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FIXED POINTS FOR MAPPINGS OF ASYMPTOTICALLY NONEXPANSIVE TYPE

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Abstract. We prove the existence of fixed points for mappings which satisfy some asymptotic nonexpansive conditions in Banach spaces which are either nearly uniformly convex or they satisfy that asymptotic centers of bounded sequences are compact. Nominally, we consider pointwise eventually nonexpansive mappings, pointwise asymptotically nonexpansive mappings and asymptotically type nonexpansive mappings. We do not assume the existence of a continuous iterate, solving some longstanding open questions about existence of a fixed point for these mappings in absence of continuity [7].
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1. Nonexpansive and eventually nonexpansive mappings

Let $(X, \|\cdot\|)$ be a Banach space and C a nonempty subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* if for each $x, y \in C$, $\|Tx - Ty\| \leq \|x - y\|$. The Banach space X satisfies *the fixed point property for nonexpansive mappings* (in short, FPP) if every nonexpansive mapping defined on a nonempty weakly compact convex subset C of X into C has a fixed point. One of the central goals in metric fixed point theory is to characterize those Banach spaces which have the FPP. In the last 60 years a large number of papers have appeared finding out some geometrical properties that imply the FPP (for instance, see the monographs [6, 11] and the references therein).

A natural way to relax the nonexpansiveness assumption is to assume that the mapping T is eventually nonexpansive, i.e.

Definition 1.1. Let X be a Banach space and C a nonempty subset of X. A mapping $T : C \to C$ is said to be eventually nonexpansive if there exists $N \in \mathbb{N}$ such that for every $n \geq N$

$$\|T^n x - T^n y\| \leq \|x - y\|, \text{ for every } x, y \in X.$$

It should be noted that an eventually nonexpansive mapping does not need to be nonexpansive, nor even continuous.

Example 1.2. Let $C = [0,1]$ and $T : [0,1] \to [0,1]$ defined by $T(x) = 0$ if $x < 1$ and $T(1) = 1/2$. It is clear that T is discontinuous at $x = 1$ but $T^n \equiv 0$ for every $n \geq 2$.

Looking at this example, we could guess that the fixed point theory for eventually nonexpansive mappings should be quite different of the corresponding theory for nonexpansive mapping. However, noting that T^n and T^{n+1} are two commuting mappings, the equivalence between both theories is a direct consequence of the following result:

Theorem 1.3 ([2]). *Let X be a Banach space which satisfies the FPP, C a weakly compact convex subset of X and $\{T_i : i \in I\}$ an arbitrary family of commuting self-nonexpansive mappings of C. Then, the common fixed point set for this family is a nonempty nonexpansive retract of C.*

Since $T^n(x)$ is a fixed point of T whenever x is a common fixed point of T^n and T^{n+1}, the following result easily follows.

Theorem 1.4 ([9]). *Let X be a Banach space which satisfies the FPP. Then, every eventually nonexpansive mapping T defined from a weakly compact convex set C into C has a fixed point.*

We can also consider mappings which are eventually nonexpansive in a weaker sense, namely: for each $x, y \in C$ there exists $N = N(x, y) \in \mathbb{N}$ such that

$$\|T^n x - T^n y\| \leq \|x - y\|$$

for $n \geq N$. The following easy example shows the difficulty to obtain a fixed point in this case.

Example 1.5. Define $T : [0,1] \to [0,1]$ by $T(x) = x/2$ if $0 < x \leq 1$ and $T(0) = 1$. It is clear that T is fixed point free and for each $x \in [0,1]$ we have $T^n x \leq 2^{1-n}$. Thus, $T^n(x) \to 0$ as $n \to \infty$ for all $x \in [0,1]$ and $\lim_n \|T^n x - T^n y\| = 0, \text{ for all } x, y \in [0,1]$.

In this paper we will consider three scaled notions of asymptotic non-expansivity which have been considered in the literature: pointwise eventually nonexpansive mappings, pointwise asymptotically nonexpansive mappings and asymptotically nonexpansive type mappings. We will revise some known results about existence of fixed points for these classes of mappings and we will state some new results under more general assumptions. In particular, we solve several long-standing open problems concerning the existence of fixed points for these mappings in absence of continuity [7].

In Section 2, we recall some geometric properties that we will need in order to state our theorems, namely, uniform normal structure, weak uniform normal structure, uniform convexity and nearly uniform convexity.

Section 3 is dedicated to introduce the different asymptotic nonexpansive conditions which have been considered in the literature. We revise the most relevant fixed point results in this setting [8, 10, 12, 14, 16] where it is assumed that the mapping T has a continuous iterate and we solve several questions which appear as open problems in [7, 10]. We include a quite technical example which shows that a pointwise eventually nonexpansive mapping does not have, in general, a continuous iterate, even if the domain is a compact set.

In Section 4 we prove several lemmas that will be needed to state our main theorems. Lemma 4.2 is taken from [16]. This lemma, jointly with Lemma 4.4 , yield to Lemma 4.5 which is a basic tool in order to prove our fixed point results.

