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A MULTIPLICITY RESULTS TO A $p-q$ LAPLACIAN SYSTEM WITH A CONCAVE AND SINGULAR NONLINEARITIES

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Abstract. In this paper we study the existence of multiple nontrivial positive weak solutions to the following system of problems.

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda f(x)|u|^{r-2}u + \nu\frac{1-\alpha}{2-\alpha-\beta}h(x)|u|^{-\alpha}|v|^{1-\beta} \text{ in } \Omega, \\ -\Delta_p v - \Delta_q v &= \mu g(x)|v|^{r-2}v + \nu\frac{1-\beta}{2-\alpha-\beta}h(x)|u|^{1-\alpha}|v|^{-\beta} \text{ in } \Omega, \\ u, v &> 0 \text{ in } \Omega, \\ u = v &= 0 \text{ on } \partial\Omega \end{aligned}$$

where $0<\alpha<1,\ 0<\beta<1,\ 2-\alpha-\beta<q<\frac{N(p-1)}{N-p}<p<r<p^*$, with $p^*=\frac{Np}{N-p}$.
We will guarantee the existence of a solution in the Nehari manifold. Further by using the Lusternik-Schnirelman category we will prove the existence of at least $\text{cat}(\Omega)+1$ number of solutions.

Key Words and Phrases: Nehari manifold, Lusternik-Schnirelman category, singularity, multiplicity, fixed point.
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1. Introduction

As mentioned in the abstract we will attempt the following problem.

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda f(x)|u|^{r-2}u + \nu\frac{1-\alpha}{2-\alpha-\beta}h(x)|u|^{-\alpha}|v|^{1-\beta} \text{ in } \Omega,\\ -\Delta_p v - \Delta_q v &= \mu g(x)|v|^{r-2}v + \nu\frac{1-\beta}{2-\alpha-\beta}h(x)|u|^{1-\alpha}|v|^{-\beta} \text{ in } \Omega,\\ u,v &> 0 \text{ in } \Omega,\\ u=v &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

where (C): $0<\alpha<1,\ 0<\beta<1,\ 2-\alpha-\beta<q<\frac{N(p-1)}{N-p}<p<r<p^*$, with $p^*=\frac{Np}{N-p}$, $\lambda,\mu,\nu>0$, $0<\alpha,\beta<1$.
The domain Ω is bounded subset of $\mathbb{R}^N$ with a lipschitz continuous boundary $\partial\Omega$. The measurable functions $f,g,h\geq 0$, $f+g\neq 0$ over a subset of Ω of positive measure and are bounded almost everywhere in Ω, i.e. $f,g,h\in L^\infty(\Omega)$. The operator $(-\Delta_s)$ acting on a function say U is the s-Laplacian operator which is defined as

$$-\Delta_s U(x) = -\nabla\cdot(|\nabla U|^{s-2}\nabla U)$$

for all $s\in[1,\infty)$. We will be assuming that $p<N$, $1<r<q<\frac{N(p-1)}{N-1}<p<p^*$ throughout the article. Off-late, a huge attention has been given to elliptic problems involving two Laplacian operators viz.

$$\begin{aligned} (-\Delta_p)u-(-\Delta_q)u &= \lambda|u|^{r-2}u+|u|^{p^*-2}u \text{ in } \Omega,\\ u &= 0 \text{ in } \partial\Omega. \end{aligned}$$

