Fixed Point Theory, 24(2023), No. 2, 611-626
DOI: 10.24193/fpt-ro.2023.2.10
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

OSCILLATION AND NONOSCILLATION FOR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

NASSIM GUERRAICHE*, SAMIRA HAMANI** AND JOHNNY HENDERSON***

*Université Constantine 2
Département d'Informatique Fondamentale et ses Applications
Nouvelle ville Ali Mendjeli, BP: 67A, Constantine, ALGÉRIE
E-mail: nassim.guerraiche@univ-constantine2.dz

**Laboratoire des Mathématiques Appliqués et Pures, Université de Mostaganem
B.P. 227, 27000, Mostaganem, ALGÉRIE
E-mail: hamani samira@yahoo.fr

***Department of Mathematics, Baylor University
Waco, Texas. 76798-7328, USA
E-mail:johnny_henderson@baylor.edu

Abstract. For $r \in (1, 2]$, we establish sufficient conditions for the existence of oscillatory and nonoscillatory solutions to a boundary value problem for an rth order Caputo-Hadamard fractional differential inclusion in a Banach space. Our approach is based upon the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness.
Key Words and Phrases: Existence, oscillatory, nonoscillatory, fractional differential inclusions, Caputo-Hadamard type derivative, fixed point, measure of noncompactness.
2020 Mathematics Subject Classification: 34A08, 26A33, 47H10.

1. Introduction

This paper deals with the existence of oscillatory and nonoscillatory solutions to boundary value problems (BVP for short) for fractional differential inclusions. In particular, we consider the boundary value problem

$$ {}^{H}_{C}D^r y(t) \in F(t, y(t)), \quad \text{for a.e. } \ t \in J = [1, T], 1 < r \leq 2, \tag{1.1} $$

$$ y(1) = y_1,\, y(T) = y_T \tag{1.2} $$

where ${}^{H}_{C}D^r$ is the Caputo-Hadamard fractional derivative, $(E, |\cdot|)$ is a Banach space, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $F : [1, T]\times E \to \mathcal{P}(E)$ is a multivalued map and $y_1, y_T \in E$.

Differential equations of fractional order are valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. In the monographs of Hilfer [25], Kilbas *et al.* [27], Podlubny [32], and

Momani *et al.* [30], we can find the background mathematics and various applications of fractional calculus. The existence of oscillatory and nonoscillatory solutions, to many different problems concerned the differential equations and inclusions, was studied by several researchers; as example of this researches, we refer the reader to the articles of Benchohra *et al* [1, 8, 9, 10], Graef *et al* [19] and Bohner *et al* [14].

The Caputo left-sided fractional derivative of order r, is defined by

$$({}^cD^r_{a+}h)(t) = \frac{1}{\Gamma(n-r)}\int_a^t (t-s)^{n-r-1}h^{(n)}(s)ds,$$

where $r > 0$, $n = [r] + 1$ and $[r]$ denotes the integer part of r. This derivative is very useful in many applied problems, because it satisfies its initial data which contains $y(0)$, $y'(0)$, etc., as well as the same data for boundary conditions.

The Hadamard fractional derivative was introduced by Hadamard in 1892 [23]. This derivative differs from the Caputo derivative in two ways; the first way is that its kernel contains a logarithmic function of arbitrary exponent, and the second way is that the Hadamard derivative of a constant does not equal to 0.

The Caputo-Hadamard fractional derivative given by Jarad *et al.* [26] is a modified Hadamard fractional derivative, but unlike the Hadamard fractional derivative, the Caputo-Hadamard fractional derivative of a constant is 0, which was inherited from the Caputo derivative.

In this paper, we present existence results for the problem (1.1)-(1.2), when the right hand side is convex valued, by using the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valued tool in solving fractional differential equations and inclusions in Banach spaces; for details, see the papers of Laosta *et al* [29], Rus *et al* [34, 33], Agarwal *et al.* [2] and Benchohra *et al.* [12], [11], [13]. This result extends to the multivalued case some of those previous results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $(E, |\cdot|)$ be a Banach space, and we set $C(J,E)$ as the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and $L^1(J,E)$ as the Banach space of Bohner integrable functions $y : J \longrightarrow E$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

The space $AC(J,E)$ is the space of functions $y : J \to E$ that are absolutely continuous.

Let $\delta = t\dfrac{d}{dt}$, and then we set

$$AC^n_\delta(J,E) = \{y : J \longrightarrow E, \delta^{n-1}y(t) \in AC(J,E)\},$$

and $AC^1(J,E)$ is the space of functions $y : J \to E$ that are absolutely continuous and have an absolutely continuous first derivative.

For any Banach space $(X, \|\cdot\|)$, we set

$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).
G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.
If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if:

(1) $t \to F(t, u)$ is measurable for each $u \in E$,
(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$

For each $y \in AC(J, E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1([1, T], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [1, T]\}$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. The function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$$

is known as the Hausdorff-Pompeiu metric.

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is called

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that
$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x, y \in X,$$
(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For more details on multivalued maps see the books of Aubin and Cellina [5], Aubin and Frankowska [6], Deimling [16] and Castaing and Valadier [15].

