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FIXED POINT THEOREMS FOR NONSELF OPERATORS ON A LARGE KASAHARA SPACE

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Abstract. In this paper we give some fixed point theorems for nonself operators on a large Kasahara space, which generalize some results given by I.A. Rus and M.-A. Şerban (I.A. Rus, M.-A. Şerban, *Some fixed point theorems for nonself generalized contractions*, Miskolc Math. Notes, **17**(2016), no.2, 1021-1031) and by S. Reich and A.J. Zaslavski (S. Reich, A.J. Zaslavski, *A note on Rakotch contractions*, Fixed Point Theory, **9**(2008), no.1, 267-273).

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1. Introduction and preliminaries

There are several techniques in the fixed point theory for nonself operators on a complete metric space ([6], [18], [12], [15], [14], [2], [8], [19], ...). Some results are given in the case of Kasahara spaces ([3], [4], [13]). By following the papers of S. Reich and A.J. Zaslavski [8] and I.A. Rus and M.-A. Şerban [19] we give some fixed point theorems for nonself operators on a large Kasahara space.

In this paper we will use the notations and terminology given in [3] and [19]. The notions of comparison function, L-space and large Kasahara space are recalled below.

Definition 1.1. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. If φ is monotone increasing, i.e., for all $t_1, t_2 \in \mathbb{R}_+$, $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$, and the sequence $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \in \mathbb{R}_+$, then φ is a comparison function.

If φ is a continuous comparison function satisfying $t - \varphi(t) \to \infty$ as $t \to \infty$, then φ is called strict comparison function. In this case, we can define the function

$$\theta_\varphi : \mathbb{R}_+ \to \mathbb{R}_+,\ \theta_\varphi(t) = \sup\{s \in \mathbb{R}_+ \mid s - \varphi(s) \leq t\},\ \text{for all } t \in \mathbb{R}_+$$

which is increasing and has the property $\theta_\varphi(t) \to 0$ as $t \to 0$. We will use the function θ_φ to study the data dependence of the fixed points.

If φ is a comparison function satisfying $\sum_{n\in\mathbb{N}} \varphi^n(t) < \infty$, for all $t \in \mathbb{R}_+$, then φ is called strong comparison function.

More consideration on comparison functions are given in [1] and [10].

The notion of L-space was given by M. Fréchet in 1906 (see [5]).

Definition 1.2. Let X be a nonempty set. Let $s(X) := \big\{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\big\}$. Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (denoted also by $(X, \xrightarrow{F})$) if the following conditions are satisfied:

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$.

(ii) if $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$, then for all subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, we have that $\{x_{n_i}\}_{i\in\mathbb{N}} \in c(X)$ and $Lim\{x_{n_i}\}_{i\in\mathbb{N}} = x$.

By definition, an element $\{x_n\}_{n\in\mathbb{N}}$ of $c(X)$ is a convergent sequence,

$$x = Lim\{x_n\}_{n\in\mathbb{N}}$$

is the limit of this sequence and we also write $x_n \xrightarrow{F} x$ as $n \to \infty$.

Example 1.1. Let X be a nonempty set and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X which does not have a constant subsequence. Let $c_1(X)$ be the set of all constant sequences and $c_2(X)$ be the set of all subsequences of $\{x_n\}_{n\in\mathbb{N}}$. Let $c(X) := c_1(X) \cup c_2(X)$ and $z \in X$ be an arbitrary element. If $\{x_n\}_{n\in\mathbb{N}} \in c_1(X)$ then $Lim\{x_n\}_{n\in\mathbb{N}} = x$. If $\{y_n\}_{n\in\mathbb{N}} \in c_2(X)$ then $Lim\{y_n\}_{n\in\mathbb{N}} = z$. The triple $(X, c(X), Lim)$ is an L-space.

More examples of L-spaces are given in [11].

The notion of large Kasahara space was given by I.A. Rus in [13], as follows:

Definition 1.3. Let X be a nonempty set, $\xrightarrow{F}$ be an L-space structure on X, $(G, +, \leq, \xrightarrow{G})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \xrightarrow{F}, d_G)$ is a large Kasahara space iff we have the following compatibility condition between $\xrightarrow{F}$ and d_G:

(i) $x_n \in X$, $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence (in some sense) with respect to d_G implies that $(x_n)_{n\in\mathbb{N}}$ converges in $(X, \xrightarrow{F})$.

