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EXISTENCE AND APPROXIMATING OF COMMON BEST PROXIMITY POINTS OF RELATIVELY NONEXPANSIVE MAPPINGS VIA ISHIKAWA ITERATION METHOD

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Abstract. In this article, we study the existence of a common best proximity points for a finite class of cyclic relatively nonexpansive mappings in the setting of Busemann convex spaces. In this way, we extend the main results given in Eldred and Raj (2009) [A.A. Eldred, V.S. Raj, On common best proximity pair theorems, Acta Sci. Math. (Szeged), 75, 707-721] for relatively nonexpansive mappings in Banach spaces to more general metric spaces. We then present a strong convergence theorem of a common best proximity point for a pair of cyclic mappings in uniformly convex Banach spaces by using the Ishikawa iterative process.
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1. Introduction

Let A and B be two nonempty and disjoint subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. Then $d(x, Tx) \geq \text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}$ for any $x \in A \cup B$. In this case, it is natural to ask how can we find a solution for a minimization problem

$$\min_{x \in A \cup B} d(x, Tx). \tag{1.1}$$

A point $p \in A \cup B$ is said to be a *best proximity point* for the cyclic mapping $T : A \cup B \to A \cup B$ if p is a solution for the minimization problem (1.1). Equivalently, $p \in A \cup B$ is called a best proximity point for T, whenever

$$d(p, Tp) = \text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Inspired of cyclic mappings, we say that the mapping $T : A \cup B \to A \cup B$ is *noncyclic* whenever $T(A) \subseteq A, T(B) \subseteq B$. For noncyclic case, we can consider the following minimization problem:

$$\min_{x\in A} d(x, Tx), \quad \min_{y\in B} d(y, Ty), \quad \min_{(x,y)\in A\times B} d(x, y). \tag{1.2}$$

We say that a point $(x^\star, y^\star) \in A \times B$ is a *best proximity pair* for the noncyclic mapping T if it is a solution of (1.2), or equivalently, $x^\star = Tx^\star, y^\star = Ty^\star$ and $d(x^\star, y^\star) = \mathrm{dist}(A, B)$.

We recall that the mapping $T : A \cup B \to A \cup B$ is *relatively nonexpansive* if $d(Tx, Ty) \le d(x, y)$ for any $(x, y) \in A \times B$.

The following existence theorem was established in [5].

Theorem 1.1. *Let (A, B) be a nonempty, weakly compact and convex pair in a strictly convex Banach space with proximal normal structure. Let $\mathcal{S} = \{T_1, T_2, ..., T_n\}$ be a commuting family of cyclic relatively nonexpansive mappings on $A \cup B$. Then $\mathcal{S}$ has a common best proximity point, that is, there exists a point $p \in A \cup B$ such that*

$$\|p - T_j p\| = \mathrm{dist}(A, B), \quad \forall j \in \{1, 2, ..., n\}.$$

This article is organized as follows: in Section 2, we recall some notions and notations which will be used throughout this paper. In Section 3, we generalize Theorem 1.1 from strictly convex Banach spaces to reflexive Busemann convex spaces and by considering the projection mappings, we establish a same result of Theorem 1.1 for noncyclic relatively nonexpansive mappings. In Section 4, by using the Ishikawa iteration process, we prove a strong convergence theorem of common best proximity points for a pair of cyclic mappings in the setting of uniformly convex Banach spaces.

2. Preliminaries

A metric space (X, d) is said to be a *(uniquely) geodesic space* if every two points x and y of X are joined by a *(unique) geodesic*, i.e., a map $c : [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. A subset A of a geodesic space X is said to be *convex* if the image of any geodesic that joins each pair of points x and y of A (geodesic segment $[x, y]$) is contained in A. A point z in X belongs to a geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$ and we write $z = (1 - t)x \oplus ty$ for simplicity. Notice that this point may not be unique. Any Banach space is for instance a geodesic space with usual segments as geodesic segments.

Definition 2.1. A geodesic metric space X is said to be *reflexive* if for every decreasing chain $\{C_\alpha\} \subseteq X$ with $\alpha \in I$ such that C_α is nonempty, bounded, closed and convex for all $\alpha \in I$ we have that $\bigcap_{\alpha\in I} C_\alpha \neq \emptyset$.

We mention that reflexive Banach spaces can be considered as reflexive geodesic spaces. Also, it is well-known that every complete uniformly convex metric space with either a monotone or lower semicontinuous from the right modulus of uniform convexity is reflexive (see [19] for more details).

Let (X,d) be a uniquely geodesic space. The *metric* $d\colon X\times X\to\mathbb{R}$ is said to be *convex* if for any $x,y,z\in X$ one has

$$d(x,(1-t)y\oplus tz)\le(1-t)d(x,y)+td(x,z)\text{ for all }t\in[0,1].$$

Definition 2.2. ([3]) A geodesic space (X,d) is called *convex in the sense of Busemann* if given any pair of geodesics $c_1:[0,l_1]\to X$ and $c_2:[0,l_2]\to X$ one has

$$d(c_1(tl_1),c_2(tl_2))\le(1-t)d(c_1(0),c_2(0))+td(c_1(l_1),c_2(l_2))\text{ for all }t\in[0,1].$$

Equivalently, a geodesic metric space (X,d) is convex in the sense of Busemann provided that

$$d((1-t)x\oplus ty,(1-t)z\oplus tw)\le(1-t)d(x,z)+td(y,w),$$

for all $x,y,z,w\in X$ and $t\in[0,1]$.

A reflexive and Busemann convex space is complete (see [9, Lemma 4.1]).

Definition 2.3. ([1]) A metric space is said to be strictly convex if X is a geodesic space and for every $r>0$, a,x and $y\in X$ with $d(x,a)\le r$, $d(y,a)\le r$ and $x\ne y$, it is the case that $d(a,p)<r$, where p is any point between x and y such that $p\ne x$ and $p\ne y$, i.e., p is any point in the interior of a geodesic segment that joins x and y.