For convenience, we first recall the definitions of the Kuratowski measure of noncompactness and summarize the main properties of this measure.

Definition 2.3. [4, 7] Let E be a Banach space and let Ω_E be the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\zeta : \Omega_E \to [0, \infty)$ defined by

$$\zeta(B)=\inf\{\epsilon>0\,:B\subset\bigcup_{j=1}^{m}B_j \text{ and } diam(B_j)\leq\epsilon\}\,;\text{here } B\in\Omega_E.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [4, 7])

(1) $\zeta(B)=0\Leftrightarrow\overline{B}$ is compact (B is relatively compact).
(2) $\zeta(B)=\zeta(\overline{B})$.
(3) $A\subset B\Rightarrow\zeta(A)\leq\zeta(B)$.
(4) $\zeta(A+B)\leq\zeta(A)+\zeta(B)$.
(5) $\zeta(cB)=|c|\zeta(B), c\in\mathbb{R}$.
(6) $\zeta(conB)=\zeta(B)$.

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively.

For a given set V of functions $u:J\to E$, we set

$$V(t)=\{u(t):u\in V\},t\in J,$$

and

$$V(J)=\{u(t):u\in V(t),t\in J\}.$$

Theorem 2.4. [7] *Let E be a Banach space and $C\subset L^1(J,E)$ be countable with $\mid u(t)\mid\leq h(t)$ for a.e. $t\in J$ and every $u\in C$, where $h\in L^1(J,\mathbb{R}_+)$. Then the function $\phi(t)=\alpha(C(t))$ belong to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\zeta\left(\left\{\int_0^T u(s)ds, u\in C\right\}\right)\leq 2\int_0^T\zeta(C(s))ds.$$

Let us now recall Mönch's fixed point theorem.

Theorem 2.5. [31] *Let K be a closed, convex subset of a Banach space E, U a relatively open subset of K, and $N:\overline{U}\mapsto\mathcal{P}(K)$. Assume graphN is closed, N maps compact sets into relatively compact sets, and for some $x_0\in U$, the following two conditions are satisfied:*

$$\begin{array}{c} M\subset\overline{U}\;\; M\subset conv(x_0\cup N(M))\\ \overline{M}=\overline{U}\;\; \text{with}\;\; C\;\; \text{a countable subset of}\;\; M\;\; \text{implies}\\ \overline{M}\;\; \text{is compact,}\end{array}\tag{2.1}$$

$$x\notin(1-\lambda)x_0+\lambda N(x)\text{ foe all } x\in\overline{U}/U,\;\;\lambda\in(0,1).\tag{2.2}$$

Then there exists $x\in\overline{U}$ with $x\in N(x)$.

Definition 2.6. [27] The Hadamard fractional integral of order r for a function $h:[1,+\infty)\to\mathbb{R}$ is defined as

$$^{H}I^{r}h(t)=\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{h(s)}{s}ds,\;\;r>0,$$

provided the integral exists.

Definition 2.7. [27] For a function h given on the interval $[1,+\infty)$, the r Hadamard fractional-order derivative of h, is defined by

$$({}^{H}D^{r}h)(t)=\frac{1}{\Gamma(n-r)}\left(t\frac{d}{dt}\right)^{n}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{n-r-1}\frac{h(s)}{s}ds,$$

$n-1<r<n,\ n=[r]+1$, where $[r]$ denotes the integer part of r and $\log(\cdot)=\log_{e}(\cdot)$.

Definition 2.8. [26] For a given function h which belongs to $AC_{\delta}^{n}([a,b],E)$, such that $a>0$, we define the Caputo-type modification of the left-sided Hadamard fractional derivative by

$${}_{C}^{H}D^{r}y(t)={}^{H}D^{r}\left[y(s)-\sum_{k=0}^{n-1}\frac{\delta^{k}y(a)}{k!}\left(\log\frac{s}{a}\right)^{k}\right](t),$$

where $Re(r)\geq 0$ and $n=[Re(r)]+1$.

Lemma 2.9. [26] *Let y belong to $AC_{\delta}^{n}([a,b],E)$ or to C_{δ}^{n} and $r\in\mathbb{C}$. Then*

$${}^{H}I^{r}({}_{C}^{H}D^{r})y(t)=y(t)-\sum_{k=0}^{n-1}\frac{\delta^{k}y(a)}{k!}\left(\log\frac{t}{a}\right)^{k}$$

3. Main result

In this section, we study the existence of solutions to a boundary value problem (1.1)-(1.2).

Definition 3.1. A function $\alpha\in AC_{\delta}^{2}([1,T],E)$ is said to be a lower solution of (1.1)-(1.2), if there exists a function $v_1\in L^{1}([1,T],E)$ with $v_1(t)\in F(t,\alpha(t))$, for a.e $t\in[1,T]$, such that ${}_{C}^{H}D^{r}\alpha(t)\leq v_1(t)$ and the function α satisfies the conditions $\alpha(1)\leq y_1$ and $\alpha(T)\leq y_T$. Similarly a function $\beta\in AC_{\delta}^{2}([1,T],E)$ is said to be an upper solution of (1.1)-(1.2), if there exists a function $v_2\in L^{1}([1,T],E)$ with $v_2(t)\in F(t,\beta(t))$, for a.e $t\in[1,T]$, such that ${}_{C}^{H}D^{r}\beta(t)\geq v_2(t)$ and the function β satisfies the conditions $\beta(1)\geq y_1$ and $\beta(T)\geq y_T$.