The notion of large Kasahara space which will be used in this paper, is the following:

Definition 1.4. Let X be a nonempty set, $\xrightarrow{F}$ be an L-space structure on X and $d : X \times X \to \mathbb{R}_+$ be a metric on X. The triple $(X, \xrightarrow{F}, d)$ is a large Kasahara space iff we have the following compatibility conditions between $\xrightarrow{F}$ and d:

(i) $\{x_n\}_{n\in\mathbb{N}}$ is a fundamental sequence in $(X, d) \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ converges in $(X, \xrightarrow{F})$;

(ii) $x_n \xrightarrow{F} x^*$, $y_n \xrightarrow{F} y^*$ and $d(x_n, y_n) \to 0$ as $n \to \infty \Rightarrow x^* = y^*$.

Example 1.2. (See [9], [17], [4]). Let (X, ρ) be a complete metric space and (X, d) be a metric space. We suppose that there exists $c > 0$ such that $\rho(x, y) \leq c d(x, y)$, for all $x, y \in X$. Then, $(X, \xrightarrow{\rho}, d)$ is a large Kasahara space.

Example 1.3. We give here a counterexample of large Kasahara space, showing that the condition (ii) of the Definition 1.4. is necessary.

Let $X := \mathbb{R}$, $c(\mathbb{R}) := c_1(\mathbb{R}) \cup c_2(\mathbb{R}) \cup c_3(\mathbb{R})$, where $c_1(\mathbb{R})$ is the set of all convergent sequences with respect to the metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, defined by $d(x, y) = |x - y|$, for all $x, y \in \mathbb{R}$ and, on $c_1(\mathbb{R})$, we consider $\xrightarrow{F} := \xrightarrow{d}$; $c_2(\mathbb{R})$ is the set of all subsequences

$\{x_n\}_{n\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ with $Lim\{x_n\}_{n\in\mathbb{N}} = 0$; $c_3(\mathbb{R})$ be the set of all subsequences $\{y_n\}_{n\in\mathbb{N}}$ of $\{n + \frac{1}{n+1}\}_{n\in\mathbb{N}}$ with $Lim\{y_n\}_{n\in\mathbb{N}} = 1$. Notice that $(\mathbb{R}, c(\mathbb{R}), Lim)$ is an L-space. But the triple $(\mathbb{R}, \stackrel{F}{\to}, d)$ is not a large Kasahara space. The condition (i) of Definition 1.4. is satisfied, but the condition (ii) is not. Indeed, let

$$x_n := n \text{ and } y_n := n + \frac{1}{n+1},$$

for all $n \in \mathbb{N}$. For these two sequences we have $x_n \stackrel{F}{\to} 0$, $y_n \stackrel{F}{\to} 1$ and $d(x_n, y_n) \to 0$ as $n \to \infty$.

Remark 1.1. Let $(X, \stackrel{F}{\to}, d)$ be a large Kasahara space. Then for any sequence $\{x_n\}_{n\in\mathbb{N}^*} \subset X$ with $x_n \stackrel{d}{\to} x^*$ as $n \to \infty$, we have $x_n \stackrel{F}{\to} x^*$ as $n \to \infty$. This implies that for any subset $A \subset X$, with A closed in $(X, \stackrel{F}{\to})$, A is closed in (X, d).

2. The case of l-contractions

We give here one of the main results of this paper, concerning the existence and uniqueness of fixed points for nonself l-contractions, in the context of a large Kasahara space. The data dependence of the fixed point is also discussed (see [16]).