The problem draws its motivation from the fundamental reaction-diffusion equation

$$\frac{\partial}{\partial t}u = \nabla\cdot[H(u)\nabla u]+c(x,u). \tag{1.2}$$

where $H(u)=|\nabla u|^{p-2}+|\nabla u|^{q-2}$. The problem is important owing to its manifold applications in Physics and other applied sciences such as in biophysics to model the cells, chemical reaction design, plasma physics, drug delivery mechanism to name a few. The reaction term has a polynomial form with respect to u. In the recent years the problem

$$-\nabla\cdot[H(u)\nabla u]=c(x,u)$$

has been studied in [4, 6, 29, 32, 16, 17]. One may refer to Yin and Yang [35] who studied the problem in (1.2) when $p^2<N$, $1<q<p<r<p*$. The authors proved the existence of cat(Ω) number of positive solutions using simple variational techniques. When $p=q$, $r=2$ the problem (1.2) reduces to the well-known *Brezis-Nirenberg problem* which has been further studied for the case of critical growth in bounded and unbounded domains by many researchers (Refer [2, 3, 5, 26]) and the references therein. A common issue which intrigued the researchers was to figure out a way to overcome the lack of compactness in the continuous embedding $W_0^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$. Two noteworthy contributions can be found in [10, 23].

Meanwhile, the elliptic systems have also gained much attention, especially for the system

$$\begin{aligned} &-(\Delta_p)u = \lambda|u|^{r-2}u + \frac{2a}{a+b}|u|^{a-2}u|v|^b \text{ in } \Omega,\\ &-(\Delta_p)v = \mu|v|^{r-2}v + \frac{2b}{a+b}|u|^{a}|v|^{b-2}u \text{ in } \Omega,\\ &\quad u = v = 0 \text{ in } \partial\Omega \end{aligned} \tag{1.3}$$

where $a+b = p^*$. Ding and Xiao [11] studied (1.3) with the $p-$superlinear perturbation of $2 \leq p \leq r < p^*$ an extension of which can be found in Yin [33]. Both the works in [33] and [11] have obtained the existence of cat(Ω) number of solutions using the Lusternik-Scnirelman category. Similar results for elliptic equations driven by the p-Laplacian or the double phase operator can be found in [22, 24]. For the sublinear perturbation, Hsu [15] obtained the existence of two positive solutions for the problem (1.3). Few years back, Fan [14] studied the problem (1.3) for $p = 2$ and $1 < r < p$. Using the Nehari manifold and the Lusternik-Schnirelman category the author has proved the admittance of at least cat$(\Omega) + 1$ positive solutions. Motivated from the work of Li, Yang [18] we extend the results of the above problem with local operators and added singular nonlinearities. As far as we know there has not been any contribution in this direction whatsoever and is entirely novel. We now state the main result of this work.

Theorem 1.1. *Assume the condition* (C) *holds. Then there exists* $\Lambda^* > 0$ *such that if* $\nu \in (0, \Lambda^*)$, *problem* (1.1) *admits at least cat*$(\Omega) + 1$ *number of distinct solutions.*

2. Preliminaries

Let $\Omega \subset \mathbb{R}^N$, then the space $(W_0^{1,p}(\Omega), \|.\|_p)$ is defined by

$$W_0^{1,p}(\Omega) \quad = \quad \{u : Du \in L^p(\Omega), u|_{\partial\Omega} = 0\}$$

equipped with the norm

$$\|u\|_p \quad = \quad \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}.$$

We will refer to $|u|_p$ as the L^p-norm of u and is defined as $(\int_\Omega |u|^p dx)^{\frac{1}{p}}$. We further define the space Clearly, $X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ is a Banach space. We define the norm of any member of X as

$$\|(u,v)\|_p = (\|u\|_p^p + \|v\|_p^p)^{\frac{1}{p}}.$$

The best Sobolev constant is defined as

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_p^p}{\left(\int_\Omega |u|^{p^*} dx\right)^{\frac{p}{p_s^*}}}. \tag{2.1}$$

and further define

$$S_{\alpha,\beta} = \inf_{(u,v)\in X\setminus\{(0,0)\}} \frac{\|(u,v)\|_p^p}{(\int_\Omega |u|^{p^*} + |v|^{p^*} dx)^{\frac{p}{p^*}}}. \tag{2.2}$$

Also, we will denote $M = \|h\|_\infty$, $M' = \max\{\|f\|_\infty, \|g\|_\infty\}$, where $\|\cdot\|_\infty$ denotes the essential supremum norm (or more commonly the L^∞-norm) of a function. We now define the associated energy functional to the problem (1.1) which is as follows.