It is worth noticing that Busemann convex spaces are strictly convex with convex metric ([10]).

We shall say that a pair (A,B) of subsets of a geodesic space X satisfies a property if both A and B satisfy that property. For example, (A,B) is convex if and only if both A and B are convex; $(A,B)\subseteq(C,D)\Leftrightarrow A\subseteq C,$ and $B\subseteq D$. We shall also adopt the notation

$$\delta_x(A)=\sup\{d(x,y)\colon y\in A\}\text{ for all }x\in X,$$
$$\delta(A,B)=\sup\{\delta_x(B):x\in A\},$$
$$\operatorname{diam}(A)=\delta(A,A).$$

From now on, $\mathcal{B}(a;r)$ will denote the closed ball in the space X centered at $a\in X$ with radius $r>0$.

For a nonempty pair (A,B) in a metric space (X,d), we define

$$A_0=\{x\in A\colon d(x,y')=\operatorname{dist}(A,B)\text{ for some }y'\in B\},$$
$$B_0=\{y\in B\colon d(x',y)=\operatorname{dist}(A,B)\text{ for some }x'\in A\}.$$

The pair (A_0,B_0) is called the *proximal pair* of the pair (A,B). It is worth mentioning that proximal pairs may be empty. It is easy to see that the proximal pair of every nonempty, weakly compact and convex pair in a Banach space is also nonempty, weakly closed and convex. In more general, we have the following result in the setting of Busemann convex spaces.

Proposition 2.1. (Proposition 3.1 of [11]) *If (A,B) is a nonempty, closed and convex pair in a reflexive and Busemann space X such that B is bounded, then (A_0,B_0) is nonempty, bounded, closed and convex.*

Definition 2.4. A pair (A, B) in a Banach space is said to be proximinal if $A = A_0$ and $B = B_0$.

We mention that if A is a nonempty subset of a geodesic space X, then the (closed) convex hull of A is the smallest (closed) convex set containing the set A or equivalently, the intersection of all (closed) convex sets containing the set A. The convex hull and closed convex hull of the set A will be denoted by $\mathrm{con}(A)$ and $\overline{\mathrm{con}}(A)$, respectively.

Bridson and Haefliger ([2]) presented an interesting equivalent concept of the convex hull of a set as follows:

Lemma 2.1. *Let A be a nonempty subset of a geodesic space X. Let $\mathcal{G}_1(A)$ denote the union of all geodesic segments with endpoints in A. Recursively, for $n \geq 2$ put $\mathcal{G}_n(A) = \mathcal{G}_1(\mathcal{G}_{n-1}(A))$. Then*

$$\mathrm{con}(A) = \bigcup_{n=1}^{\infty} \mathcal{G}_n(A).$$

It is worth noticing that in a Busemann convex space X the closure of $\mathrm{con}(A)$ is convex and so, coincides with $\overline{\mathrm{con}}(A)$ ([8]).

The next lemmas play important roles in our main results.

Lemma 2.2. (Lemma 3.7 of [15]) *Let (A, B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space (X, d). Assume that $(E, F) \subseteq (A, B)$ is a nonempty and proximinal pair with* $\mathrm{dist}(E, F) = \mathrm{dist}(A, B)$. *Then the pair $(\overline{\mathrm{con}}(E), \overline{\mathrm{con}}(F))$ is proximinal with*

$$\mathrm{dist}(\overline{\mathrm{con}}(E), \overline{\mathrm{con}}(F)) = \mathrm{dist}(A, B).$$

If A is a nonempty subset of a metric space (X, d), then the *metric projection operator* is a set-valued mapping $\mathcal{P}_A : X \to 2^A$ which is defined as

$$\mathcal{P}_A(x) := \{y \in A : d(x, y) = \mathrm{dist}(\{x\}, A)\},$$

where 2^A denotes the set of all subsets of A. It is well-known that if A is a nonempty, bounded, closed and convex subset of reflexive Busemann convex space X, then the metric projection $\mathcal{P}_A$ is single-valued from X to A.

Proposition 2.2. ([12, 13]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Busemann convex space X. Define $\mathcal{P} : A_0 \cup B_0 \to A_0 \cup B_0$ as*

$$\mathcal{P}(x) = \begin{cases} \mathcal{P}_{A_0}(x) & \text{if } x \in B_0, \\ \mathcal{P}_{B_0}(x) & \text{if } x \in A_0. \end{cases} \tag{2.1}$$

Then the following statements hold.
(i) $\mathcal{P}$ is cyclic on $A_0 \cup B_0$ and $d(x, \mathcal{P}x) = \mathrm{dist}(A, B)$ for any $x \in A_0 \cup B_0$,
(ii) If X is a Hilbert space, then $\mathcal{P}$ is relatively isometry, that is, $d(\mathcal{P}x, \mathcal{P}y) = d(x, y)$ for all $(x, y) \in A_0 \times B_0$,
(iii) $\mathcal{P}$ is affine,
(iv) $\mathcal{P}^2|_{A_0} = i_{A_0}$ and $\mathcal{P}^2|_{B_0} = i_{B_0}$, where i_{A_0} denotes the identity mapping on A_0,
(v) $\mathcal{P}|_{A_0}$ and $\mathcal{P}|_{B_0}$ are continuous,

(vi) If moreover, $T : A \cup B \to A \cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping, then (A_0, B_0) is T-invariant, that is, T is cyclic (noncyclic) on $A_0 \cup B_0$ and also T and $\mathcal{P}$ commute on $A_0 \cup B_0$.

Here, we recall a well-known geometric notion of Banach spaces, called uniformly convexity, and some of related suitable properties.