Theorem 2.1. *Let $(X, \stackrel{F}{\to}, d)$ be a large Kasahara space, $Y \subset X$ be a closed subset of $(X, \stackrel{F}{\to})$ and $f : Y \to X$ be an operator. We suppose that:*

- (i) *there exists $y_n \in Y$, for all $n \in \mathbb{N}^*$, such that the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded and $f^i(y_n)$ is defined for $i = \overline{1, n}$, $n \in \mathbb{N}^*$;*
- (ii) *f is continuous in $(X, \stackrel{F}{\to})$;*
- (iii) *f is an l-contraction w.r.t. the metric d.*

Then:

1. $F_f = \{x^*\}$;
2. $f^n(y_n) \stackrel{F}{\to} x^*$ *as* $n \to \infty$;
3. $f^n(y_n) \stackrel{d}{\to} x^*$ *as* $n \to \infty$;
4. $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, *for all* $x \in Y$;
5. *if the operator $g : Y \to X$ is such that*
   - (j) *there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for all $x \in Y$;*
   - (jj) $F_g \neq \varnothing$

   then $d(x^*, y^*) \leq \frac{\eta}{1-l}$, for all $y^* \in F_g$.

Proof. $(1) + (2)$. First, we remark that

$$\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$$

is a bounded set. Indeed, since the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded, for a given $y_0 \in Y$ there exists a constant $R > 0$ such that $d(y_0, y_n) \leq R$, for all $n \in \mathbb{N}^*$.

By the assumption (iii), we have the following estimations

$$d(y_0, f(y_n)) \leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) \leq d(y_0, f(y_0)) + lR, \text{ for all } n \in \mathbb{N}^*.$$

On the other hand, for any $i \geq 2$, we have

$$\begin{aligned} d(y_0, f^i(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) + d(f(y_n), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + lR + \sum_{j=1}^{i-1} d(f^j(y_n), f^{j+1}(y_n)) \\ &\leq d(y_0, f(y_0)) + lR + \frac{l}{1-l} d(y_n, f(y_n)), \text{ for all } n \in \mathbb{N}^* \end{aligned}$$

and

$$d(y_n, f(y_n)) \leq d(y_n, y_0) + d(y_0, f(y_n)) \leq R + lR + d(y_0, f(y_0)).$$

So, there exists

$$\tilde{R} := \frac{1}{1-l} d(y_0, f(y_0)) + \frac{2lR}{1-l} > 0$$

such that $d(y_0, f^i(y_n)) \leq \tilde{R}$, for all $n \in \mathbb{N}^*$ and $i = \overline{0, n-1}$. Thus, the set

$$\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$$

is bounded.

Let $A \in P_b(Y)$ be such that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\} \subset A$. (*The authors of the paper* [19] *informed me that in their paper, the condition* $\{y_n \mid n \in \mathbb{N}\} \subset A$ *must be replaced with* $\{f^i(y_n) \mid i = \overline{0, n-1}, n \in \mathbb{N}\} \subset A$).

Let $A_1 := f(A)$, $A_2 := f(A_1 \cap A)$, ..., $A_{n+1} := f(A_n \cap A)$, for all $n \in \mathbb{N}^*$. By this construction, we obtain the sequence of sets $\{A_n\}_{n \in \mathbb{N}^*}$ with the following properties: $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}^*$, and $f^n(y_n) \in A_n$, for all $n \in \mathbb{N}^*$.

Since the operator f is an l-contraction w.r.t. the metric d, there exists a constant $l \in [0, 1)$ such that $d(f(x), f(y)) \leq l d(x, y)$, for all $x, y \in Y$. By taking the supremum over $x, y \in Y$ in the contraction condition, we get that

$$\delta(f(B)) \leq l\delta(B), \text{ for all } B \in P_b(Y).$$

By using the properties of the diameter functional δ and the contraction condition of the operator f, we have

$$\delta(A_{n+1}) = \delta(f(A_n \cap A)) \leq \delta(f(A_n)) \leq l\delta(A_n), \text{ for all } n \in \mathbb{N}^*.$$

By mathematical induction over $n \in \mathbb{N}^*$, it follows that

$$\delta(A_{n+1}) \leq l^{n+1}\delta(A), \text{ for all } n \in \mathbb{N}^*.$$

By letting $n \to \infty$, we get that $\delta(A_{n+1}) \to 0$.