Lemma 3.2. *Let h belong to $AC_{\delta}^{2}([1,T],E)$. For $r\in(1,2]$, A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{r-1}h(s)\frac{ds}{s} \\ &+ \frac{\log t}{\log T}\left[y_T-y_1-\frac{1}{\Gamma(r)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{r-1}h(s)\frac{ds}{s}\right]+y_1 \end{aligned} \tag{3.1}$$

if y is a solution of the nonlinear fractional boundary value problem

$${}_{C}^{H}D^{r}y(t)=h(t)\ \ \textit{for a.e.}\ \ t\in J=[1,T], \tag{3.2}$$

$$y(1)=y_1,\ y(T)=y_T \tag{3.3}$$

Proof. Applying the Hadamard fractional integral of order r to both sides of (3.2), and by using Lemma 2.9, we find

$$y(t) = c_1 + c_2 \log t + {}^H I^r h(t). \tag{3.4}$$

Then by using the conditions (3.3), we get

$$c_1 = y_1$$

and

$$y(T) = \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s) \frac{ds}{s} + y_1 + c_2(\log T).$$

Hence

$$c_2 = \frac{y_T - y_1 - \dfrac{1}{\Gamma(r)} \displaystyle\int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s) \frac{ds}{s}}{\log T}.$$

Finally, we obtain the solution (3.1). □

Theorem 3.3. *Assume that:*

(H1) $F : [1, T] \times E \to \mathcal{P}_{cp,c}(E)$ *is a Carathéodory multi-valued map.*

(H2) *There exists a function* $p \in C(J, E)$ *such that*

$$\|F(t, y)\|_{\mathcal{P}} = \sup\{|v| \ : \ v(t) \in F(t, y)\} \leq p(t)$$

(H3) *There exists* $l > 0$ *such that*

$$H_d(F(t, x), F(t, \bar{x})) \leq l|x - \bar{x}| \text{ for every } x, \bar{x} \in E$$

(H4) *For each bounded set* $B \subset C(J, E)$ *and for each* $t \in J$*, we have*

$$\zeta(F(t, B)) \leq p(t)\zeta(B),$$

where ζ *is a measure of noncompactness on* E*.*

(H5) *The function* $\varphi = 0$ *is the unique solution in* $C(J, [1, 2R])$ *of the inequality*

$$\begin{aligned} \varphi(t) \ &\leq \ \frac{2}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \zeta(M(s)) \frac{ds}{s} \\ &+ \ \frac{2 \log t}{\log T} \left[y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \psi(s, \zeta(M(s)) \frac{ds}{s} \right] \end{aligned}$$

for $t \in J$*.*

(H6) *There exists* α *and* $\beta \in AC^2_\delta([1, T], E)$*, lower and upper solutions for the problem (1.1)-(1.2) such that* $\alpha \leq \beta$*.*

Then the BVP (1.1)-(1.2) has at least one solution y *in* J*.*

Proof. We wish to transform the problem (1.1)-(1.2) into a fixed point problem. Consider the following modified problem

$${}^H_C D^r y(t) \in F(t, (\tau y)(t)), \quad \text{for a.e } \ t \in J = [1, T], 1 < r \leq 2, \tag{3.5}$$

$$y(1) = y_1, \, y(T) = y_T \tag{3.6}$$

where $\tau : C(J,E) \mapsto C(J,E)$ be the truncation operator defined by

$$(\tau y)(t) = \begin{cases} \alpha(t), & y(t) < \alpha(t), \\ y(t), & \alpha(t) \le y(t) \le \beta(t), \\ \beta(t), & y(t) > \beta(t) \end{cases}$$

A solution to (3.5)-(3.6) is a fixed point of the operator $N : C([1,T],E) \mapsto \mathcal{P}(C([1,T],E))$ defined by

$$N(y) = \left\{ h \in AC^2_\delta([1,T],E) : \begin{array}{l} h(t) = y_1 \\ \quad + \dfrac{\log t}{\log T}\left[y_T - y_1 - \dfrac{1}{\Gamma(r)}\displaystyle\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\frac{g(s)}{s}ds \right] \\ \quad + \dfrac{1}{\Gamma(r)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{g(s)}{s}ds, \quad g \in \hat{S}_{F,\tau y} \end{array} \right\}$$

such that

$$\hat{S}_{F,\tau y} = \{g \in S_{F,\tau y}, g(t) \ge v_1(t) \text{ a.e. on } A_1 \text{ and } g(t) \le v_2(t) \text{ a.e. on } A_2\},$$
$$S_{F,\tau y} = \{g \in L^1([1,T],E), g(t) \in F(t,(\tau y)(t)) \text{ for a.e. } t \in J\},$$
$$A_1 = \{t \in J, y(t) < \alpha(t) \le \beta(t)\}, \quad A_2 = \{t \in J, \alpha(t) \le \beta(t) < y(t)\}.$$