Definition 2.5. A Banach space X with positive modulus of convexity $\delta_X(\varepsilon)$ is said to be uniformly convex where $0 < \varepsilon < 2$ and

$$\delta_X(\varepsilon) := \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}.$$

It is well-known that Hilbert and l^p spaces $(1 < p < \infty)$ are uniformly convex. Also, it is worth noticing that the uniform convexity can be defined on geodesic spaces as follows.

Definition 2.6. ([19]) A geodesic metric space (X, d) is said to be uniformly convex if for any $r > 0$ and any $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$, we have

$$d(m, a) \leq (1 - \delta)r,$$

where m stands for a midpoint of the points x and y.

The following lemma gives a suitable property for characterization of uniformly convex Banach spaces.

Lemma 2.3. ([22]) *A Banach space X is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) = 0 \Leftrightarrow t = 0$, such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\varphi(\|x - y\|),$$

for all $\lambda \in [0, 1]$ and $x, y \in X$, where $\|x\| \leq r$ and $\|y\| \leq r$.

Uniformly convex Banach spaces has another interesting property which states as below.

Lemma 2.4. ([6]) *Let A be a nonempty, closed and convex subsets and B be a nonempty and closed subset of a uniformly convex Banach space X. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and let $\{y_n\}$ be a sequence in B satisfying*

(i) $\|x_n - y_n\| \to \operatorname{dist}(A, B)$,

(ii) $\|z_n - y_n\| \to \operatorname{dist}(A, B)$.

Then $\|x_n - z_n\| \to 0$.

We also refer to the following auxiliary lemma.

Lemma 2.5. *Consider a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$. If a sequence $\{r_n\}$ in $[0, \infty)$ satisfies* $\lim_{n\to\infty} \phi(r_n) = 0$, *then* $\lim_{n\to\infty} r_n = 0$.

Definition 2.7. ([21]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property if and only if

$$\begin{cases} d(x_1, y_1) = \text{dist}(A, B) \\ d(x_2, y_2) = \text{dist}(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It was announced in [18] that every nonempty, bounded, closed and convex pair in a reflexive and Busemann convex space X has the P-property.

A concept of proximal normal structure was first introduced in [4]. It was then improved in [11] from Banach spaces to geodesic spaces as below.

Definition 2.8. A convex pair (K_1, K_2) in a geodesic space is said to have proximal normal structure (PNS) if for any bounded, closed, convex and proximinal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\text{dist}(H_1, H_2) = \text{dist}(K_1, K_2)$ and $\delta(H_1, H_2) > \text{dist}(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that

$$\max\{\delta_{x_1}(H_2), \delta_{x_2}(H_1)\} < \delta(H_1, H_2).$$

For instance, every nonempty, compact and convex pair in a geodesic space with convex metric, has the PNS (see Proposition 3.10 of [11]). Also, every nonempty, bounded, closed and convex pair in a uniformly convex metric space X has the PNS (see Proposition 3.5 of [11]).
We refer to [16] for some interesting characterization of PNS.

3. Existence results of a common best proximity pair

We begin our main conclusions with the following theorem.

Theorem 3.1. ([11]) *Let X be a reflexive and Busemann convex metric space and let (A, B) be a nonempty, closed and convex pair of subsets of X such that A is bounded. Suppose $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If (A, B) has the* PNS*, then T has a best proximity point.*

We are now ready to state the first existence result of a common best proximity point for a finite family of cyclic relatively nonexpansive mappings in Busemann convex spaces.

Theorem 3.2. *Let (A, B) be a nonempty, disjoint, bounded, closed and convex pair in a reflexive Busemann convex space X such that (A, B) has the* PNS. *Let $\mathcal{S} = \{T_1, T_2, \ldots, T_n\}$ be a finite commuting family of cyclic relatively nonexpansive mappings on $A \cup B$. Then $\mathcal{S}$ has a common best proximity point, that is, there exists a point $x^\star \in A \cup B$ for which*

$$d(x^\star, T_j x^\star) = \text{dist}(A, B), \quad \forall j = 1, 2, ..., n.$$

Proof. This result follows by applying similar patterns as in the proof of Theorem 3.1 in [5]. However, in this more general setting, several changes and new techniques must be considered to get the result. It follows from Proposition 2.1 that (A_0, B_0) is nonempty, closed and convex. Without loss of generality let us assume that $\mathcal{S} =$

$\{T_1, T_2, ..., T_n\}$ contains an odd number of relatively nonexpansive maps. If not, then we replace the collection $\mathcal{S}$ by $\mathcal{S}' = \{T_1, T_2, ..., T_n, T_1\}$ and proceed. From Proposition 2.2, the proximal pair (A_0, B_0) associated with (A, B) is a nonempty, closed, convex and proximinal pair with $\text{dist}(A_0, B_0) = \text{dist}(A, B)$. Let Γ denote the collection of all nonempty subsets F of $A_0 \cup B_0$ for which $F \cap A_0$ and $F \cap B_0$ are nonempty closed and convex with

$$T_j(F \cap A_0) \subseteq F \cap B_0, \quad T_j(F \cap B_0) \subseteq F \cap A_0, \quad \forall j = 1, 2..., n,$$

and $\text{dist}(F \cap A_0, F \cap B_0) = \text{dist}(A, B)$. Then Γ is nonempty since $A_0 \cup B_0 \in \Gamma$. By a standard argument using Zorn's lemma, there is a minimal element in Γ, namely K. Let $K_1 = K \cap A_0$ and $K_2 = K \cap B_0$. Then (K_1, K_2) is a closed, bounded and convex subset of (A, B) satisfying $T_j(K_1) \subseteq K_2$ and $T_j(K_2) \subseteq K_1$ for each $j = 1, 2..., n$ and $\text{dist}(K_1, K_2) = \text{dist}(A, B)$. Also,$(K_1, K_2)$ is proximinal. Define a mapping $G : K_1 \cup K_2 \to K_1 \cup K_2$ by

$$G(x) = T_1 \circ T_2 \cdots \circ T_n(x), \forall x \in K_1 \cup K_2.$$

Since $T_j(K_1) \subseteq K_2$ and $T_j(K_2) \subseteq K_1$ and n is odd, we conclude that $G : K_1 \cup K_2 \to K_1 \cup K_2$ is a cyclic relatively nonexpansive map. Put

$$W_1 = \{x \in K_1 : d(x, G(x)) = \text{dist}(K_1, K_2)\},$$

$$W_2 = \{y \in K_2 : d(y, G(y)) = \text{dist}(K_1, K_2)\}.$$

By Theorem 3.1, $(W_1, W_2) \subseteq (K_1, K_2)$ is nonempty.
Claim 1. $T_j(W_1) \subseteq W_2$ and $T_j(W_2) \subseteq W_1$ for all $j = 1, 2..., n$.