Since $f^n(y_n) \in A_n$, $f^{n-1}(y_n) \in A_{n-1} \cap A$ and $\delta(A_{n-1}) \to 0$ as $n \to \infty$, we get that $\{f^n(y_n)\}_{n \in \mathbb{N}^*}$ and $\{f^{n-1}(y_n)\}_{n \in \mathbb{N}^*}$ are fundamental sequences in (X, d).

By the condition (i) of Definition 1.4. we have that

$$f^{n-1}(y_n) \xrightarrow{F} u^* \text{ and } f^n(y_n) \xrightarrow{F} v^* \text{ as } n \to \infty.$$

On the other hand, $d(f^{n-1}(y_n), f^n(y_n)) \to 0$ as $n \to \infty$.

By the condition (ii) of Definition 1.4. we have that $u^* = v^* =: x^*$.

Since f is continuous, we have

$$f^n(y_n) = f(f^{n-1}(y_n)) \overset{F}{\to} f(x^*) \text{ as } n \to \infty.$$

So, $f(x^*) = x^*$. Hence $F_f = \{x^*\}$.

(3). By (iii),

$$d(f^n(y_n), x^*) = d(f^n(y_n), f^n(x^*)) \leq ld(f^{n-1}(y_n), f^{n-1}(x^*)) \leq \ldots \leq l^n d(y_n, x^*) \to 0$$

as $n \to \infty$. So, $f^n(y_n) \overset{d}{\to} x^*$ as $n \to \infty$.

(4). Let $x \in Y$. By using the triangle inequality of the metric d and the assumption (iii), we have

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + ld(x, x^*).$$

So, $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, for all $x \in Y$.

(5). Let $y^* \in F_g$. Then, by choosing $x := y^*$ in the conclusion (4), we get

$$d(y^*, x^*) \leq \frac{1}{1-l} d(y^*, f(y^*)) = \frac{1}{1-l} d(g(y^*), f(y^*)).$$

By the symmetry of the metric d and by condition (j), it follows that

$$d(x^*, y^*) \leq \frac{\eta}{1-l}, \text{ for all } y^* \in F_g.$$

3. The case of φ-contractions

The following theorem generalizes Theorem 1 and Theorem 3 given by I.A. Rus and M.-A. Şerban in [19], for φ-contractions in the context of complete metric spaces, and Theorem 1, given by S. Reich and A.J. Zaslavski in [8], for Rakotch contractions, in the same context.

Theorem 3.1. *Let $(X, \overset{F}{\to}, d)$ be a large Kasahara space, $Y \subset X$ be a closed subset of $(X, \overset{F}{\to})$ and $f : Y \to X$ be an operator. We suppose that:*

- (i) *there exists $y_n \in Y$, for all $n \in \mathbb{N}^*$, such that the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded and $f^i(y_n)$ is defined for $i = \overline{1, n}$, $n \in \mathbb{N}^*$;*
- (ii) *f is continuous in $(X, \overset{F}{\to})$;*
- (iii) *f is a φ-contraction w.r.t. the metric d, where φ is a strict and strong comparison function.*

Then:

1. *$F_f = \{x^*\}$;*
2. *$f^n(y_n) \overset{F}{\to} x^*$ as $n \to \infty$;*
3. *$f^n(y_n) \overset{d}{\to} x^*$ as $n \to \infty$;*
4. *$d(f^n(y_n), x^*) \leq \varphi(d(y_n, x^*))$, for all $n \in \mathbb{N}^*$;*
5. *$d(x, x^*) \leq \theta_\varphi(d(x, f(x)))$, for all $x \in Y$;*
6. *if $g : Y \to X$ is such that*
   - (j) *there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for all $x \in Y$;*
   - (jj) *$F_g \neq \varnothing$*

   then $d(x^*, y^*) \leq \theta_\varphi(\eta)$, for all $y^* \in F_g$.

Proof. (1) + (2). First, we remark that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is a bounded set. Indeed, since the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded, for a given $y_0 \in Y$ there exists a constant $R > 0$ such that $d(y_0, y_n) \leq R$, for all $n \in \mathbb{N}^*$.

Since $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, we have $\varphi(t) \leq t$, for all $t \geq 0$.