In Section 5 we present our main results. Nominally, we prove the existence of fixed a point for a pointwise asymptotically nonexpansive mapping $T : C \to C$ when C is a bounded convex closed subset of a Banach space X and one of the following conditions is satisfied:

- C is a compact set.
- X is nearly uniformly convex.

and we also prove the existence of a fixed point for a pointwise eventually nonexpansive mapping when the asymptotic center of any sequence in C is a compact set.

We include an example which shows that the result for nearly uniformly convex spaces cannot be derived from Theorem 3.17 in [3] as claimed by the authors.

A final section includes several problems concerning the existence of fixed point for pointwise eventually nonexpansive mappings which still remain open.

2. Preliminaries

Henceforth, $(X, \|\cdot\|)$ will be a Banach space with unit ball $B(0,1)$ and C a nonempty weakly compact convex subset of X.

Let $\{x_n\}$ be a bounded sequence in X, the *asymptotic radius* of $\{x_n\}$ with respect to a subset C of X is given by

$$r(C,\{x_n\}) = \inf\{\limsup_n \|x_n - x\| : x \in C\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(C,\{x_n\}) = \{x \in C : \limsup_n \|x_n - x\| = r(C,\{x_n\})\}.$$

It is known that $A(C,\{x_n\})$ is a nonempty weakly compact convex set as C is. Whenever $C = \overline{\text{co}}\,(\{x_n\})$ we write $r(\{x_n\})$ instead of $r(C,\{x_n\})$.

We recall some properties which can be satisfied by X.

Definition 2.1. It is said that X has *uniform normal structure* if $N(X) > 1$, where $N(X)$ is the normal structure coefficient of X defined by

$$N(X) = \inf\left\{\frac{\text{diam}(A)}{r(A)} : A \subset X \text{ bounded closed convex }, \text{diam}(A) > 0\right\},$$

where $\text{diam}(A) = \sup\{\|x-y\| : x, y \in A\}$ is the diameter of A and

$$r(A) = \inf\{\sup\{\|x-y\| : y \in A\} : x \in A\}$$

is the Chebyshev radius of A.

The space is said to have *weak uniform normal structure* if $WCS(X) > 1$ where

$$WCS(X) = \inf\left\{\frac{\operatorname{diam}_a(\{x_n\})}{r(\{x_n\})}\right\},$$

the infimum running over all weakly convergent sequences $\{x_n\}$ which are not norm convergent and $\operatorname{diam}_a(\{x_n\}) =: \limsup_k \sup\{\|x_n - x_m\| : n, m \geq k\}$.

The *modulus of convexity* of X is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x-y}{2}\right\| : x, y \in B(0,1) \text{ with } \|x - y\| \geq \varepsilon\right\},$$

and the *characteristic of convexity* of X is the number

$$\varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta(\varepsilon) = 0\}.$$

It is known that $\varepsilon_0(X) = 0$ if and only if X is uniformly convex, while $\varepsilon_0(X) < 1$ implies that X has uniform normal structure.

Definition 2.2. Let X be a Banach space and ϕ a measure of noncompactness on X. The modulus of noncompact convexity associated to ϕ is defined in the following way:

$$\Delta_\phi(\varepsilon) = \inf\{1 - d(0, A) : A \subset B(0,1) \text{ is convex },\phi(A) \geq \varepsilon\}.$$

The characteristic of noncompact convexity of X associated with the measure ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geq 0 : \Delta_\phi(\varepsilon) = 0\}.$$

The space X is said to be *nearly uniformly convex* if $\varepsilon_\phi(X) = 0$. We shall use the Kuratowski measure of noncompact convexity defined for a nonempty bounded subset A of X as follows:

$$\alpha(A) = \inf\{\varepsilon > 0 \quad : \quad A \text{ can be covered by a finite number of sets with diameter smaller than } \varepsilon\}$$

3. Pointwise asymptotic conditions

There is a class of mappings which lies between the class of eventually nonexpansive mappings and the class of mappings which are eventually nonexpansive in the weak sense considered in the introduction. It was introduced in [12] (see also [10]) .

Definition 3.1. A mapping $T : C \to C$ is said to be pointwise eventually nonexpansive if for every $x \in C$ there exists $N(x) \in \mathbb{N}$ such that if $n \geq N(x)$

$$\|T^n x - T^n y\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Remark 3.2. Notice that if $N = N(x)$, the mapping T^N is continuous at x. We could guess that it should be possible to find an iterate which is nonexpansive, and so simultaneously continuous for all $x \in C$. However, the following example shows a pointwise eventually nonexpansive mappings which does not have any continuous iterate.

Example 3.3. (A pointwise eventually nonexpansive mapping which does not have a continuous iterate).