$$\begin{aligned} I_{\alpha,\beta}(u,v) &= \frac{1}{p}\|(u,v)\|_p^p + \frac{1}{q}\|(u,v)\|_q^q - \frac{1}{r}\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx \\ &\quad - \frac{\nu}{2-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx. \end{aligned}$$

A function $(u,v) \in X$ is a weak solution to the problem (1.1), if

(i) $u, v > 0,\ u^{-\alpha}\phi_1, v^{-\beta}\phi_2 \in L^1(\Omega)$ and

$$\begin{aligned} (ii) &\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla \phi_1 + |\nabla v|^{p-2}\nabla v \cdot \nabla \phi_2)dx \\ &+ \int_\Omega (|\nabla u|^{q-2}\nabla u \cdot \nabla \phi_1 + |\nabla v|^{q-2}\nabla v \cdot \nabla \phi_2)dx \\ &- \int_\Omega (\lambda f(x)u^{r-1}\phi_1 + \mu g(x)v^{r-1}\phi_2)dx - \nu\frac{1-\alpha}{2-\alpha-\beta}\int_\Omega h(x)u^{-\alpha}v^{1-\beta}\phi_1 dx \\ &- \nu\frac{1-\beta}{2-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{-\beta}\phi_2 dx = 0 \end{aligned}$$

for each $\phi_2, \phi_2 \in X$. Note that the nontrivial critical points of the functional $I_{\alpha,\beta}$ are the positive weak solutions of the problem (1.1). Note that the functional $I_{\alpha,\beta}$ is not a C^1-functional and hence the classical variational methods are not applicable. One can easily verify that the energy functional $I_{\alpha,\beta}$ is not bounded below in X. However, we will show that $I_{\alpha,\beta}$ is bounded below on a *Nehari manifold* and we will extract solutions by minimizing the functional on suitable subsets. We further define the Nehari manifold as follows.

$$\mathcal{N}_{\alpha,\beta} = \{(u,v) \in Z \setminus (0,0), u, v > 0 : \langle I'_{\alpha,\beta}(u,v), (u,v) = 0\rangle\}.$$

For a detailed study on the method of Nehari manifolds we refer the readers to [30]. It is not difficult to see that a pair $(u,v) \in \mathcal{N}_{\alpha,\beta}$ if and only if

$$\|(u,v)\|_p^p + \|(u,v)\|_q^q - \int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx - \nu\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx = 0.$$

Furthermore, it is customary to see, as for any problem which has an involvement of a Nehari manifold, that

$$\begin{aligned} I_{\alpha,\beta}(u,v) &= \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p + \left(\frac{1}{q}-\frac{1}{r}\right)\|(u,v)\|_q^q \\ &\quad + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx. \\ &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\left(\|(u,v)\|_p^p + \|(u,v)\|_q^q\right) + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx. \\ &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \end{aligned}$$

$$\geq \left(\frac{1}{p} - \frac{1}{r}\right) \|(u,v)\|_p^p - \nu \left(\frac{1}{2-\alpha-\beta} - \frac{1}{r}\right) \|(u,v)\|_p^{2-\alpha-\beta}.$$

Since $2-\alpha-\beta < p$, therefore $I_{\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\alpha,\beta}$. Therefore the functional is coercive and is bounded below in $\mathcal{N}_{\alpha,\beta}$. In fact $I_{\alpha,\beta}(u,v) \geq 0$ for sufficiently small $\nu > 0$ and for all $(u,v) \in \mathcal{N}_{\alpha,\beta}$. We define for $t \geq 0$ the fiber maps

$$\begin{aligned}\Phi_{\alpha,\beta}(t) = I_{\alpha,\beta}(tu,tv) = & \frac{t^p}{p}\|(u,v)\|_p^p + \frac{t^q}{q}\|(u,v)\|_q^q \\ & - \frac{t^r}{r}\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx \\ & - \nu \frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.\end{aligned}$$