Proof. If $x \in W_1$, then

$$\begin{aligned} d(x, G(x)) = \text{dist}(K_1, K_2) &\leq d(T_j(x), G(T_j(x))) \\ = d(T_j(x), T_j(G(x))) &\quad \text{(by the commuting condition of } \mathcal{S}) \\ &\leq d(x, G(x)) = \text{dist}(K_1, K_2). \end{aligned}$$

This implies $d(T_j(x), G(T_j(x))) = \text{dist}(K_1, K_2)$, for any $j = 1, 2, ..., n$. Hence $T_j(x) \in W_2$ which implies $T_j(W_1) \subseteq W_2$, for all $j = 1, 2, ..., n$. Similarly $T_j(W_2) \subseteq W_1$ for any $j = 1, 2, ..., n$.

Claim 2. $G(W_1) \subseteq W_2$ and $G(W_2) \subseteq W_1$.
Proof. It follows from the assumption that n is odd and Claim 1.
Claim 3. For each $j \in \{1, 2, ..., n\}$, $T_j(W_2) = W_1$ and $T_j(W_1) = W_2$.
Proof. Fix $x \in W_1$ and $j \in \{1, 2, ..., n\}$. Then $d(x, G(x)) = \text{dist}(K_1, K_2)$. Also since $G(x) \in K_2$ and G is relatively nonexpansive, we have

$$d(G(x), G^2(x)) \leq d(x, G(x)) = \text{dist}(K_1, K_2).$$

Therefore, $d(G(x), G^2(x)) = \text{dist}(K_1, K_2)$. Using the fact that X is strictly convex, $x = G^2(x)$. Since $\mathcal{S}$ is a commuting family, we have

$$x = T_j(G(T_1 \circ T_2 \circ \cdots \circ T_{j-1} \circ T_{j+1} \circ \cdots \circ T_n))(x).$$

Take $y_j = G(T_1 \circ T_2 \circ \cdots \circ T_{j-1} \circ T_{j+1} \circ \cdots \circ T_n)(x)$. Then $x = T_j(y_j)$. Since the map

$$G(T_1 \circ T_2 \circ \cdots \circ T_{j-1} \circ T_{j+1} \circ \cdots \circ T_n),$$

consists of an odd $(2n-1)$ numbers of relatively nonexpansive maps and together with Claim 1 and Claim 2, we conclude that y_j is in W_2. Hence $T_j(W_2) = W_1$. Equivalently, we can prove that $T_j(W_1) = W_2$.

Claim 4. (W_1, W_2) is proximinal and $\operatorname{dist}(W_1, W_2) = \operatorname{dist}(K_1, K_2)$.

Proof. Since $(W_1, W_2) \subseteq (K_1, K_2)$, for any $x \in W_1$ we have

$$\operatorname{dist}(K_1, K_2) \leq \operatorname{dist}(W_1, W_2) \leq d(x, G(x)) = \operatorname{dist}(K_1, K_2).$$

Hence $\operatorname{dist}(W_1, W_2) = \operatorname{dist}(K_1, K_2)$. Also, for any $(x, y) \in W_1 \times W_2$, we have $(G(y), G(x)) \in W_1 \times W_2$ and

$$d(x, G(x)) = d(y, G(y)) = \operatorname{dist}(K_1, K_2) = \operatorname{dist}(W_1, W_2),$$

which ensures that (W_1, W_2) is a proximinal pair in (K_1, K_2). Notice that if $\operatorname{dist}(W_1, W_2) = \delta(W_1, W_2)$, then

$$d(x, T_j(x)) = \operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B), \quad \forall x \in W_1,\ \forall j \in \{1, 2, ..., n\},$$

and the proof is completed. So we assume that

$$\operatorname{dist}(W_1, W_2) < \delta(W_1, W_2).$$

Set

$$H_1 = \overline{\operatorname{con}}(W_1), \quad H_2 = \overline{\operatorname{con}}(W_2).$$

By Lemma 2.2 we have (H_1, H_2) is a nonempty, bounded, closed, convex and proximinal pair in (K_1, K_2) so that

$$\begin{aligned}\operatorname{dist}(H_1, H_2) &= \operatorname{dist}(W_1, W_2)\\ &< \delta(W_1, W_2) = \delta(H_1, H_2) \quad \text{(Lemma 2.6 of [17])}.\end{aligned}$$