Let $X = \ell_2$, $u_k = e_k/2^k$ and C the convex compact set $\overline{\text{co}}\{u_k : k \in \mathbb{N}\}$. Note that every $x = (x(k)) \in C$ has the representation $x = \sum_{i=1}^{\infty} \lambda_k u_k$ where $0 \leq \lambda_k \leq 1$, $\sum_{k=1}^{\infty} \lambda_k \leq 1$ and $x(k) = \lambda_k/2^k$. Denote $P = \{n \in N : n = 2^j \text{ for some } j \in \mathbb{N}\}$ and $Q = \mathbb{N} \setminus P$. We write $\mu_k = 0$ if $\lambda_k < 1$ and $\mu_k = 1/2$ if $\lambda_k = 1$. Define

$$S\left(\sum_{k=1}^{\infty} \lambda_k u_k\right) = \sum_{k \in Q} \lambda_k u_{k+1} + \sum_{k \in P} \mu_k u_{k+1}.$$

We will prove that for every $x \in C$, there exists $n(x) \in \mathbb{N}$ such that for every $n \geq n(x)$ there is a number $\alpha_n(x) \in [0, \infty)$ satisfying $\lim_n \alpha_n(x) = 0$ and

$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\|.$$

In particular, T is pointwise eventually nonexpansive.

We will distinguish two cases:

(1) Assume $x \notin \{u_k : k \in \mathbb{N}\}$. We split again in two cases.

(1a) Assume $y \notin \{u_k : k \in \mathbb{N}\}$. For any $k \in \mathbb{N}$, we have that either $(T^n x)(k) = (T^n y)(k) = 0$ or $(T^n x)(k) = (1/2^n)x(k)$ and $(T^n y)(k) = (1/2^n)y(k)$. Thus, $\|T^n x - T^n y\| \leq (1/2^n)\|x - y\|$.

(1b) Assume $y = u_k$ for some $k \in \mathbb{N}$. Denote $a = \min\{1 - 2^i x(i) : i \in \mathbb{N}\}$. We have $|y(k) - x(k)| \geq a/(2^k)$. For $j \neq k$ we have
$|(T^n x)(j) - (T^n y)(j)| = |(T^n x)(j)| \leq (1/2^n)|x(j)| \leq (1/2^n)|x(j) - y(j)|$.
For $j = k$ we have

$$\begin{aligned} |(T^n x)(k) - (T^n y)(k)| &\leq |(T^n x)(k)| + |(T^n y)(k)| \leq \frac{2}{2^{n+k}} \\ &= \frac{a}{a2^{n+k-1}} \leq \frac{1}{2^{n-1}a}|x(k) - y(k)|. \end{aligned}$$

Thus,

$$\|T^n x - T^n y\| \leq \frac{1}{2^{n-1}a}\|x - y\|.$$

(2) Assume $x = u_k$. Let q be an integer in P greater than k. For $n > q$ we have $T^n(u_k) = 0$. Thus, $(T^n x)(k) = (T^n y)(k) = 0$. For $j \neq k$ we have

$$|(T^n x)(j) - (T^n y)(j)| = |(T^n y)(j)| \leq \frac{1}{2^n}|y(j)| = \frac{1}{2^n}|x(j) - y(j)|.$$

Thus, $\|Tx - Ty\| \leq (1/2^n)\|x - y\|$.

To finish, we will show that for each $n \in \mathbb{N}$, T^n is discontinuous at infinitely many points of C. Indeed, for a given $n \in \mathbb{N}$, choose $k = 2^j \in P$ such that $n < 2^j$. We have $T^n(u_k) = 2^{-(k+1)} u_{k+n}$ and $T(\lambda u_k) = 0$ for every $\lambda \in (0,1)$.

Related to the notion of asymptotically nonexpansive mapping [5], the following concept has been considered:

Definition 3.4 ([10])**.** A mapping $T : C \to C$ is said to be pointwise asymptotically nonexpansive if for each $x \in C$ there exist $N(x) \in \mathbb{N}$ and a real sequence $\alpha_n(x)$ such that if $n \geq N(x)$

$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\| \quad \text{for all } y \in C,$$

where $\lim_n \alpha_n(x) = 1$.

A more general class of mappings has been considered by Kirk [8]:

Definition 3.5. A mapping $T : C \to C$ is said to be an asymptotically nonexpansive type mapping if for each $x \in C$,

$$\limsup_{n\to\infty} \{\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}\} \leq 0.$$

Remark 3.6. It is easy to check that if C is bounded then a pointwise asymptotically nonexpansive mapping T is an asymptotically nonexpansive type mapping.

Although it is not yet known whether the FPP is equivalent to the FPP for pointwise eventually nonexpansive mappings, some classical existence results for fixed points of nonexpansive mappings have been extended to some classes of asymptotically nonexpansive type mappings. Most of them assume, in addition, the continuity of an iterate.

In 1974 by Kirk [8] proved the following fixed point theorem in a class of Banach spaces satisfying a property weaker than uniform convexity.

Theorem 3.7 ([8])**.** *Let X be a Banach space for which $\varepsilon_0(X) < 1$, C a nonempty bounded closed and convex subset of X, and $T : C \to C$ an asymptotically nonexpansive type mapping. Suppose that there exists an integer $N \geq 1$ such that T^N is continuous. Then T has a fixed point.*

In 2000, Kim and Xu [7] demonstrated that the uniform normal structure of the space X implies the existence of fixed points for asymptotically nonexpansive mappings. Shortly after, this result was extended to the class of asymptotically nonexpansive type mappings [14].