(It is clear that $\hat{S}_{F,\tau y}$ is nonempty, and this is for the reason that $S_{F,\tau y}$ is nonempty). We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in \hat{S}_{F,\tau y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{g_i(s)}{s}ds \\ &+ \frac{\log t}{\log T}\left[y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\frac{g_i(s)}{s}ds \right] + y_1. \end{aligned}$$

For $i = 1, 2$, let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{[dg_1 + (1-d)g_2]}{s}ds \\ &+ \frac{\log t}{\log T}\left[\frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\frac{[dg_1 + (1-d)g_2]}{s}ds \right]. \end{aligned}$$

Since $\hat{S}_{F,\tau y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *$N(M)$ is relatively compact for each compact $M \subset C(J,E)$.*

Let $M \subset C(J,E)$ be a compact set and let (h_n) by any sequence of elements of $N(M)$. We show that (h_n) has a convergent subsequence by using the Arzela-Ascoli criterion of compactness in $C(J,E)$. Since $h_n \in N(M)$ there exist $y_n \in M$ and $g_n \in \hat{S}_{F,\tau y_n}$ such that

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{g_n(s)}{s}ds \\ &+ \frac{\log t}{\log T}\left[y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\frac{g_n(s)}{s}ds\right] + y_1, \end{aligned}$$

for $n \geq 1$. Using Theorem 2.5 and the properties of the measure of noncompactness of Kuratowski, we have

$$\begin{aligned} \zeta(\{h_n(t)\}) &\leq \frac{2}{\Gamma(r)}\int_1^t \zeta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1} g_n(s)\frac{ds}{s} : n \geq 1\right\}\right) \\ &+ \frac{2\log t}{\log T}\left[y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T \zeta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1} g_n(s)\frac{ds}{s} : n \geq 1\right\}\right)\right]. \end{aligned} \tag{3.7}$$

On the other hand, since $M(s)$ is compact in E, the set $\{g_n(s) : n \geq 1\}$ is compact. Consequently, $\zeta(g_n(s) : n \geq 1) = 0$ for a.e. $s \in J$. Furthermore

$$\zeta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1} g_n(s)\frac{ds}{s}\right\}\right) = \left(\log\frac{T}{s}\right)^{r-1}\frac{1}{s}\zeta(\{g_n(s) : n \geq 1\}) = 0,$$

and

$$\zeta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1} g_n(s)\frac{ds}{s}\right\}\right) = \left(\log\frac{T}{s}\right)^{r-1}\frac{1}{s}\zeta(\{g_n(s) : n \geq 1\}) = 0,$$

for a.e. $t, s \in J$. Now (3.7) implies that $\{h_n(t) : n \geq 1\}$ is relatively compact in B, for each $t \in J$. In addition, for each t_1 and t_2 from J, $t_1 < t_2$, we have

$$\begin{aligned} |h_n(t_2) - h_n(t_1)| &= \Big|\frac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1} - \left(\log\frac{t_1}{s}\right)^{r-1}\right] g_n(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1} g_n(s)\frac{ds}{s} \\ &+ \frac{(\log t_2 - \log t_1)}{\log T}\left[y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1} g_n(s)\frac{ds}{s}\right]\Big| \\ &\leq \frac{p(t)}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1} - \left(\log\frac{t_2}{s}\right)^{r-1}\right]\frac{ds}{s} \\ &+ \frac{p(t)}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}\frac{ds}{s} \\ &+ \frac{(\log t_2 - \log t_1)}{\log T}\left[\left|y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1} g_n(s)\frac{ds}{s}\right|\right]. \end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that$\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C(J,E)$.

Step 3: *The graph of N is closed .*

Let $y_n \to y_*, h_n \in N(y_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{g_n(s)}{s}ds \\ &+ \frac{\log t}{\log T}\left[y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\frac{g_n(s)}{s}ds\right] + y_1,, \quad g_n \in \hat{S}_{F,\tau y_n}. \end{aligned}$$

We must show that there exists $v_* \in S_{F,y_*}$ such that for each $t \in J$

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{g_*(s)}{s}ds \\ &+ \frac{\log t}{\log T}\left[y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\frac{g_*(s)}{s}ds\right] + y_1,, \quad g_* \in \hat{S}_{F,\tau y_*}. \end{aligned}$$

Since $F(t,\cdot,\cdot)$ is upper semicontinuous, for every $\epsilon > 0$, there exists $n_0(x)$ such that for every $n \geq n_0$, we have $v_n \in F(t,y(t),x(t)) \subset F(t,y_*(t),x_*(t)) + \epsilon B(0,1)$ a,e. $t \in J$. And since F has compact values, there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(.) \to v_* \text{ as } m \to \infty,$$

$$v_* \in F(t,y_*(t)) \text{ as } t \in J.$$

For every $w(t) \in F(t,y_*(t))$, we have

$$|v_{n_m} - v_*| \leq |v_{n_m} - w(t)| + |w(t) - v_*|$$

and so

$$|v_{n_m} - v_*| \leq d(v_{n_m}(t), F(t,y_*(t))).$$

By an analogous relation obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$\begin{aligned} |v_{n_m} - v_*| &\leq H_d(F(t,y_{n_m}(t),F(t,y_*(t))) \\ &\leq l|y_{n_m} - y_*|. \end{aligned}$$