In view of the fact that (A, B) has the PNS, there exist $(x_1, x_2) \in H_1 \times H_2$ and $\beta \in (0, 1)$ such that

$$\max\{\delta_{x_1}(H_2), \delta_{x_2}(H_1)\} \leq \beta\delta(H_1, H_2).$$

Since (H_1, H_2) is a proximinal pair, there exists $(x_1', x_2') \in H_1 \times H_2$ such that $d(x_1, x_2') = d(x_1', x_2) = \operatorname{dist}(H_1, H_2)$. Now for any $z \in H_2$, using the convexity of the metric,

$$\begin{aligned}d(\frac{1}{2}x_1 \oplus \frac{1}{2}x_1', z) &\leq \frac{1}{2}d(x_1, z) + \frac{1}{2}d(x_1', z)\\ &\leq \frac{\beta}{2}\delta(H_1, H_2) + \frac{1}{2}\delta(H_1, H_2) = \alpha\delta(H_1, H_2),\end{aligned}$$

where $\alpha := \frac{(1+\beta)}{2} \in (0, 1)$. Suppose $x_0 := \frac{1}{2}x_1 \oplus \frac{1}{2}x_1'$ and $y_0 := \frac{1}{2}x_2' \oplus \frac{1}{2}x_2$. Then $(x_0, y_0) \in H_1 \times H_2$ and

$$\delta_{x_0}(H_2) \leq \alpha\delta(H_1, H_2), \quad \delta_{y_0}(H_1) \leq \alpha\delta(H_1, H_2),$$

and that $d(x_0, y_0) = \operatorname{dist}(H_1, H_2)$. Define

$$\begin{aligned}L_1 &= \{x \in H_1 : \delta_x(H_2) \leq \alpha\delta(H_1, H_2)\},\\ L_2 &= \{y \in H_2 : \delta_y(H_1) \leq \alpha\delta(H_1, H_2)\}.\end{aligned}$$

Then (L_1, L_2) is a nonempty, closed, and convex subset of (H_1, H_2) and since $(x_0, y_0) \in L_1 \times L_2$, we have $\operatorname{dist}(L_1, L_2) = \operatorname{dist}(H_1, H_2)(= \operatorname{dist}(W_1, W_2) = \operatorname{dist}(A, B))$.

We show that if $x \in L_1$, then $T_j(x) \in L_2$ or equivalently, $\delta_{T_j(x)}(H_1) \leq \alpha\delta(H_1, H_2)$, for all $j \in \{1, 2, ..., n\}$. Let $w \in W_1$ be an arbitrary element. Then there exists $w_j \in W_2$ such that $w = T_j(w_j)$. We now have

$$d(T_j(x), w) = d(T_j(x), T_j(w_j)) \leq d(x, w_j) \leq \delta_x(W_2) \leq \delta_x(H_2) \leq \alpha\delta(H_1, H_2).$$

Thereby, $\delta_{T_j(x)}(W_1) \leq \alpha\delta(H_1, H_2)$, that is, $T_j(L_1) \subseteq L_2$. By a similar manner, we can see that $T_j(L_2) \subseteq L_1$ for any $j = 1, 2..., n$. Hence, $L_1 \cup L_2 \in \Gamma$. On the other hand, $\delta(L_1, L_2) \leq \alpha\delta(H_1, H_2) < \delta(K_1, K_2)$, which is a contradiction by the minimality of (K_1, K_2).

The following results can be concluded, immediately.

Corollary 3.1. *Let (A, B) be a nonempty, compact and convex pair in a Busemann convex (X, d). If $\mathcal{S} = \{T_1, T_2, ..., T_n\}$ is a commuting family of cyclic relatively non-expansive mappings, then $\mathcal{S}$ has a common best proximity point.*

Corollary 3.2. *Let (A, B) be a nonempty, bounded, closed and convex pair in a Busemann convex (X, d) which is uniformly convex in the sense of Definition 2.2. If $\mathcal{S} = \{T_1, T_2, ..., T_n\}$ is a commuting family of cyclic relatively nonexpansive mappings, then $\mathcal{S}$ has a common best proximity point.*

Example 3.1. Let $X = \mathbb{R}^2$ and d be the *river metric* on X defined with

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |x_1 - x_2| + |y_1| + |y_2|, & \text{if } x_1 \neq x_2. \end{cases}$$

It is well known that $(\mathbb{R}^2, d)$ is a complete $\mathbb{R}$-tree and so, is a reflexive and Busemann convex space (see [7]). Suppose $A = \{(0, y) : 0 \leq y \leq 1\}$ and $B = \{(1, y) : 0 \leq y \leq 1\}$. Then (A, B) is compact and convex and so has the PNS with $\operatorname{dist}(A, B) = 1$. Define a pair of mappings $T, S : A \cup B \to A \cup B$ by

$$T(x, y) = \begin{cases} (1, \frac{y}{2}) & \text{if } \ x = 0, \ y \in \mathbb{Q} \cap [0, 1], \\ (1, 0) & \text{if } \ x = 0, \ y \in \mathbb{Q}^c \cap [0, 1], \\ (0, \frac{y}{2}) & \text{if } \ x = 1, \ y \in \mathbb{Q} \cap [0, 1], \\ (0, 0) & \text{if } \ x = 1, \ y \in \mathbb{Q}^c \cap [0, 1], \end{cases}$$

$$Sx = \begin{cases} (1, \frac{y}{4}) & \text{if } \ x = 0, \ y \in \mathbb{Q} \cap [0, 1], \\ (1, 0) & \text{if } \ x = 0, \ y \in \mathbb{Q}^c \cap [0, 1], \\ (0, \frac{y}{4}) & \text{if } \ x = 1, \ y \in \mathbb{Q} \cap [0, 1], \\ (0, 0) & \text{if } \ x = 1, \ y \in \mathbb{Q}^c \cap [0, 1]. \end{cases}$$

Then T, S are commuting mappings that are cyclic and relatively nonexpansive. Therefore, all of the assumptions of Theorem 3.2 hold. It is worth noticing that the points $(0, 0)$ and $(1, 0)$ are best proximity points for the mappings T and S.

We can apply Theorem 3.2 to establish the existence of a common best proximity pairs for *noncyclic relatively nonexpansive mappings* as follows. To this end, we need the following geometric concept.