Then

$$\begin{aligned}\Phi'_{\alpha,\beta}(t) \quad = \quad & t^{p-1}\|(u,v)\|_p^p + t^{q-1}\|(u,v)\|_q^q - t^{r-1}\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx \\ & -\nu t^{1-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx\end{aligned}$$

and

$$\begin{aligned}\Phi''_{\alpha,\beta}(t) = & (p-1)t^{p-2}\|(u,v)\|_p^p + (q-1)t^{q-2}\|(u,v)\|_q^q \\ & - (r-1)t^{r-2}\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx \\ & - \nu(1-\alpha-\beta)t^{-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.\end{aligned}$$

A simple observation shows that $(u,v) \in \mathcal{N}_{\alpha,\beta}$ if and only if $\Phi'_{\alpha,\beta}(1) = 0$. Furthermore, in general we have that $(u,v) \in \mathcal{N}_{\alpha,\beta}$ if and only if $\Phi'_{\alpha,\beta}(t) = 0$. Therefore for $(u,v) \in \mathcal{N}_{\alpha,\beta}$ we have

$$\begin{aligned}\Phi''_{\alpha,\beta}(1) = & (p-1)\|(u,v)\|_p^p + (q-1)\|(u,v)\|_q^q - (r-1)\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx \\ & - \nu(1-\alpha-\beta)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\ = & (p-r)\|(u,v)\|_p^p + (q-r)\|(u,v)\|_q^q + \nu(r+\alpha+\beta-2)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\ = & (p+\alpha+\beta-2)\|(u,v)\|_p^p + (q+\alpha+\beta-2)\|(u,v)\|_q^q \\ & + (2-\alpha-\beta-r)\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx.\end{aligned}$$

Therefore we split the Nehari manifold into three parts, namely

$$\begin{aligned}\mathcal{N}^+_{\alpha,\beta} &= \{(u,v) \in \mathcal{N}_{\alpha,\beta} : \Phi''_{\alpha,\beta}(1) > 0\}, \\ \mathcal{N}^-_{\alpha,\beta} &= \{(u,v) \in \mathcal{N}_{\alpha,\beta} : \Phi''_{\alpha,\beta}(1) < 0\}, \\ \mathcal{N}^0_{\alpha,\beta} &= \{(u,v) \in \mathcal{N}_{\alpha,\beta} : \Phi''_{\alpha,\beta}(1) = 0\}\end{aligned}$$

which corresponds to the collection of local minima, maxima and points of inflection respectively. We now prove a lemma which falls back on the proof due to Hsu [15] (refer Theorem 2.2).

Lemma 2.1. *For $(u,v) \in \mathcal{N}_{\alpha,\beta}$, there exists a positive constant A_0, that depends on $p, S, N, \alpha, \beta, |\Omega|$ such that $I_{\alpha,\beta}(u,v) \geq -\nu A_0 \left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]$.*

Proof. We use

$$I_{\alpha,\beta}(u,v) \quad \geq \quad \left(\frac{1}{p} - \frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) + \nu\left(\frac{1}{r} - \frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx. \tag{2.3}$$

By the Hölder inequality, the Young's inequality, and the Sobolev embedding theorem to (2.3), we have

