x}\| \le \|x_n - \bar{x}\|.$$

Hence,

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^q \\
&\le \alpha_n\langle f(z_n) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle + \beta_n\langle z_n - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle \\
&\quad + \gamma_n\langle Res_{r_n}^{P}(x_n + y_n - r_n Qx_n) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle \\
&\le \alpha_n\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle + (\kappa\alpha_n + \beta_n)\|z_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1} \\
&\quad + \gamma_n\|(x_n + y_n - r_n Qx_n) - (\bar{x} - r_n Q\bar{x})\|\|x_{n+1} - \bar{x}\|^{q-1} \\
&\le \alpha_n\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle + (\kappa\alpha_n + \beta_n)\|z_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1} \\
&\quad + \gamma_n\|(x_n - r_n Qx_n) - (\bar{x} - r_n Q\bar{x})\|\|x_{n+1} - \bar{x}\|^{q-1} + \gamma_n\|y_n\|\|x_{n+1} - \bar{x}\|^{q-1} \\
&\le \alpha_n\langle f(\bar{x}) - \bar{x}, \mathfrak{J}_q(x_{n+1} - \bar{x})\rangle + (1 - \alpha_n(1-\kappa))\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1} \\
&\quad + \|x_{n+1} - \bar{x}\|^{q-1}\|y_n\|.
\end{aligned}$$

Using the inequality in Lemma 1.8 yields that

$$\|x_{n+1}-\bar{x}\|^q \leq q\alpha_n\langle f(\bar{x})-\bar{x}, \mathfrak{J}_q(x_{n+1}-\bar{x})\rangle + (1-\alpha_n(1-\kappa))\|x_n-\bar{x}\|^q + q\|x_{n+1}-\bar{x}\|^{q-1}\|y_n\|.$$

Using Lemma 1.4, one concludes $x_n \to \bar{x}$ as $n \to \infty$ immediately. □

From Theorem 2.1, the following result can be derived immediately.

Corollary 2.1. *Let E be a real uniformly convex and q-uniformly smooth Banach space with smooth constant K_q. Let f be a contractive mapping on E with the contractive coefficient $\kappa \in (0,1)$. Let P be an m-accretive operator, and let Q be a η-inverse strongly accretive operator. Let $\{x_n\}$ be a sequence generated via the following scheme*

$$x_1 \in E,\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Res^P_{r_n}(x_n + y_n - r_n Q x_n), \quad \forall n \geq 1,$$

where $\{y_n\}$ is a sequence in E such that

$$\sum_{n=1}^{\infty}\|y_n\| < \infty,$$

$\{r_n\}$ is a real sequence with

$$0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \left(\frac{q\eta}{K_q}\right)^{\frac{1}{q-1}},\ \text{and} \sum_{n=1}^{\infty}|r_n - r_{n+1}| < \infty,$$

$\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences satisfying the condition:

$$\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=1}^{\infty}\alpha_n = \infty,\ \sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty,\ \sum_{n=1}^{\infty}|\beta_n - \beta_{n+1}| < \infty,$$

$$\liminf_{n\to\infty}\beta_n\gamma_n > 0,\ \sum_{n=1}^{\infty}|\gamma_n - \gamma_{n+1}| < \infty,\ \sum_{n=1}^{\infty}|\delta_n - \delta_{n+1}| < \infty,$$

and $\alpha_n + \beta_n + \gamma_n = 1$. If $(P+Q)^{-1}(0)$ is nonempty, then the sequence $\{x_n\}$ generated above converges strongly to $\bar{x} = Proj^E_{(P+Q)^{-1}(0)} f(\bar{x})$, where $\bar{x} = Proj^E_{(P+Q)^{-1}(0)}$ is the unique sunny nonexpansive retraction of E onto $(P+Q)^{-1}(0)$.

Remark 2.1. The framework of the spaces in the theorem and corollary above is applicable to L_p, where $1 < p < \infty$.

Finally, we apply our iterative algorithm to a convex minimization problem. In the setting of Hilbert spaces, the class of accretive operators coincides with the class of maximally monotone operators. Let $\omega : H \to (-\infty, \infty]$, where H denotes a Hilbert space, be a proper, lower semicontinuous, and convex function. Recall that the subdifferential mapping, $\partial\omega$, of ω is defined by

$$\partial\omega(x) = \{x^* \in H : \omega(x) + \langle y - x, x^*\rangle \leq \omega(y), \forall y \in H\}, \quad \forall x \in H.$$

From Rockafellar [18], one sees that the subdifferential mapping, $\partial\omega$, is a maximal monotone operator. One also easily verify $\omega(v) = \min_{x\in H}\omega(x)$ if and only if $0 \in \partial\omega(v)$.