Definition 3.1. [14] Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that A_0 is nonempty. We say that the pair (A, B) has the diagonal property provided that

$$\begin{cases} d(x_1, y_1) = \operatorname{dist}(A, B), \\ d(x_2, y_2) = \operatorname{dist}(A, B), \end{cases} \Rightarrow d(x_1, y_2) = d(x_2, y_1),$$

for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

For instance if (A, B) is a pair of nonempty subsets of a metric space (X, d) such that $\operatorname{dist}(A, B) = 0$, then (A, B) has the diagonal property.
Also, every two parallel segments in the Euclidian plan $\mathbb{R}^2$ has the diagonal property. In more general, every nonempty, closed and convex pair in Hilbert spaces has the diagonal property.

Theorem 3.3. *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Busemann convex space (X, d) such that (A, B) has both the* PNS *and the diagonal property. Suppose $\mathcal{S} = \{H_1, H_2, ..., H_n\}$ is a finite commuting family of noncyclic relatively nonexpansive mappings on $A \cup B$. Then $\mathcal{S}$ has a common best proximity pair, that is, there is a point $(p, q) \in A \times B$ such that*

$$H_j p = p, \quad H_j q = q, \quad d(p, q) = \operatorname{dist}(A, B), \quad \forall j \in \{1, 2, ..., n\}.$$

Proof. It follows from Proposition 2.1 that (A_0, B_0) is nonempty and it is also closed and convex. For any $j \in \{1, 2, ..., n\}$ put

$$T_j := H_j \mathcal{P} : A_0 \cup B_0 \to A_0 \cup B_0,$$

where $\mathcal{P}$ is a projection mapping defined by (2.1). Since $\mathcal{P}$ is cyclic on $A_0 \cup B_0$, we obtain T_j is cyclic for any $j \in \{1, 2, ..., n\}$. Moreover, by the statement (vi) from the Proposition 2.2, the mappings H_j and $\mathcal{P}$ commute on $A_0 \cup B_0$. In view of the fact that the family $\mathcal{S}$ is commuting, for any $i, j \in \{1, 2, ..., n\}$ we have

$$\begin{aligned} T_i \circ T_j = (H_i \mathcal{P}) \circ (H_j \mathcal{P}) &= H_i \circ (\mathcal{P} \circ H_j) \circ \mathcal{P} = H_i \circ (H_j \circ \mathcal{P}) \circ \mathcal{P} \\ &= H_j \circ (H_i \circ \mathcal{P}) \circ \mathcal{P} = (H_j \circ \mathcal{P}) \circ (H_i \circ \mathcal{P}) = T_j \circ T_i, \end{aligned}$$

that is, T_i and T_j commutes. Furthermore, for any $j \in \{1, 2, ..., n\}$,

$$T_j(A_0) = H_j \mathcal{P}(A_0) = H_j(B_0) \subseteq B_0,$$
$$T_j(B_0) = H_1 \mathcal{P}(B_0) = H_j(B_0) \subseteq A_0,$$

and so T_j is cyclic on $A_0 \cup B_0$. By Proposition 1.3 of [14] that $\mathcal{P}$ is relatively isometry, and so for any $(x, y) \in A_0 \times B_0$ we have

$$d(T_j x, T_j y) = d(H_j \mathcal{P} x, H_j \mathcal{P} y) \le d(\mathcal{P} x, \mathcal{P} y) = d(x, y),$$

which deduces that T_j is a relatively nonexpansive mapping for any $j \in \{1, 2, ..., n\}$. It now follows from Theorem 3.2 that the family $\{T_j\}_{j=1}^n$ has a common best proximity point, that is, there exists a point $p \in A_0$ for which

$$d(p, T_j p) = \operatorname{dist}(A, B), \quad \forall j \in \{1, 2, ..., n\}.$$

Therefore,

$$\operatorname{dist}(A, B) = d(p, H_j \mathcal{P} p) = d(p, \mathcal{P} H_j p) = d(H_j p, \mathcal{P} H_j p),$$

and by the fact that (A, B) has the P-property, we must have $H_j p = p$ for any $j \in \{1, 2, ..., n\}$. This implies that

$$H_j \mathcal{P} p = \mathcal{P} H_j p = \mathcal{P} p.$$

Hence, $(p, \mathcal{P}p) \in A_0 \times B_0$ is a common best proximity pair of the family $\mathcal{S}$ and the proof is completed.

The following corollaries are straightforward consequence of Theorem 3.3.

Corollary 3.3. *Let (A, B) be a nonempty, compact and convex pair in a Busemann convex (X, d). If $\mathcal{S} = \{H_1, H_2, ..., H_n\}$ is a commuting family of noncyclic relatively nonexpansive mappings, then $\mathcal{S}$ has a common best proximity pair.*

Corollary 3.4. *Let (A, B) be a nonempty, bounded, closed and convex pair in a Busemann convex (X, d) which is uniformly convex in the sense of Definition 2.2. If $\mathcal{S} = \{H_1, H_2, ..., H_n\}$ is a commuting family of noncyclic relatively nonexpansive mappings, then $\mathcal{S}$ has a common best proximity pair.*

4. Convergence of common best proximity points

In this section, we prove a strong convergence theorem for a common best proximity point for two cyclic mappings by using Ishikawa iterative method ([20]).

Let (A, B) be a nonempty pair in a metric space X and T be a cyclic mapping on $A \cup B$. In what follows the set of all best proximity points of T in A is denoted by $\mathrm{Best}_A(T)$ and the set of all fixed points of the self-mapping $T^2 : A \to A$ well be denoted with $\mathrm{Fix}_A(T^2)$.