Theorem 3.8 ([14])**.** *Let X be a Banach space with uniform normal structure, C a nonempty bounded closed and convex subset of X, and $T : C \to C$ an asymptotically nonexpansive type mapping such that T is continuous. Then T has a fixed point.*

The same result had already been proved by Xu [16] in a nearly uniformly convex Banach space. At this point, it should be remembered that nearly uniform convexity implies normal structure but does not imply uniform normal structure (see pages 78-79 in [6]).

Theorem 3.9 ([16]). *Let X be nearly uniformly convex Banach space, C a nonempty bounded closed and convex subset of X, and $T : C \to C$ an asymptotically nonexpansive type mapping. Suppose that there exists an integer $N \geq 1$ such that T^N is continuous. Then T has a fixed point.*

Bearing in mind Example 1.5, the continuity assumption cannot be dropped in the preceding theorems. Notice that the mapping in Example 1.5 is an asymptotically nonexpansive type mapping but it is not a pointwise eventually asymptotically nonexpansive mapping. Motivated by this fact, some natural questions have been raised (see [10, 12]): Does a pointwise eventually nonexpansive mapping (or more generally, a pointwise asymptotically nonexpansive mapping) defined on a weakly compact convex subset C of a Banach space X have a fixed point if X satisfies some of the following conditions

- (1) X is a Banach space with the FPP
- (2) X has uniform normal structure
- (3) X is nearly uniformly convex
- (4) The asymptotic center relative to C of each sequence in C is compact?

A positive answer to Question (2) is given in [15, Theorem 3.4] for pointwise asymptotically nonexpansive mappings. Nevertheless, we will see that this theorem is actually a consequence of the proof carried out to prove Theorem 2.1 in [14].

Our main results in the last section of this paper will give a positive answer to Questions (3) and (4). Note that both questions are independent. Indeed, in [13], an example is shown of a Banach space which is nearly uniformly convex but asymptotic centers of bounded sequences are not, in general, compact. On the other hand, the following example shows a Banach spaces which fails to be nearly uniformly convex, but it is uniformly rotund in every direction (see [17]) which implies that any asymptotic center of a bounded sequence is a singleton.

Example 3.10. Assume that X is the ℓ_2 product of the space ℓ_k, $k \geq 2$, i.e. $X = \{(x(k)) : x(k) \in \ell_k$ such that $\sum_{k=2}^{\infty} \|x(k)\|_k^2 < \infty\}$ where $\|x(k)\|_k$ is the norm of the vector $x(k)$ in ℓ_k. This space can be equipped with the norm

$$\|(x(k))\| = \left(\sum_{k=2}^{\infty} \|x(k)\|_k^2 \right)^{1/2}.$$

Since X contains isometrically ℓ_k for every $k \geq 2$ and $WCS(\ell_k) = 2^{1/k}$ we have $WCS(X) = 1$ which implies that X fails to be nearly uniformly convex (see, for instance, [1, Chapter 6]). However, X is uniformly rotund in every direction. Indeed, let $\{x_n\}$ be a bounded sequence and z a normalized vector in X. Assume that

$$\lim_n 2 \left(\|x_n + z\|^2 + \|x_n\|^2 \right) - \|2x_n + z\|^2 = 0.$$

Choose $k \geq 2$ such that $z(k)$ is a non-null vector of ℓ_k. Denote $\tilde{z} = (\tilde{z}(j))$ where $\tilde{z}(j) = z(j)$ if $j \neq k$ and $\tilde{z}(k) = 0$. Define analogously $\tilde{x}_n$. We have

$$\|x_n + z\|^2 = \|\tilde{x}_n + \tilde{z}\|^2 + \|x_n(k) + z(k)\|_k^2;$$

$$\|2x_n + z\|^2 = \|2\tilde{x}_n + \tilde{z}\|^2 + \|2x_n(k) + z(k)\|_k^2;$$

$$\|x_n\|^2 = \|\tilde{x}_n\|^2 + \|x_n(k)\|_k^2;$$
$$2\left(\|\tilde{x}_n + \tilde{z}\|^2 + \|\tilde{x}_n\|^2\right) - \|2\tilde{x}_n + \tilde{z}\|^2 \geq 0.$$

Thus,

$$\lim_n 2\left(\|x_n(k) + z(k)\|_k^2 + \|x_n(k)\|_k^2\right) - \|2x_n(k) + z(k)\|_k^2 = 0$$

which is a contradicition because ℓ_k is uniformly convex (see [17, Proposition 1]).

4. Previous lemmas

Let us establish some lemmas which will be used in our main result. Assume that C is a nonempty weakly compact convex subset of a Banach space X and $T : C \to C$ an asymptotically nonexpansive type mapping.

For each $x \in C$, let us denote by $\omega(x)$ the cluster point set of the sequence $\{T^n x\}$ for the norm topology. Similarly, $\omega_w(x)$ will be the cluster point set of the sequence $\{T^n x\}$ for the weak topology.

Remark 4.1. Note that the cluster point set A of a sequence $\{x_n\}$ is a closed set in any metric space. Indeed if $\{u_k\}$ is a sequence in A convergent to u, we can inductively choose integers $n_k > n_{k-1}$ such that $d(x_{n_k}, u_k) < 1/k$. The sequence $\{x_{n_k}\}$ converges to u and $u \in A$.