Therefore,

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} l|v_{n_m} - v_*|ds \\ &+ \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} l|v_{n_m} - v_*|ds \\ &\leq \frac{l(\log T)^r}{\Gamma(r+1)}\|y_{n_m} - y_*\|_{L^1}. \end{aligned}$$

Hence

$$\|h_n(t) - h_*(t)\|_\infty \to 0, \text{as} \quad m \to \infty.$$

Step 4: *M is relatively compact in $C(J,E)$*

Suppose $M \subset C(J,E)$, $M \subset conv(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an estimation similar to the one used in Step 2, we can see that $N(M)$ is equicontinuous. Then from $M \subset conv(\{0\} \cup N(M))$, we deduce that M is equicontinuous as well. To apply the Arzéla-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup N(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $g_n \in \hat{S}_{F,\tau y_n}$ such that

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} g_n(s) \frac{ds}{s} \\ &+ \frac{\log t}{\log T} \left[y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} g_n(s) \frac{ds}{s} \right] + y_1. \end{aligned}$$

From $\overline{M} \subset \overline{C} \subset \overline{conv}(\{0\} \cup H))$, and according to Theorem 2.5, we have

$$\zeta(M(t)) \leq \zeta(\overline{C(t)} \leq \zeta(H(t)) = \zeta(\{h_n((t) : n \geq 1\}).$$

Using (3.7) and the fact that $g_n(s) \in M(s)$, we obtain

$$\begin{aligned} \zeta(M(t)) &\leq \frac{2}{\Gamma(r)} \int_1^t \zeta \left(\left\{ \left(\log \frac{t}{s}\right)^{r-1} g_n(s) \frac{ds}{s} : n \geq 1 \right\} \right) \\ &+ \frac{2 \log t}{\log T} \left[y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \zeta \left(\left\{ \left(\log \frac{T}{s}\right)^{r-1} g_n(s) \frac{ds}{s} : n \geq 1 \right\} \right) \right] \\ &\leq \frac{2}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \zeta(M(s)) \frac{ds}{s} \\ &+ \frac{2 \log t}{\log T} \left[y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \zeta(M(s)) \frac{ds}{s} \right] \\ &\leq \frac{2}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \zeta(M(s)) \frac{ds}{s} \\ &+ \frac{2 \log t}{\log T} \left[y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \psi(s, \zeta(M(s)) \frac{ds}{s} \right] \end{aligned}$$

Also, the function φ given by $\varphi(t) = \zeta(M(t))$ belongs to $C(J,E)$. Consequently by (H4), $\varphi = 0$; that is, $\zeta(M(t)) = 0$ for all $t \in J$. Now, by the Arzéla-Ascoli theorem, M is relatively compact in $C(J,E)$.

Step 5: *The priori estimate*

Let $h \in C(J,E)$ such that $y \in \lambda N(y)$ for some $0 < \lambda < 1$.

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds \\ &+ \frac{\log t}{\log T} \left[y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \frac{g(s)}{s} ds \right] + y_1 \ g \in \hat{S}_{F,\tau y}, \end{aligned}$$

and

$$
\begin{aligned}
\|N(y)\|_{\mathcal{P}} &\le \left| \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \frac{g(s)}{s} ds \right| \\
&+ \left| y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \frac{g(s)}{s} ds \right| + |y_1| \\
&\le \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \frac{|g(s)|}{s} ds + |y_T| + |y_1| \\
&+ \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \frac{|g(s)|}{s} ds + |y_1| \\
&\le \frac{2(\log T)^r}{\Gamma(r+1)} \int_1^T p(s) ds + |y_1| + |y_T| \\
&\le \frac{2p^*(\log T)^r}{\Gamma(r+1)} + |y_1| + |y_T|.
\end{aligned}
$$

where

$$p^* = \sup\{|p(t)| : t \in J\}.$$

Then

$$\|N(y)\|_{\mathcal{P}} \le \frac{2p^*(\log T)^r}{\Gamma(r+1)} + |y_1| + |y_T| := R.$$

Set

$$U = \{u \in C(J, E) : \|N(y)\|_{\mathcal{P}} \le 1 + R\}.$$

Hence the condition (2.2) is satisfied. As a consequence of Steps $1-5$ and Theorem (2.5), we conclude that N has a fixed point $y \in C(J, E)$ which is a solution of problem (1.1)-(1.2).