By (iii) we have the following estimations

$$\begin{aligned} d(y_0, f(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) \leq d(y_0, f(y_0)) + \varphi(d(y_0, y_n)) \\ &\leq d(y_0, f(y_0)) + R, \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

On the other hand, for any $i \geq 2$, we have

$$\begin{aligned} d(y_0, f^i(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) + d(f(y_n), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + R + \sum_{j=1}^{i-1} d(f^j(y_n), f^{j+1}(y_n)) \\ &\leq d(y_0, f(y_0)) + R + \sum_{j=1}^{\infty} \varphi^j(d(y_n, f(y_n))), \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

Since φ is a strong comparison function, it follows that

$$\sum_{j=1}^{\infty} \varphi^j(d(y_n, f(y_n))) < \infty.$$

So, there exists a number $\Phi \in \mathbb{R}_+$ such that $d(y_0, f^i(y_n)) \leq d(y_0, f(y_0)) + R + \Phi$, for all $n \in \mathbb{N}^*$. Thus, the set $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is bounded.

Let $A \in P_b(Y)$ be such that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\} \subset A$.

Let $A_1 := f(A)$, $A_2 := f(A_1 \cap A)$, …, $A_{n+1} := f(A_n \cap A)$, for all $n \in \mathbb{N}^*$. By this construction, we obtain the sequence of sets $\{A_n\}_{n \in \mathbb{N}^*}$ with the following properties: $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}^*$, and $f^n(y_n) \in A_n$, for all $n \in \mathbb{N}^*$.

Since the operator f is a φ-contraction w.r.t. the metric d,

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in Y.$$

By taking the supremum over $x, y \in Y$ in the contraction condition, we get that

$$\delta(f(B)) \leq \varphi(\delta(B)), \text{ for all } B \in P_b(Y).$$

By using the properties of the diameter functional δ and by taking into account the assumption (iii), we have

$$\delta(A_{n+1}) = \delta(f(A_n \cap A)) \leq \delta(f(A_n)) \leq \varphi(\delta(A_n)), \text{ for all } n \in \mathbb{N}^*.$$

By mathematical induction over $n \in \mathbb{N}^*$, it follows that

$$\delta(A_{n+1}) \leq \varphi^{n+1}(\delta(A)), \text{ for all } n \in \mathbb{N}^*.$$

By letting $n \to \infty$, we get that $\delta(A_{n+1}) \to 0$.

Since $f^n(y_n) \in A_n$, $f^{n-1}(y_n) \in A_{n-1} \cap A$ and $\delta(A_{n-1}) \to 0$ as $n \to \infty$, we get that $\{f^n(y_n)\}_{n \in \mathbb{N}^*}$ and $\{f^{n-1}(y_n)\}_{n \in \mathbb{N}^*}$ are fundamental sequences in (X, d).

By the condition (i) of Definition 1.4. we have that

$$f^{n-1}(y_n) \xrightarrow{F} u^* \text{ and } f^n(y_n) \xrightarrow{F} v^* \text{ as } n \to \infty.$$

On the other hand, $d(f^{n-1}(y_n), f^n(y_n)) \to 0$ as $n \to \infty$.
By the condition (ii) of Definition 1.4. we have that $u^* = v^* =: x^*$.
Since f is continuous, we have

$$f^n(y_n) = f(f^{n-1}(y_n)) \xrightarrow{F} f(x^*) \text{ as } n \to \infty.$$

So, $f(x^*) = x^*$. Hence $F_f = \{x^*\}$.
$(3) + (4)$. By using the assumption (iii), we have

$$d(f^n(y_n), x^*) = d(f^n(y_n), f^n(x^*)) \le \varphi(d(f^{n-1}(y_n), f^{n-1}(x^*))) \le \ldots \le \varphi^n(d(y_n, x^*)),$$

for all $n \in \mathbb{N}^*$. Since $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, we have that $\varphi^n(d(y_n, x^*)) \to 0$ as $n \to \infty$. So, $f^n(y_n) \xrightarrow{d} x^*$ as $n \to \infty$. On the other hand, φ is an increasing function on $\mathbb{R}_+$ and $\varphi(t) \le t$, for all $t \in \mathbb{R}_+$, with equality when $t = 0$. By mathematical induction over $n \in \mathbb{N}^*$ we have

$$d(f^n(y_n), x^*) \le \varphi^n(d(y_n, x^*)) \le \ldots \le \varphi(d(y_n, x^*)), \text{ for all } n \in \mathbb{N}^*.$$