According to Remark 1 in [16], there exists a closed convex nonempty subset K of C such that $\omega_w(x) \subset K$ for every $x \in K$ and it is minimal under these conditions. Indeed, denote $\mathfrak{F}$ the collection formed by all closed convex nonempty subsets D of C which contain $\omega_w(x)$ for every $x \in D$. Let $\mathfrak{F}$ be ordered by inclusion. Then, $C \in \mathfrak{F}$ and for every chain $\{D_i : I \in I\}$ in $\mathfrak{F}$ we have that $\cap_{i\in I} D_i \in \mathfrak{F}$. By Zorn's lemma, we obtain a minimal set K in $\mathfrak{F}$.

To prove our main result (see Lemma 4.5 in this paper) we will use the following lemma from [16].

Lemma 4.2 ([16])**.** *Let C be a weakly compact convex subset of a Banach space X, $T : C \to C$ an asymptotically nonexpansive type mapping and K a closed convex nonempty subset of C such that the cluster point set $\omega_w(x)$ of the sequence $\{T^n x\}$ is contained in K for every $x \in K$ and the set K is minimal under these conditions. Then there exists $\rho \geq 0$ such that*

$$\limsup_n \|T^n x - y\| = \rho$$

for every $x, y \in K$.

Lemma 4.3. *Let C be a weakly compact convex subset of a Banach space X, $T : C \to C$ an asymptotically nonexpansive type mapping. Assume that H is a closed subset of C such that for every $x \in H$, every subsequence of $\{T^n x\}$ has a further (norm)-convergent subsequence. Then, for every $x \in H$, $\omega(x)$ is a compact set. Furthermore, if $x_0 \in H$ and $x \in \omega(x_0)$, for every increasing sequence $\{h_k\}$ of positive integers, there exist a subsequence, again denoted $\{h_k\}$, and a sequence $\{v_k\}$ in $\omega(x_0)$ such that $lim_k T^{h_k} v_k = x$.*

Proof. Indeed, if $\omega(x)$ were not compact, we could find a d-separated sequence $\{x_k\}$ in $\omega(x)$. By induction, we choose $n_k > n_{k-1}$ such that $\|x_k - T^{n_k}x\| < d/3$. Then, $\{T^{n_k}x\}$ is a $d/3$-separated subsequence of $\{T^n x\}$ contradicting the assumptions. Assume that $\{T^{n_i}x_0\} \to x$. For any fixed k, we can take a subsequence $\{n_i(k)\}_i$ of $\{n_i\}$ such that the sequence $\{T^{n_i(k)-h_k}x_0\}_i$ converges, say to v_k. By compactness, there is a convergent subsequence of $\{v_k\}$, denoted again $\{v_k\}$, say to v. For an arbitrary $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for all positive integer n greater than or equal to n_0, we have

$$\sup\{\|T^n v - T^n u\| - \|v - u\| : u \in C\} < \varepsilon/6.$$

We can choose k, large enough, such that $h_k \geq n_0$ and $\|v - v_k\| < \varepsilon/6$. Fixing such a k, we can choose $n_i(k)$, large enough, such that

$$\|T^{n_i(k)}x_0 - x\| < \varepsilon/6,\ \|T^{n_i(k)-h_k}x_0 - v_k\| < \varepsilon/6.$$

Thus, we have

$$\begin{aligned}
\|x - T^{h_k}v_k\| &\leq \|x - T^{n_i(k)}x_0\| + \|T^{n_i(k)}x_0 - T^{h_k}v\| + \|T^{h_k}v - T^{h_k}v_k\| \\
&\leq \varepsilon/6 + \|T^{n_i(k)-h_k}x_0 - v\| + \varepsilon/6 + \|v - v_k\| + \varepsilon/6 \\
&\leq \|T^{n_i(k)-h_k}x_0 - v_k\| + 2\|v - v_k\| + 3\varepsilon/6 \\
&\leq \varepsilon.
\end{aligned}$$

Since ε is arbitrary we have $T^{h_k}v_k = x$. □

Lemma 4.4. *Let C be a weakly compact convex subset of a Banach space X, $T : C \to C$ an asymptotically nonexpansive type mapping. Let x_0 be an arbitrary point in C. Then, for every $z \in \omega(x_0)$ we have $\omega(z) \subset \omega(x_0)$.*

Proof. Assume $\lim_k T^{n_k}x_0 = z$ and $\lim_i T^{n_i}z = y$. For an arbitrary $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all positive integer n greater than or equal to n_0, we have

$$\sup\{\|T^n z - T^n u\| - \|z - u\| : u \in C\} < \varepsilon/3.$$

Choose k_0 such that $\|T^{n_k}x_0 - z\| < \varepsilon/3$ for each $k \geq k_0$ and for every $k \geq k_0$ choose $n_i(k) \geq n_0$ such that $n_k + n_i(k) > n_{k-1} + n_i(k-1)$ and $\|T^{n_i(k)}z - y\| < \varepsilon/3$.