Theorem 4.1. *Let (A, B) be a nonempty, disjoint, bounded, closed and convex pair in a uniformly convex Banach space X and $S, T : A \cup B \to A \cup B$ be two commuting cyclic mappings such that*

$$\|Tx - Sy\| \leq \|x - y\|, \quad \forall (x, y) \in (A \times B) \cup (B \times A).$$

Let $x_0 \in A$ and define

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^2 y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n S^2 x_n, \end{cases} \tag{4.1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\alpha_n, \beta_n \in (\varepsilon, 1 - \varepsilon)$ and $\varepsilon \in (0, \frac{1}{2})$. If moreover, T satisfies the condition

$$\|T^2 x - Tx\| < \|x - Tx\| \text{ whenever } \|x - Tx\| > \mathrm{dist}(A, B), \quad \forall x \in A_0 \cup B_0, \tag{4.2}$$

then T and S has a common best proximity point, and $\|x_n - S^2 x_n\| \longrightarrow 0$. Also, if $S^2(A)$ lies in a compact set and $\mathrm{dist}(x_n, A_0) \longrightarrow 0$, then $\{x_n\}$ converges strongly to an element, namely $x^\star$, for which

$$x^\star \in \mathrm{Fix}_A(T^2) \cap \mathrm{Fix}_A(S^2) \cap \mathrm{Best}_A(T) \cap \mathrm{Best}_A(S).$$

Proof. Define $H_1, H_2 : A \cup B \to A \cup B$ with

$$H_1 x = \begin{cases} T^2 x & \text{if } x \in A, \\ S^2 x & \text{if } x \in B, \end{cases} \qquad \& \quad H_2 x = \begin{cases} S^2 x & \text{if } x \in A, \\ T^2 x & \text{if } x \in B, \end{cases}$$

Then for any $(x,y) \in A \times B$ we have

$$H_1(A) = T^2(A) \subseteq A, \quad H_1(B) = S^2(B) \subseteq B,$$
$$H_2(A) = S^2(A) \subseteq A, \quad H_2(B) = T^2(B) \subseteq B.$$

Also,

$$\|H_1x - H_1y\| = \|T^2x - S^2y\| \le \|x-y\|, \quad \|H_2x - H_2y\| = \|S^2x - T^2y\| \le \|x-y\|.$$

This implies that both H_1 and H_2 are noncyclic relatively nonexpansive mappings. Since T and S commutes, it is easy to see that H_1 and H_2 commutes too. It now follows from corollary 3 that H_1 and H_2 have a common best proximity pair, namely $(p,q) \in A \times B$, for which

$$\begin{cases} p = H_1p = T^2p, \ q = H_1q = S^2q, \\ p = H_2p = S^2p, \ q = H_2q = T^2q, \end{cases} \quad \& \quad \|p-q\| = \operatorname{dist}(A,B).$$

Notice that if $\|p - Tp\| > \operatorname{dist}(A,B)$, then by the condition (4.2) we obtain

$$\|p - Tp\| = \|T^2p - Tp\| < \|p - Tp\|,$$

which is impossible. Thus $p \in A$ is a best proximity point for the mapping T. Besides,

$$\|p - Tp\| = \|S^2p - Tp\| \le \|Sp - p\| = \|Sp - T^2p\| \le \|p - Tp\|,$$

which concludes that p is a best proximity point for the mapping S. Therefore,

$$p \in \operatorname{Best}_A(T) \cap \operatorname{Best}_A(S).$$

In view of the fact that $\|p - Tp\| = \operatorname{dist}(A,B) = \|p-q\|$ and that (A,B) has the P-property, we must have $Tp = q$. Hence

$$\|q - Sq\| = \|Tp - Sq\| \le \|p - q\| = \operatorname{dist}(A,B),$$

that is, $q \in B$ is a best proximity point for the mapping S. Thereby,

$$\|q - Tq\| = \|S^2q - Tq\| \le \|Sq - q\| = \operatorname{dist}(A,B),$$

which ensures that

$$q \in \operatorname{Best}_B(T) \cap \operatorname{Best}_B(S).$$

We now have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)x_n + \alpha_n T^2 y_n - (1-\alpha_n)q - \alpha_n T^2 q\| \\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n \|T^2y_n - T^2q\| \\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n \|y_n - q\| \\ &= (1-\alpha_n)\|x_n - q\| + \alpha_n \|(1-\beta_n)x_n + \beta_n S^2 x_n - q\| \\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n(1-\beta_n)\|x_n - q\| + \alpha_n\beta_n\|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

This deduces that the sequence $\{\|x_n - q\|\}$ is decreasing. Assume that

$$\lim_{n\to\infty} \|x_n - q\| = r \ge \operatorname{dist}(A,B).$$

Using Lemma 2.3 there exists a strictly increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, for which

$$\begin{aligned}\|y_n - q\|^2 &= \|(1-\beta_n)x_n + \beta_n S^2 x_n - (1-\beta_n)q - \beta_n T^2 q\|^2 \\ &= \|(1-\beta_n)(x_n - q) + \beta_n(S^2 x_n - T^2 q)\|^2 \\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n\|S^2 x_n - T^2 q\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n - S^2 x_n\|) \\ &\le \|x_n - q\|^2 - \beta_n(1-\beta_n)\varphi(\|x_n - S^2 x_n\|).\end{aligned}$$

Thus

$$\varepsilon^2\varphi(\|x_n - S^2x_n\|) < \beta_n(1-\beta_n)\varphi(\|x_n - S^2x_n\|) \le \|x_n - q\|^2 - \|y_n - q\|^2. \quad (4.3)$$

Also for all $n \in \mathbb{N}$

$$\|x_{n+1} - q\| - \|x_n - q\| + \alpha_n(\|x_n - q\| - \|y_n - q\|) \le 0,$$

and so,

$$\limsup_{n\to\infty}(\|x_n - q\| - \|y_n - q\|) \le 0.$$

It now follows from the inequality (4.3) and the property of the function φ that

$$\lim_{n\to\infty}\|x_n - S^2x_n\| = 0.$$

Since $S^2(A)$ lies in a compact set, $\{S^2x_n\}_{n\ge1}$ has a convergent subsequence $\{S^2x_{n_k}\}_{k\ge1}$, converging to a point $x^\star \in A$. Therefore,

$$\|x_{n_k} - x^\star\| \le \|x_{n_k} - S^2x_{n_k}\| + \|S^2x_{n_k} - x^\star\| \to 0,$$