Step 6: the solution y of (3.5)-(3.6) satisfies

$$\alpha(t) \le y(t) \le \beta(t), \text{ for all } t \in J$$

Let y be a solution to (3.5)-(3.6). We prove that

$$y(t) \le \beta(t), \text{ for all } t \in J.$$

Suppose not. Then there exists t_1, $t_2 \in J$, $t_1 < t_2$ such that $y(t_1) = \beta(t_1)$ and

$$y(t) > \beta(t) \qquad \text{for all } t \in [t_1, t_2]. \tag{3.8}$$

In view of the definition of τ one has

$$y(t) \in \int_{t_1}^{t} F(s, \beta(s)) ds \qquad \text{for all } t \in (t_1, t_2).$$

Thus there exists $g(s) \in F(s,\beta(s))$ for all $s \in (t_1,t_2)$ with $g \leq v_2(t)$ for all $s \in (t_1,t_2)$

$$y(t) = \frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\log\frac{t}{s}\right)^{r-1} g(s)\frac{ds}{s} + \frac{\log t}{\log t_2}\left[y(t_2) - \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1} g(s)\frac{ds}{s}\right] + y(t_1)\left(1 - \frac{1}{\log t_2}\right) \tag{3.9}$$

An integration on $(t_1,t]$, with $t \in (t_1,t_2)$ and there exists $g(.) \in F(.,\beta(.))$ yields

$$\begin{aligned} y(t) - y(t_1) = {} & \frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\log\frac{t}{s}\right)^{r-1} g(s)\frac{ds}{s} \\ & + \frac{\log t}{\log t_2}\left[y(t_2) - \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1} g(s)\frac{ds}{s}\right] \\ & - \frac{y(t_1)}{\log t_2}. \end{aligned}$$

Using the fact that β is an upper solution to (3.5)-(3.6) we find

$$\begin{aligned} \beta(t) - \beta(t_1) \geq {} & \frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\log\frac{t}{s}\right)^{r-1} g(s)\frac{ds}{s} \\ & + \frac{\log t}{\log t_2}\left[y(t_2) - \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1} g(s)\frac{ds}{s}\right] \\ & - \frac{y(t_1)}{\log t_2}, \qquad t \in (t_1,t_2). \end{aligned}$$

It follows from $y(t_1) = \beta(t_1)$ that

$$\beta(t) \geq y(t), \qquad \text{for all } t \in (t_1,t_2),$$

which is a contradiction, since $y(t) > \beta(t)$ for all $t \in (t_1,t_2)$. Consequently

$$\beta(t) \geq y(t), \qquad \text{for all } t \in J.$$

Analogously, we can prove that

$$\alpha(t) \leq y(t), \qquad \text{for all } t \in J.$$

This shows that

$$\alpha(t) \leq y(t) \leq \beta(t), \text{ for all } t \in J. \qquad \square$$

4. Nonoscillation and oscillation of solutions

The following theorem gives sufficient conditions to ensure the nonoscillation of solutions of problem (1.1)-(1.2).

Theorem 4.1. *Let α and β be lower and upper solutions, respectively, of (1.1)-(1.2) with $\alpha \leq \beta$ and assume that*

$(H5)$ *α is eventually positive nondecreasing, or β is eventually negative nonincreasing. Then every solutions y of (1.1)-(1.2) such that $y \in [\alpha,\beta]$ is nonoscillatory.*

Proof. Assume that α is eventually positive. Thus there exists $T_\alpha > 1$ such that

$$\alpha(t) > 0, \qquad \text{for all } t > T_\alpha.$$

Hence $y(t) > 0$ for all $t > T_\alpha$. From $(H4)$ we get $y(t) \geq \alpha(t)$. Since for each $h > 0$, $\alpha(t+h) \geq \alpha(t) > 0$, then $y(t) > 0$ for all $t > T_\alpha$, which means that y is nonoscillatory. Analogously, if β eventually negative, then there exists $T_\beta > 1$ such that

$$y(t) < 0, \qquad \text{for all } t > T_\beta,$$

which means that y is nonoscillatory. □

The following theorem discusses the oscillatory of the solutions of the problem (1.1)-(1.2).

Theorem 4.2. *Let α and β be lower and upper solutions, respectively, of (1.1)-(1.2) with $\alpha \leq \beta$ and assume that $\alpha(t)$ and $\beta(t)$ are oscillatory then every solution y of (1.1)-(1.2) such that $y \in [\alpha, \beta]$ is oscillatory.*

Proof. Suppose on the contrary that y is nonoscillatory solutions of (1.1)-(1.2). Then there exists $T_y > 1$ such that $y(t) > 0$ for all $t > T_y$ or $y(t) < 0$ for all $t > T_y$. In the case $y(t) > 0$ for all $t > T_y$ we have $\beta(t) > 0$ for all $t > T_y$, which is a contradiction since $\beta(t)$ is an oscillatory upper solution. Analogously in the case $y(t) < 0$ for all $t > T_y$ we have $\alpha(t) < 0$ for all $t > T_y$, which is also a contradiction since $\alpha(t)$ is an oscillatory lower solution. □