We have

$$\begin{aligned}
\|T^{n_k+n_i(k)}x_0 - y\| &\leq \|T^{n_k+n_i(k)}x_0 - T^{n_i(k)}z\| + \|T^{n_i(k)}z - y\| \\
&\leq \|T^{n_k}x_0 - z\| + \varepsilon/3 + \varepsilon/3 < \varepsilon,
\end{aligned}$$

for every $k \geq k_0$ which implies that $\lim_k T^{n_k+n_i(k)}x_0 = y$ and $y \in \omega(x_0)$. □

The following lemma will play a key role in the proof of our main theorems.

Lemma 4.5. *Let C be a weakly compact convex subset of a Banach space X, $T : C \to C$ an asymptotically nonexpansive type mapping. Assume that there exists a closed convex nonempty subset H of C which satisfies*

(i) For each $x \in H$, $\omega_w(x) \subset H$.

(ii) For each $x \in H$, every subsequence of $\{T^n x : n \in \mathbb{N}\}$ has a further convergent subsequence.

Then there exists $z \in H$ such that $\{T^n z\}$ is norm convergent to z.

Proof. As stated above, we can find a closed convex nonempty subset K of H which satisfies $\omega_w(x) \subset K$ for every $x \in K$ and it is minimal under these conditions. Fix $x_0 \in K$ and denote $S = \omega(x_0)$ which, by Lemma 4.3, is a nonempty compact set. By Lemma 4.4 we have $\omega(z) \subset S$ for every $z \in S$. By applying Zorn's lemma, we can find a closed nonempty subset S_0 of S which satisfies $\omega(z) \subset S_0$ for each $z \in S_0$ and it is minimal under these conditions. Note that Lemma 4.4 and the minimality of S_0 imply that $S_0 = \omega(x_1)$ for some (any) $x_1 \in S_0$. We will prove that diam $S_0 = 0$. Indeed, since any compact set has normal structure, if diam $S_0 > 0$, the Chebyshev radius of S_0 is smaller that the diameter of S_0. Thus, there exists a point z in $\overline{\text{co}}(S_0) \subset K$ such that $r = \sup\{\|z - x\| : x \in S_0\} < \text{diam } (S_0)$. Let

$$D = \{x \in K : \sup_{y \in S_0} \|x - y\| \leq r\}.$$

It is clear that D is a nonempty closed convex proper subset of K. We will prove that $\omega_w(x) \subset D$ for every $x \in D$ contradicting the minimality of K. Indeed, assume that $y = w - \lim_i T^{n_i}x$ and $z \in S_0$. By Lemma 4.3, there exists an subsequence of $\{n_i\}$, denoted again $\{n_i\}$, and a sequence $\{v_i\}$ in $\omega(x_1) = S_0$ such that $T^{n_i}v_i = z$. Hence,

$$\begin{aligned}\|z - y\| &\leq \liminf_i \|z - T^{n_i}x\| = \liminf_i \|T^{n_i}v_i - T^{n_i}x\| \\ &\leq \limsup_i \|T^{n_i}v_i - T^{n_i}x\|.\end{aligned}$$

Since T is an asymptotically nonexpansive type mapping and $\|v_i - x\| \leq r$ for every i we have

$$\limsup_i \|T^{n_i}v_i - T^{n_i}x\| - r \leq \limsup_i \{\|T^{n_i}v_i - T^{n_i}x\| - \|v_i - x\|\} \leq 0,$$

which implies that $\|z - y\| \leq r$. Thus, $y \in D$ and diam $S_0 = 0$. Write $S_0 = \{z\}$. Since $\omega(z) \subset S_0$ we have that any convergent subsequence of $\{T^n z\}$ must converge to z. From condition (ii) the whole sequence $\{T^n z\}$ converges to z. □

5. Fixed points for pointwise eventually asymptotically nonexpansive mappings

We start this section with a very simple and general result.

Theorem 5.1. *Let X be an arbitrary topological space, M a nonempty subset of X and T a mapping from M into X. Assume that there exists $x \in M$ such that $\lim_n T^n x = x$ and there exists $N \in \mathbb{N}$ such that T^N is continuous at x. Then, $Tx = x$.*

Proof. Since $T^n x \to x$ and T^N is continuous at x, we have $T^N x = x$. Thus

$$T^{nN+1}x = Tx$$

for all $n \geq 1$, which implies that $Tx = x$. □

As we have before mentioned, a Banach space such that $\varepsilon_0(X) < 1$ satisfies that any asymptotically nonexpansive type mapping T has a fixed point in a weakly compact convex T-invariant set if the continuity of some iterate of the mapping is assumed. The same is true in the more general setting of a Banach space with uniform normal structure [14]. Looking in detail at their proofs, we find that, in both cases, it is proved that there exists $x \in C$ such that $\{T^n x\}$ converges to x. Therefore, we obtain the following theorems as an immediately consequence of Theorem 5.1 together with Remark 3.2.

Theorem 5.2. *Let C be a convex bounded subset of a Banach space X, $T : C \to C$ an asymptotically nonexpansive type mapping. Assume that $\varepsilon_0(X) < 1$. Then, there exists $x \in C$ such that $T^n x \to x$. In particular, T has a fixed point if it is pointwise asymptotically eventually nonexpansive.*

Theorem 5.3. *Let C be a convex bounded subset of a Banach space X, $T : C \to C$ an asymptotically nonexpansive type mapping. Assume that X has uniform normal structure. Then, there exists $x \in C$ such that $T^n x \to x$. In particular, T has a fixed point if it is pointwise asymptotically eventually nonexpansive.*

It should be noted that the above theorem recovers and extends Theorem 3.4 in [15].