that is, $x_{n_k} \to x^\star$. By the assumption since $\operatorname{dist}(x_n, A_0) \to 0$, there exists a sequence $\{a_n\} \subseteq A_0$ for which $\|x_n - a_n\| \to 0$ and so, $a_{n_k} \to x^\star$. Closedness of the set A_0 implies that $x^\star \in A_0$. On the other hand, for each $x \in A_0$ we have

$$\operatorname{dist}(A,B) \le \|S\mathcal{P}x - Tx\| \le \|\mathcal{P}x - x\| = \operatorname{dist}(A,B) = \|\mathcal{P}Tx - Tx\|,$$

and by this reality that (A,B) has the P-property we must have

$$\mathcal{P}Tx = S\mathcal{P}x, \quad \forall x \in A_0.$$

Similarly, we can see that $T\mathcal{P}x = \mathcal{P}Sx$ for any $x \in A_0$. Furthermore, $T^2\mathcal{P}x = \mathcal{P}S^2x$ for all $x \in A_0$. Hence,

$$\|S^2x_{n_k} - \mathcal{P}S^2x^\star\| = \|S^2x_{n_k} - T^2\mathcal{P}x^\star\| \le \|x_{n_k} - \mathcal{P}x^\star\|.$$

If $k \to \infty$, in above relation, then

$$\|x^\star - \mathcal{P}S^2x^\star\| \le \|x^\star - \mathcal{P}x^\star\| = \operatorname{dist}(A,B).$$

In view of the fact that (A,B) has the P-property and $\|S^2x^\star - \mathcal{P}S^2x^\star\| = \operatorname{dist}(A,B)$, we obtain $S^2x^\star = x^\star$. We now have

$$\operatorname{dist}(A,B) = \|x^\star - \mathcal{P}x^\star\| = \|S^2x^\star - \mathcal{P}S^2x^\star\| = \|x^\star - T^2\mathcal{P}x^\star\|,$$

which ensures that $\mathcal{P}x^\star = T^2\mathcal{P}x^\star$, that is, $\mathcal{P}x^\star \in B_0$ is a fixed point of T^2 and so,

$$\lim_{n\to\infty}\|x_n - \mathcal{P}x^\star\|$$

exists. Thus

$$\lim_{n\to\infty}\|x_n - \mathcal{P}x^\star\| = \lim_{k\to\infty}\|x_{n_k} - \mathcal{P}x^\star\| = \|x^\star - \mathcal{P}x^\star\| = \operatorname{dist}(A,B).$$

It now follows from Lemma 2.4 that $x_n \to x^\star$. Here we prove that $x^\star$ is a common best proximity point for the mappings S and T. Since $\mathcal{P}x^\star = T^2\mathcal{P}x^\star$, we conclude that

$$\|\mathcal{P}x^\star - T\mathcal{P}x^\star\| = \text{dist}(A,B) = \|\mathcal{P}x^\star - x^\star\|$$

which implies that $T\mathcal{P}x^\star = x^\star$. Thereby,

$$\mathcal{P}x^\star = T^2\mathcal{P}x^\star = T(T\mathcal{P}x^\star) = Tx^\star,$$

and then

$$\|x^\star - Tx^\star\| = \|x^\star - \mathcal{P}x^\star\| = \text{dist}(A,B),$$

that is, $x^\star$ is a best proximity point for the mapping T.
Moreover, since $\mathcal{P}Tx^\star = S\mathcal{P}x^\star$,

$$\text{dist}(A,B) = \|Sx^\star - \mathcal{P}Sx^\star\| = \|Sx^\star - T\mathcal{P}x^\star\| = \|Sx^\star - x^\star\|,$$

i.e. $x^\star$ is a best proximity point for the mapping S. Therefore,

$$x^\star \in \text{Best}_A(T) \cap \text{Best}_A(S).$$

Finally, from the above discussion we have

$$T^2x^\star = T(Tx^\star) = T\mathcal{P}x^\star = x^\star.$$

Thus

$$x^\star \in \text{Fix}_A(T^2) \cap \text{Fix}_A(S^2) \cap \text{Best}_A(T) \cap \text{Best}_A(S),$$

and this completes the proof.

Let us illustrate this reality with the following example.

Example 4.1. Consider $X = \mathbb{R}$ with the Euclidean norm and let $A = [-1,0]$, $B = [0,1]$. Suppose $S,T : A \cup B \to A \cup B$ are defined as

$$Tx = \begin{cases} \frac{-x}{2} & \text{if} \quad x \in \mathbb{Q} \cap (A \cup B), \\ 0 \quad \text{if} & x \in \mathbb{Q}^c \cap (A \cup B), \end{cases} \qquad \& \quad Sx = \begin{cases} \frac{-x}{3} & \text{if} \quad x \in \mathbb{Q}^c \cap (A \cup B), \\ 0 \quad \text{if} & x \in \mathbb{Q} \cap (A \cup B). \end{cases}$$

It is easy to check that S and T are cyclic and commuting mappings which satisfy $|Tx - Sy| \leq |x - y|$ for any $(x,y) \in (A \times B) \cup (B \times A)$. Moreover, for any $x \neq 0$, we have

$$\begin{aligned} |T^2x - Tx| &= \begin{cases} \frac{-3x}{4} & \text{if} \quad x \in \mathbb{Q} \cap (A \cup B), \\ 0 \quad \text{if} & x \in \mathbb{Q}^c \cap (A \cup B) \end{cases} \\ &< \begin{cases} \frac{-3x}{2} & \text{if} \quad x \in \mathbb{Q} \cap (A \cup B), \\ -x & \text{if} \quad x \in \mathbb{Q}^c \cap (A \cup B) \end{cases} \\ &= |x - Tx|, \end{aligned}$$

which ensures that the relation (4.2) satisfies. We note that the point $x^\star = 0$ is a common best proximity point of the mapping T and S which is a common fixed point of T and S in this case.

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