5. An example

We conclude this paper with an example to illustrate our main result. Let

$$E = l^1 = \{(y_1, y_2, ..., y_n, ..), \ \sum_1^\infty |y_n| < \infty\},$$

be our Banach space with norm

$$\|y\|_E = \sum_1^\infty |y_n|.$$

We apply the Theorems (3.3),(4.1) and (4.2) to the the following fractional differential inclusion

$$ {}^H_C D^r y(t) \in F_n(t, y(t)), \quad \text{for a.e } \ t \in J_1 = [1, e], 1 < r \leq 2, \tag{5.1}$$

$$y(1) = y_1,\ y(e) = y_2 \tag{5.2}$$

We set

$$F_n(t, y) = \{v \in \mathbb{R} : f_n(t, y) \leq v \leq h_n(t, y)\}$$

where $f_n, h_n : [1, e] \times E \mapsto E$. We assume that for each $t \in [1, e]$, $f_n(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in E : f_n(t, y) > \mu\}$ is open for each $\mu \in E$), and assume that for each $t \in [1, e]$, $h_n(t, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in E : f_2(t, y) < \mu\}$ is open for each $\mu \in E$) with $y = (y_1, y_2, \ldots, y_n, \ldots)$.

Set $F = (F_1, F_2, \ldots, F_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$, $g = (g_1, g_2, \ldots, g_n, \ldots)$.. Assume that there exists $p \in C([1,e], \mathbb{R}^+)$ such that,

$$\begin{array}{rcl} \|F(t,u)\|_{\mathcal{P}} & = & \sup\{|v|, v(t) \in F(t, y(t))\} \\ & = & \max(|f_n(t, y(t))|, |g_n(t, y(t))|) \\ & \leq & p(t), \text{ for each } t \in [1,e],\ y \in E. \end{array}$$

It is clear that F is compact and convex-valued, and it is upper semi-continuous, and furthermore, we assume that there exists $h_1(.)$, $h_2(.) \in L^1(J,E)$ such that

$$h_1(t) \leq \max(|f_n(t,y)|, |h_n(t,y)|) \leq h_2(t), \quad \text{for all } t \in J, \text{ and } y \in E,$$

and for each $t \in J$

$$\int_1^t h_1(s)\frac{ds}{s} \leq y_1 \text{ and } \int_1^t h_1(s)\frac{ds}{s} \leq y_T,$$

$$\int_1^t h_2(s)\frac{ds}{s} \geq y_1 \text{ and } \int_1^t h_2(s)\frac{ds}{s} \geq y_T,$$

Consider the functions

$$\alpha(t) = \int_1^t g_1(s)\frac{ds}{s}, \quad \beta(t) = \int_1^t g_2(s)\frac{ds}{s}.$$

Clearly, α and β are lower and upper solutions of the problem (5.1)-(5.2), respectively; that is,

$${}_C^H D^r \alpha(t) \leq f_n(t, \alpha(t)), \quad \text{for all } t \in J_1 \text{ and all } y \in E,$$

and

$${}_C^H D^r \beta(t) \geq h_n(t, \beta(t)), \quad \text{for all } t \in J_1 \text{ and all } y \in E,$$

We also assume that for each bounded set $B \subset C(J,E)$ and for each $t \in J$, we have

$$\zeta(F(t,B) \leq p(t)\alpha(B),$$

where α is a measure of noncompactness on E, and the function $\phi = 0$ is the unique solution in $C(J,E)$ of

$$\begin{array}{rcl} \varphi(t) & \leq & \frac{2}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \varphi(s, \phi(s))\frac{ds}{s} \\ & + & \frac{2\log t}{\log T}\left[y_T - y_1 - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} \varphi(s,\phi(s))\frac{ds}{s}\right] \end{array}$$

for $t \in J$.
Since all the conditions of the Theorem (3.3) are satisfied, problem (5.1)-(5.2) has at least one solution y on J_1 with $\alpha < y < \beta$. If $h_1(t) > 0$ then α is positive and nondecreasing, thus $y(t)$ is nonoscillatory. If $h_2(t)$ then β is negative and nonincreasing, thus $y(t)$ is nonoscillatory. If the functions $\alpha(t)$ and $\beta(t)$ are both oscillatory, then $y(t)$ is oscillatory.

References

[1] S. Abbas, M. Benchohra, *The method of upper and lower solutions for partial hyperbolic fractional order differential inclusions with impulses,* Discuss. Math. Differ. Incl. Control Optim., **30**(2010), no. 1, 141-161.

[2] R.P. Agarwal, M. Benchohra, D. Seba, *An the application of measure of noncompactness to the existence of solutions for fractional differential equations,* Results Math., **55**(2009), no. 3-4, 221-230.

[3] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations,* Kluwer Academic Publishers, Dordrecht, 1999.

[4] R.R. Akhmerov, M.I. Kamenski, A.S. Patapov, A.E. Rodkina, B.N. Sadovski, *Measure of Noncompactness and Condensing Operators*, Translated from the 1986 Russian original by A. Iacop, Operator theory: Advances and Applications, **55**, Birkhäuser Verlag, Bassel, 1992.

[5] J.P. Aubin, A. Cellina, *Differential Inclusions, Springer-Verlag*, Berlin-Heidelberg, New York, 1984.

[6] J.P. Aubin, H. Frankowska, *Set-Valued Analysis,* Birkhäuser, Boston, 1990.