Our next fixed point theorems are a direct consequence of Lemma 4.5. When C is a compact set, the assumptions of Lemma 4.5 are immediately satisfied for $H = C$. Therefore, Theorem 5.1 leads to the following result.

Theorem 5.4. *Let C a nonempty, compact, convex subset of a Banach space and $T : C \to C$ an asymptotically nonexpansive type mapping. Assume that for each $x \in C$ the mapping $T^{N(x)}$ is continuous at x for some $N(x) \in \mathbb{N}$. Then T has a fixed point.*

In particular, the above theorem extends [8, Theorem 4], removing the continuity assumption for an iterate T^N. As a consequence we have:

Corollary 5.5. *Let C a nonempty, compact, convex subset of a Banach space and $T : C \to C$ a pointwise asymptotically nonexpansive mapping. Then T has a fixed point.*

Having in mind Schauder's Fixed Point Theorem, it is interesting to note that we can obtain a fixed point for a discontinuous mapping defined on a convex closed bounded set.

The following theorem solves a long-standing open question (see [7, Question 2] and [10, Question 2]).

Theorem 5.6. *Let C be a weakly compact convex Banach space X and $T : C \to C$ a pointwise eventually nonexpansive mapping. Suppose that each sequence in C has compact asymptotic center relative to C. Then T has a fixed point.*

Proof. Choose an arbitrary point $x_0 \in C$ and let $A = A(C, \{T^n x_0\})$. It is easy to check that for each $y \in A$, $T^m y \in A$ for all large m. Therefore, the weak (in fact, strong) cluster point set of the $\{T^m y\}$ is a subset of A. Hence A satisfies condition (i) in Lemma 4.5. Since A is compact, it also meets condition (ii). These facts together with Theorem 5.1 lead to the desired result. □

The main result in this section extends Xu's result in [16] and gives a positive answer to another question raised in [7, 10].

Theorem 5.7. *Let X be a nearly uniformly convex Banach space, C a closed convex bounded nonempty subset of X, $T : C \to C$ an asymptotically nonexpansive type mapping and K a closed convex nonempty subset of C such that the cluster point set $\omega_w(x)$ of the sequence $\{T^n x\}$ is contained in K for every $x \in K$ and the set K is minimal under these conditions. Then, K is a singleton $\{x_0\}$ and $\{T^n x_0\}$ is norm convergent to x_0. In particular, every pointwise asymptotically nonexpansive mapping $T : C \to C$ has a fixed point.*

Proof. From Lemma 4.2 it suffices to prove that $\rho = 0$. Assume that $\rho > 0$. By construction, K satisfies (i) in Lemma 4.5. We will prove that it also satisfies (ii). Otherwise, assume that there are a point $x \in K$ and a subsequence of $\{T^n x\}$ which does no admit a further convergent subsequence. In this case, there exist a number $d > 0$ and a d-separated subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$. For an arbitrary $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for all positive integer n greater than or equal to n_0, we have

$$\|T^n x - x\| < \rho + \varepsilon/2$$

and

$$\sup\{\|T^n x - T^n u\| - \|x - u\| : u \in C\} < \varepsilon/2.$$

Fix such an $n \in \mathbb{N}$. Since the sequence $\{T^{n_i} x - T^n x\}_i$ is d-separated we have

$$\alpha(\{T^{n_i} x - T^n x\}_i) \geq d.$$

Taking a subsequence, we can assume that $\{T^{n_i} x\}$ converges weakly, say to $x_\infty \in K$. On the other hand, there exists i_0 large enough such that $n_i \geq n + n_0$ for $i \geq i_0$. Thus, for $i \geq i_0$, we have

$$\|T^n x - T^{n_i} x\| = \|T^n x - T^n T^{n_i - n} x\| \leq \varepsilon/2 + \|x - T^{n_i - n} x\| \leq r + \varepsilon.$$

Hence, by the lower weakly semicontinuity of the norm,

$$\frac{\|T^n x - x_\infty\|}{r + \varepsilon} \leq 1 - \Delta_\alpha \left(\frac{d}{r + \varepsilon} \right)$$

which implies $r = \limsup_n \|T^n x - x_\infty\| \leq (r + \varepsilon)(1 - \Delta_\alpha(d/(r + \varepsilon)))$. Letting $\varepsilon \to 0$ we obtain the contradiction $r \leq r(1 - \Delta_\alpha(d/r^-)) < r$. Thus, K is a singleton $\{x_0\}$ and satisfies (i) and (ii). Hence $\{T^n x_0\}$ is norm convergent to x_0. By Theorem 5.1 and Lemma 4.2, every pointwise asymptotically nonexpansive mapping $T : C \to C$ has a fixed point. □

Remark 5.8. Butsan et al. [3, Corollary 3.18] claim that they have proved Theorem 5.7 solving the mentioned question. However, the proof is strongly based on certain connection between the asymptotic center of a sequence and the modulus of noncompact convexity which are stated in [6]. Actually, they assert that in a nearly uniformly convex space X, there exists a $\lambda \in [0, 1)$ such that for each closed bounded convex subset C of X and for each sequence $\{x_n\} \subset C$ the following inequality holds

$$r(C, \{y_n\}) \leq \lambda r(C, \{x_n\}),$$

for each sequence $\{y_n\}$ in $A(C, \{x_n\})$.