Theorem 2.2. *Let $\omega : H \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function with $(\partial\omega)^{-1}(0) \neq \emptyset$. Let f be a contractive mapping on H the contractive coefficient $\kappa \in (0,1)$. Let $N_1, N_2, \cdots N_i, \cdots$ be nonexpansive mappings with a common fixed point. Let $\{x_n\}$ be a sequence generated via the following scheme*

$$\begin{cases} x_1 \in H, \ z_n = (1-\delta_n)W_n x_n + \delta_n x_n, \\ \xi_n = \arg\min_{z\in H}\{\omega(z) + \frac{\|z-x_n-y_n\|^2}{2r_n}\}, \\ x_{n+1} = \alpha_n f(z_n) + \beta_n z_n + \gamma_n \xi_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is defined by (1.1), $\{y_n\}$ is a sequence in H such that

$$\sum_{n=1}^{\infty} \|y_n\| < \infty,$$

$\{r_n\}$ $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are real sequences satisfying the condition:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty, \ \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty, \ \sum_{n=0}^{\infty} |\beta_n - \beta_{n+1}| < \infty,$$

$$\liminf_{n\to\infty} \beta_n\gamma_n > 0, \ \sum_{n=0}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty, \ \sum_{n=0}^{\infty} |\delta_n - \delta_{n+1}| < \infty,$$

and $\alpha_n + \beta_n + \gamma_n = 1$. If $\cap_{i=1}^{\infty}\mathrm{Fix}(N_i) \cap (\partial\omega)^{-1}(0)$ is nonempty, then the sequence $\{x_n\}$ generated above converges strongly to $\bar{x} = Proj^H_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(\partial\omega)^{-1}(0)} f(\bar{x})$, where $\bar{x} = Proj^H_{\cap_{i=1}^{\infty}\mathrm{Fix}(N_i)\cap(\partial\omega)^{-1}(0)}$ is the nearest point projection of H onto

$$\cap_{i=1}^{\infty}\mathrm{Fix}(N_i) \cap (\partial\omega)^{-1}(0).$$

Proof. Set $Q = 0$. From the fact that ω is proper, convex, and lower semicontinuous function, one sees that the subdifferential, $\partial\omega$, is maximal monotone. Observe that

$$\xi_n = \arg\min_{z\in H}\left\{\frac{\|z-x_n-y_n\|^2}{2r_n} + \omega(z)\right\}$$

is equivalent to

$$0 \in \frac{1}{r_n}(\xi_n - x_n - y_n) + \partial\omega(\xi_n).$$

It follows that

$$x_n + y_n \in \xi_n + r_n\partial\omega(\xi_n).$$

From Theorem 2.1, we obtain the desired conclusion immediately. □

If $N_i = Id$, the identity mapping, then the above theorem is reduced to the following.

Corollary 2.2. *Let $\omega : H \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function with $(\partial\omega)^{-1}(0) \neq \emptyset$. Let f be a contractive mapping on H the contractive coefficient $\kappa \in (0,1)$. Let $\{x_n\}$ be a sequence generated via the following scheme*

$$\begin{cases} x_1 \in H, \ \xi_n = \arg\min_{z\in H}\{\omega(z) + \frac{\|z-x_n-y_n\|^2}{2r_n}\}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \xi_n, \quad \forall n \geq 1, \end{cases}$$

where $\{y_n\}$ *is a sequence in* H *such that*

$$\sum_{n=1}^{\infty} \|y_n\| < \infty,$$

$\{r_n\}$ $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$, *are real sequences satisfying the condition:*

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

$$\sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty,\ \liminf_{n\to\infty} \beta_n\gamma_n > 0,\ \sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty,$$

and $\alpha_n + \beta_n + \gamma_n = 1$. *Then the sequence* $\{x_n\}$ *generated above converges strongly to* $\bar{x} = Proj^H_{(\partial\omega)^{-1}(0)} f(\bar{x})$, *where* $\bar{x} = Proj^H_{(\partial\omega)^{-1}(0)}$ *is the nearest point projection of* H *onto* $(\partial\omega)^{-1}(0)$.

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Received: September 2, 2022; Accepted: December 8, 2022.