[7] J. Banas, K. Goebel, *Measure of Noncompactness in Banach Spaces*, *In lecture Notes in Pure and Applied Mathematics,* **60,** Marcel Dekker, New York.

[8] M. Benchohra, S. Hamani, *The method of upper and lower solution and impulsive fractional differential inclusions,* Nonlinear Anal. Hybrid Syst., **3**(2009), 433-440.

[9] M. Benchohra, S. Hamani, J. Henderson, *Oscillatory and nonoscillatory of solutions for first order impulsive dynamic inclusions on time scales*, Arch. Math., **36**(2000), 159-169.

[10] M. Benchohra, S. Hamani, Y. Zhou, *Oscillation and nonoscillation for Caputo-Hadamard impulsive fractional differential inclusions*, Adv. Differ. Equ., **74**(2019).

[11] M. Benchohra, J. Henderson, D. Seba, *Measure of noncompactenes and fractional differential equations in Banach Space*, Commun. Appl. Anal., **12**(2008), no. 4, 419-428.

[12] M. Benchohra, J. Henderson, D. Seba, *Boundary value problems for fractional differential inclusions in Banach space*, Frac. Diff. Cal., **2**(2012), 99-108.

[13] M. Benchohra, J.J. Nieto, D. Seba, *Measure of noncompactnes and fractional and hyperbolic partial fractional differential equations in Banach space,* Panamer. Math. J., **20**(2010), no. 3, 27-37.

[14] E.A. Bohner, M. Bohner, S.H. Saker, *Oscillation criteria for a certain class of second order Emden-Fowler dynamic equations,* Electronic Trans. Numerical Anal., **27**(2007), 1-12.

[15] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics*, **580**, Springer-Verlag, Berlin - Heidelberg - New York, 1977.

[16] K. Deimling, *Multivalued Differential Equations,* Walter De Gruyter, Berlin - New York, 1992.

[17] L. Erbe, *Oscillation criteria for second order linear equations on a time scale*, Canad. Appl. Math. Quart., **9**(2001) 1-31.

[18] J.R. Graef, N. Guerraiche, S. Hamani, *Boundary value problems for fractional differential inclusions with Hadamard type derivatives in Banach spaces,* Stud. Univ. Babeş-Bolyai. Math., **62**(2017), no. 4, 427-438.

[19] J.R. Graef, J. Karsai, *On the oscillation of impulsively damped halflinear oscillators,* Proc. Sixth. Colloquium Qual. Theory Differential Equations, Electron. J. Qual. Theory Differential Equations, **14**(2000) 1-12.

[20] N. Guerraiche, S. Hamani, J. Henderson, *Initial value problems for fractional functional differential inclusions with Hadamard type derivative,* Archivum Math., **52**(2016), 263-273.

[21] N. Guerraiche, S. Hamani, J. Henderson, *Boundary value problems for differential inclusions with integral and anti-periodic conditions,* Commun. Appl. Nonlinear Anal., **23**(2016), no. 3, 33-46.

[22] N. Guerraiche, S. Hamani, J. Henderson, *Nonlinear boundary value Problems for Hadamard fractional differential inclusions with integral boundary conditions,* Adv. Dynam. Systems Appl., **12**(2017), no. 2, 107-121.

[23] J. Hadamard, *Essai sur l'etude des fonctions donnees par leur development de Taylor,* Mat. Pure Appl., **8**(1892), 101-186.

[24] H.P. Heinz, *On the behaviour of measure of noncompactness with respect of differentiation and integration of vector-valued function*, Nonlinear Anal., **7**(1983), 1351-1371.
[25] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[26] F. Jarad, T. Abdeljawad, D. Beleanu, *Caputo-type modification of the Hadamard fractional derivative*, Adv. Difference Equ., **142**(2012), 8 pp.
[27] A.A. Kilbas, H. M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies*, **204**, Elsevier Science B.V., Amsterdam, 2006.
[28] K.Q. Lan, W. Lin, *Positive solutions of systems of Caputo fractional differential equations*, Communications in Applied Analysis, **17**(2013), no. 1, 61-86.
[29] A. Lasota, Z. Opial, *An application of the Kakutani-ky Fan theorem in the theory of ordinary differential equation*, Bull. Accd. Pol. Sci., Ser. Sci. Math. Astronom. Phys., **13**(1965), 781-786.
[30] S.M. Momani, S.B. Hadid, Z.M. Alawenh, *Some analytical properties of solutions of differential equations of noninteger order*, Int. J. Math. Math. Sci., **2004**(2004), 697-701.
[31] D. O'Regan, R. Precup, *Fixed point theorems for set-valued maps and existence principles for integral inclusions*, J. Math. Anal. Appl., **245**(2000), 594-612.
[32] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.
[33] I.A. Rus, *Relevant classes of weakly Picard operators*, An. Univ. Vest Timis. Ser. Mat.-Inform., **54**(2016), no. 2, 131-147.
[34] I.A. Rus, M.A. Şerban, *Basic problems of the metric fixed point theory and the relevance of a metric fxed point theorem*, Carpathian J. Math., **29**(2013), no. 2, 239-258.

Received: April 4, 2021; Accepted: June 28, 2022.