However, it should be emphasized that this inequality is not true unless the regularity of the sequence $\{x_n\}$ is assumed (see [4, Remark 3.5]), as the following example shows. Since the proof in [3] does not allow to take a subsequence, Corollary 3.15 does not follow from Theorem 3.14 in [3] and the question still remained open. For completeness, we include the counterexample of [4].

Example 5.9. Consider the product space $X = Y \bigotimes \ell_2$ where $Y = (\mathbb{R}^2, \|\cdot\|_\infty)$ with the norm

$$\|(x,y)\| = (\|x\|_\infty^2 + \|y\|_2^2)^{\frac{1}{2}},\ x \in Y, y \in \ell_2.$$

As it is proved in [4]

$$\Delta_\alpha(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}},$$

following that X is a nearly uniform convex space. If $z_n \in \mathbb{R}^2$ is the sequence defined by $z_{2n-1} = (-1, 0)$ and $z_{2n} = (1, 0)$ for each $n \in \mathbb{N}$, we consider the sequence $x_n = (z_n, 0) \in X$. Denote B the unit ball of Y and let $C = B \times \{0\}$. Clearly C is a weakly compact (and compact) convex subset of X which contains $\{x_n\}$. It is not difficult to see that $r(C, \{x_n\}) = 1$ and $A(C, \{x_n\}) = \{((0, y), 0) : y \in [-1, 1]\}$. Define the sequence $u_n \in \mathbb{R}^2$ by $u_{2n-1} = (0, -1)$ and $u_{2n} = (0, 1)$ for each $n \in \mathbb{N}$ and $y_n = (u_n, 0)$. Then $y_n \in A(C, \{x_n\})$ and $r(C, \{y_n\}) = r(C, \{x_n\}) = 1$.

6. Open questions

Although our results solve several long-standing open problems, as far as we know, the following questions are still open:

Questions 6.1. Does a pointwise eventually nonexpansive mapping, defined from a weakly compact convex subset C into C, have a fixed point if X satisfies one of the following conditions

- (a) $\varepsilon_\alpha(X) < 1$ where $\varepsilon_\alpha(X)$ is the characteristic of noncompact convexity for the Kuratowski measure of noncompactness.
- (b) X has uniform weak normal structure (i.e $WCS(X) > 1$).
- (c) X has normal structure.
- (d) X has the FPP for nonexpansive mappings?

References

[1] J.M. Ayerbe, T. Domínguez Benavides, G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory: Advances and Applications, 99. Birkhäuser Verlag, Basel, 1997.

[2] R.E. Bruck, *A common fixed point theorem for a commuting family of nonexpansive mappings*, Pacific J. Math., **53**(1974), 59-71.

[3] T. Butsan, S. Dhompongsa, W. Takahashi, *A fixed point theorem for eventually nonexpansive mappings in nearly uniformly convex Banach spaces*, Nonlinear Anal., **74**(2011), 1694-1701.

[4] T. Domínguez Benavides, P. Lorenzo Ramírez, *Fixed point theorems for multivalued nonexpansive mappings without uniform convexity*, Abstr. Appl. Anal., (2003), no. 6, 375-386.

[5] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972), 171-174.

[6] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, **28**, Cambridge University Press, Cambridge, 1990.

[7] T.H. Kim, H.K. Xu, *Remarks on asymptotically nonexpansive mappings*, Nonlinear Anal., **41**(2000), 405-415.

[8] W.A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math., **17**(1974), 339-346.
[9] W.A. Kirk, *The fixed point property and mappings which are eventually nonexpansive*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., 178, Dekker, New York, 1996, 141-147.
[10] W.A. Kirk, *Remarks on nonexpansive mappings and related asymptotic conditions*, J. Nonlinear Convex Anal., **18**(2017), no. 1, 1-15.
[11] W.A. Kirk, B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.
[12] W.A. Kirk, H.K. Xu, *Asymptotic pointwise contractions*, Nonlinear Anal., **69**(2008), 4706-4712.
[13] T. Kuczumow, S. Prus, *Compact asymptotic centers and fixed points of multivalued nonexpansive mappings*, Houston J. Math., **16**(1990), 465-468.
[14] G. Li, B. Sims, *Fixed point theorems for mappings of asymptotically nonexpansive type*, Nonlinear Anal., **50**(2002), 1085-1091.
[15] M. Radhakrishnan, S. Rajesh, *Existence of fixed points for pointwise eventually asymptotically nonexpansive mappings*, Appl. Gen. Topol., **20**(2019), no. 1, 119-133.
[16] H.K. Xu, *Existence and convergence of fixed points for mappings of asymptotically nonexpansive type*, Nonlinear Anal., **16**(1991), 1139-1146.
[17] V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math. Rozprawy Mat., **87**(1971), 33 pp.

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