(5). Since $d(x, x^*) \le d(x, f(x)) + d(f(x), x^*) \le d(x, f(x)) + \varphi(d(x, x^*))$, for all $x \in Y$, it follows that $d(x, x^*) - \varphi(d(x, x^*)) \le d(x, f(x))$, for all $x \in Y$. We have next

$$d(x, x^*) \le \sup_{d(x,x^*) - \varphi(d(x,x^*)) \le d(x, f(x))} d(x, x^*) = \theta_\varphi(d(x, f(x))), \text{ for all } x \in Y.$$

(6). Let $y^* \in F_g$. Then, from the conclusion (5) we have

$$d(x^*, y^*) = d(y^*, x^*) \le \theta_\varphi(d(y^*, f(y^*))) = \theta_\varphi(d(g(y^*), f(y^*))) \le \theta_\varphi(\eta).$$

4. The case of Kannan operators

Let us recall first the notion of α-Kannan nonself operator.

Definition 4.1. Let (X, d) be a metric space and $Y \in P(X)$. The operator $f : Y \to X$ is an α-Kannan operator, if there exists a constant $\alpha \in [0, \frac{1}{2})$ such that

$$d(f(x), f(y)) \le \alpha[d(x, f(x)) + d(y, f(y))], \text{ for all } x, y \in Y.$$

In the proof of the main result of this section, we will use the maximal displacement functional, also recalled below.

Definition 4.2. Let (X, d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous nonself operator. By the maximal displacement functional corresponding to f, we understand the functional $E_f : P(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$E_f(A) := \sup\{d(x, f(x)) \mid x \in A\}, \text{ for all } A \in P(Y).$$

The maximal displacement functional has the following properties:
(i) $A, B \in P(Y)$, $A \subset B$ imply $E_f(A) \le E_f(B)$;
(ii) $E_f(A) = E_f(\overline{A})$, for all $A \in P(Y)$.

Lemma 4.1. Let (X, d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous α-Kannan operator. Then:

(1) $E_f(f(A)) \le \frac{\alpha}{1-\alpha} E_f(A)$, for all $A \in Y$ with $f(A) \subset Y$;

(2) $E_f(f(A)\cap Y)\leq \frac{\alpha}{1-\alpha}E_f(A)$, for all $A\in Y$ with $f(A)\cap Y\neq\varnothing$.

Proof. (1). From the definition of the maximal displacement functional corresponding to f we have that

$$E_f(f(A))=\sup\{d(x,f(x))\ |x\in f(A)\}=\sup\{d(f(u),f^2(u))\ |u\in A\}.$$

Since f is an α-Kannan operator, we have

$$d(f(u),f^2(u))\leq\alpha[d(u,f(u))+d(f(u),f^2(u))],\ \text{for all } u\in A.$$

It follows that $d(f(u),f^2(u))\leq\frac{\alpha}{1-\alpha}d(u,f(u))$, for all $u\in A$. Hence

$$E_f(f(A))\leq\frac{\alpha}{1-\alpha}\sup\{d(u,f(u))\mid u\in A\}=\frac{\alpha}{1-\alpha}E_f(A),$$

for all $A\in Y$ with $f(A)\subset Y$.

(2). Taking into consideration that f is an α-Kannan operator, we have that

$$\begin{aligned}E_f(f(A)\cap Y)&=\sup\{d(x,f(x))\mid x\in f(A)\cap Y\}\\&=\sup\{d(f(u),f^2(u))\mid u\in A,\ f(u)\in Y\}\\&\leq\frac{\alpha}{1-\alpha}\sup\{d(u,f(u))\mid u\in A\}\\&=\frac{\alpha}{1-\alpha}E_f(A),\end{aligned}$$

for all $A\in Y$ with $f(A)\cap Y\neq\varnothing$.