$$\begin{aligned} I_{\alpha,\beta}(u,v) &\geq \left(\frac{1}{p} - \frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) - \nu\left(\frac{1}{2-\alpha-\beta} - \frac{1}{r}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\ &\geq \left(\frac{1}{p} - \frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) \\ &- \nu M|\Omega|^{1-\frac{2-\alpha-\beta}{p^*}} \\ &\times \left(\frac{1}{2-\alpha-\beta} - \frac{1}{r}\right)\int_\Omega \left(\frac{1-\alpha}{2-\alpha-\beta}|u|_{p^*}^{2-\alpha-\beta} + \frac{1-\beta}{2-\alpha-\beta}|v|_{p^*}^{2-\alpha-\beta}\right)dx \\ &\geq \left(\frac{1}{p} - \frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) \\ &- \nu M|\Omega|^{1-\frac{2-\alpha-\beta}{p^*}} S^{\frac{\alpha+\beta-2}{p}} \\ &\times \left(\frac{1}{2-\alpha-\beta} - \frac{1}{r}\right)\int_\Omega \left(\frac{1-\alpha}{2-\alpha-\beta}|\nabla u|_p^{2-\alpha-\beta} + \frac{1-\beta}{2-\alpha-\beta}|\nabla v|_p^{2-\alpha-\beta}\right)dx \\ &\geq -\nu A_0(p,S,N,\alpha,\beta,|\Omega|)\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]. \end{aligned}$$

□

Lemma 2.2. *There exists $\Lambda^* > 0$ such that if*

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right] \in (0, \Lambda^*),$$

then $\mathcal{N}_{\alpha,\beta}^0 = \phi$.

Proof. Let us choose

$$\Lambda^* = \left((p-2+\alpha+\beta)\frac{1}{M'(\lambda+\mu)}\right)^{\frac{p}{r-p}} \frac{(r-p)S_{\alpha,\beta}^{\frac{rp}{N(r-p)}+\frac{2-\alpha-\beta}{p}}}{\nu M(r-2+\alpha+\beta)^{\frac{r}{r-p}}|\Omega|^{1-\frac{2-\alpha-\beta}{p^*}}}.$$

The proof follows by contradiction. □

From the lemma (2.2), we have that if

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\in(0,\Lambda^*),$$

then $\mathcal{N}_{\alpha,\beta}=\mathcal{N}^+_{\alpha,\beta}\bigcup\mathcal{N}^-_{\alpha,\beta}$.
We can define $i^+=\inf_{(u,v)\in\mathcal{N}^+_{\alpha,\beta}} I_{\alpha,\beta}$ and $i^-=\inf_{(u,v)\in\mathcal{N}^-_{\alpha,\beta}} I_{\alpha,\beta}$ since the functional $I_{\alpha,\beta}$ is bounded below in $\mathcal{N}_{\alpha,\beta}$.

Remark 2.3. We will denote the norm convergence by $\to$, the weak convergence by $\rightharpoonup$ and Λ (or Λ^*) as any small parameter we will encounter or any cumbersome representation in short form.

Lemma 2.4. *There exists $\Lambda^*>0$ such that if*

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\in(0,\Lambda^*),$$

then

(1) $i^+<0$,
(2) $i^-\geq D_0$ for some $D_0>0$.

Proof. (1) Let $(u,v)\in\mathcal{N}^+_{\alpha,\beta}\subset\mathcal{N}_{\alpha,\beta}$. Then we have

$$\begin{aligned}0&<(r-p)\|(u,v)\|_p^p+(r-q)\|(u,v)\|_q^q\\&<\nu(r+\alpha+\beta-2)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx\end{aligned}\tag{2.4}$$

Further,

$$\begin{aligned}I_{\alpha,\beta}(u,v)&=\left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p+\left(\frac{1}{q}-\frac{1}{r}\right)\|(u,v)\|_q^q\\&\quad+\nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.\\&<\left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p+\left(\frac{1}{q}-\frac{1}{r}\right)\|(u,v)\|_q^q\\&\quad-\frac{(r-p)}{r(2-\alpha-\beta)}\|(u,v)\|_p^p-\frac{(r-q)}{r(2-\alpha-\beta)}\|(u,v)\|_q^q\\&=\frac{(r-p)}{r}\left(\frac{1}{p}-\frac{1}{2-\alpha-\beta}\right)\|(u,v)\|_p^p+\frac{(r-p)}{r}\left(\frac{1}{q}-\frac{1}{2-\alpha-\beta}\right)\|(u,v)\|_q^q\\&<0.\end{aligned}$$

Therefore, $i^+=\inf_{(u,v)\in\mathcal{N}^+_{\alpha,\beta}} I_{\alpha,\beta}(u,v)<0$.