The following result generalizes Theorem 4 given by I.A. Rus and M.-A. Şerban in [19] for Kannan operators in the context of complete metric spaces.

Theorem 4.1. *Let $(X,\stackrel{F}{\to},d)$ be a large Kasahara space, $Y\subset X$ be a closed subset of $(X,\stackrel{F}{\to})$ and $f:Y\to X$ be an operator. We suppose that:*

- (i) *there exists $y_n\in Y$, for all $n\in\mathbb{N}^*$, such that the set $\{y_n\mid n\in\mathbb{N}^*\}$ is bounded and $f^i(y_n)$ is defined for $i=\overline{1,n}$, $n\in\mathbb{N}^*$;*
- (ii) *f is continuous in $(X,\stackrel{F}{\to})$;*
- (iii) *f is an α-Kannan operator w.r.t. the metric d;*
- (iv) *$E_f(Y)<\infty$.*

Then:

1. *$F_f=\{x^*\}$;*
2. *$f^n(y_n)\stackrel{F}{\to}x^*$ as $n\to\infty$;*
3. *$f^n(y_n)\stackrel{d}{\to}x^*$ as $n\to\infty$;*
4. *$d(x,x^*)\leq(1+\alpha)d(x,f(x))$, for all $x\in Y$;*
5. *$d(f^{n-1}(x_n),x^*)\leq(1+\alpha)\left(\frac{\alpha}{1-\alpha}\right)^{n-1}d(x_n,f(x_n))$, for all $n\in\mathbb{N}^*$;*
6. *if $g:Y\to X$ is such that*
   - (j) *there exists $\eta>0$ such that $d(f(x),g(x))\leq\eta$, for all $x\in Y$;*
   - (jj) *$F_g\neq\varnothing$*

   then $d(x^*,y^*)\leq(1+\alpha)\eta$, for all $y^*\in F_g$.

Proof. (1) + (2). First, we remark that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is a bounded set. Indeed, since the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded, for a given $y_0 \in Y$ there exists a constant $R > 0$ such that $d(y_0, y_n) \leq R$, for all $n \in \mathbb{N}^*$.

By (iii) we have the following estimations

$$\begin{aligned} d(y_0, f(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) \\ &\leq d(y_0, f(y_0)) + \alpha[d(y_0, f(y_0)) + d(y_n, f(y_n))] \\ &\leq (1+\alpha)d(y_0, f(y_0)) + \alpha d(y_n, y_0) + \alpha d(y_0, f(y_n)),\ \text{for all } n \in \mathbb{N}^*. \end{aligned}$$

It follows that

$$d(y_0, f(y_n)) \leq \frac{1+\alpha}{1-\alpha} d(y_0, f(y_0)) + \frac{\alpha}{1-\alpha} R,\ \text{for all } n \in \mathbb{N}^*.$$

On the other hand, for any $i \geq 2$, we have

$$\begin{aligned} d(y_0,&f^i(y_n)) \leq d(y_0, f(y_0)) + d(f(y_0), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) + d(f(y_n), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + \alpha[d(y_0, f(y_0)) + d(y_n, f(y_n))] + \sum_{j=1}^{i-1} d(f^j(y_n), f^{j+1}(y_n)) \\ &\leq (1+\alpha)d(y_0, f(y_0)) + \alpha[d(y_n, y_0) + d(y_0, f(y_n))] + \sum_{j=1}^{i-1} \left(\frac{\alpha}{1-\alpha}\right)^j d(y_n, f(y_n)) \\ &\leq (1+\alpha)d(y_0, f(y_0)) + \alpha R + \alpha d(y_0, f(y_n)) + \frac{\alpha}{1-2\alpha}[d(y_n, y_0) + d(y_0, f(y_n))] \\ &\leq (1+\alpha)d(y_0, f(y_0)) + 2\alpha\frac{1-\alpha}{1-2\alpha}[R + d(y_0, f(y_n))] \\ &\leq \frac{1+\alpha}{1-2\alpha} d(y_0, f(y_0)) + \frac{2\alpha}{1-2\alpha} R,\ \text{for all } n \in \mathbb{N}^*. \end{aligned}$$

Thus, the set $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is bounded.

Let $A \in P_b(Y)$ be such that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\} \subset A$.

Let $A_1 := f(A)$, $A_2 := f(A_1 \cap A)$, …, $A_{n+1} := f(A_n \cap A)$, for all $n \in \mathbb{N}^*$. By this construction, we obtain the sequence of sets $\{A_n\}_{n \in \mathbb{N}^*}$ with the following properties: $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}^*$, and $f^n(y_n) \in A_n$, for all $n \in \mathbb{N}^*$.

By the definitions of the diameter functional δ and maximal displacement functional E_f and taking into account the properties stated in Lemma 4.1., we have

$$\begin{aligned} \delta(A_{n+1}) = \delta(f(A_n \cap A)) &\leq 2\alpha E_f(A_n \cap A) = 2\alpha E_f(f(A_{n-1} \cap A) \cap A) \\ &\leq \frac{2\alpha^2}{1-\alpha} E_f(A_{n-1} \cap A) \leq \ldots \leq \frac{2\alpha^{n+1}}{(1-\alpha)^n} E_f(A) \to 0 \text{ as } n \to \infty. \end{aligned}$$

By following the proof of Theorem 2.1., the conclusions follow.

(3). Follows from the proof of (5).

(4). Let $x \in Y$. By using the assumption (iii), we have

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + \alpha[d(x, f(x)) + d(x^*, f(x^*))].$$

It follows that $d(x, x^*) \leq (1+\alpha)d(x, f(x))$, for all $x \in Y$.

(5). Let $x := f^{n-1}(x_n)$ in the conclusion (4). Then we have

$$\begin{aligned} d(f^{n-1}(x_n), x^*) &\leq (1+\alpha)d(f^{n-1}(x_n), f^n(x_n)) \\ &\leq (1+\alpha)\frac{\alpha}{1-\alpha}d(f^{n-2}(x_n), f^{n-1}(x_n)) \\ &\vdots \\ &\leq (1+\alpha)\left(\frac{\alpha}{1-\alpha}\right)^{n-1} d(x_n, f(x_n)), \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

(6). Let $y^* \in F_g$. By letting $x := y^*$ in the conclusion (4) we have

$$d(y^*, x^*) \leq (1+\alpha)d(y^*, f(y^*)).$$

Hence $d(x^*, y^*) \leq (1+\alpha)d(g(y^*), f(y^*)) \leq (1+\alpha)\eta$, for all $y^* \in F_g$.

5. Some problems

The above considerations give rise to the following problems.

Problem 5.1. Let (X, d) be a metric space, $Y \subset X$ be a nonempty closed subset and $\{y_n\}_{n\in\mathbb{N}} \subset Y$ be a bounded sequence. For which generalized contractions ([10], [18], [1], [7]) $f : Y \to X$, the following implication holds: $f^i(y_n)$ is defined for $i = \overline{1,n}$, $n \in \mathbb{N}^*$ implies that the set $\{f^i(y_n) \mid i = \overline{1,n},\ n \in \mathbb{N}^*\}$ is bounded ?

Problem 5.2. If f is a solution of Problem 5.1. in which conditions we have that:

(*i*) $F_f = \{x^*\}$;
(*ii*) $f^n(y_n) \to x^*$ as $n \to \infty$?

Problem 5.3. Let $(X, \stackrel{F}{\to}, d)$ be a large Kasahara space, $Y \subset X$ be a closed subset of $(X, \stackrel{F}{\to})$ and $f : Y \to X$ be an operator. We suppose that:

(*i*) there exists a bounded sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$ such that $f^i(y_n)$ is defined for $i = \overline{1,n}$, $n \in \mathbb{N}^*$;
(*ii*) f is continuous in $(X, \stackrel{F}{\to})$.

The problem is to find those generalized contractions f, satisfying the above conditions, for which we have that:

(1) $F_f = \{x^*\}$;
(2) $f^n(y_n) \stackrel{F}{\to} x^*$ as $n \to \infty$.

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