(2) Likewise, let us choose $(u,v) \in \mathcal{N}^-_{\alpha,\beta}$. We again appeal to the following inequality

$$\begin{aligned}(p+\alpha+\beta-2)\|(u,v)\|_p^p &< (p+\alpha+\beta-2)\|(u,v)\|_p^p + (q+\alpha+\beta-2)\|(u,v)\|_q^q \\ &< (r+\alpha+\beta-2)\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx \\ &\leq (r+\alpha+\beta-2)CM'(\lambda^{\frac{r}{r-p}} + \mu^{\frac{r}{r-p}})\|(u,v)\|_p^r. \end{aligned} \tag{2.5}$$

by virtue of the fact that $(u,v) \in \mathcal{N}_{\alpha,\beta}$. Therefore

$$\|(u,v)\|_p \geq \left[\left(\frac{p+\alpha+\beta-2}{r+\alpha+\beta-2}\right)\frac{1}{CM'(\lambda^{\frac{r}{r-p}} + \mu^{\frac{r}{r-p}})}\right]^{\frac{1}{r-p}}.$$

We will call this cumbersome looking constant as Λ. Therefore on proceeding further we have

$$\begin{aligned} I_{\alpha,\beta}(u,v) &= \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p + \left(\frac{1}{q}-\frac{1}{r}\right)\|(u,v)\|_q^q \\ &\quad + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\ &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p \\ &\quad - \nu M|\Omega|^{1-\frac{2-\alpha-\beta}{p^*}} S^{\frac{\alpha+\beta-2}{p}} \\ &\quad \times \left(\frac{1}{2-\alpha-\beta}-\frac{1}{r}\right)\int_\Omega \left(\frac{1-\alpha}{2-\alpha-\beta}|\nabla u|_p^{2-\alpha-\beta} + \frac{1-\beta}{2-\alpha-\beta}|\nabla v|_p^{2-\alpha-\beta}\right)dx \\ &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p \\ &\quad - \nu A_0(p,s,N,\alpha,\beta,|\Omega|)\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right] \\ &\quad \times \|(u,v)\|_p^{2-\alpha-\beta} \\ &= \|(u,v)\|_p^{2-\alpha-\beta}\left[\left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^{p+\alpha+\beta-2}\right. \\ &\quad \left. -\nu A_0(p,s,N,\alpha,\beta,|\Omega|)\left\{\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right\}\right]. \\ &\geq \Lambda^{2-\alpha-\beta}\left[\left(\frac{1}{p}-\frac{1}{r}\right)\Lambda^{p+\alpha+\beta-2}\right. \\ &\quad \left. -\nu A_0(p,s,N,\alpha,\beta,|\Omega|)\left\{\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right\}\right]. \end{aligned}$$

Then for a sufficiently small $\Lambda^* > 0$ and $D_0 > 0$ such that

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p+\alpha+\beta-2}{p}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p+\alpha+\beta-2}{p}}\right] \in (0,\Lambda^*),$$

we have $i^- \geq D_0 > 0$. □

Remark 2.5. For a better understanding of the Nehari manifold and the fiber maps, we define the function

$$F_{u,v}(t) = t^{p-r}\|(u,v)\|_p^p + t^{q-r}\|(u,v)\|_q^q - \nu t^{2-\alpha-\beta-r}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.$$

Then

$$\Phi'(t) = t^{r-1}[F_{u,v}(t) - \int_\Omega(\lambda f(x)u^r + \beta g(x)v^r)dx].$$

Observe that $\lim_{t\to\infty} F_{u,v}(t) = 0$ and $\lim_{t\to 0^+} F_{u,v}(t) = -\infty$. Further,

$$\begin{aligned} F'_{u,v}(t) &= (p-r)t^{p-r-1}\|(u,v)\|_p^p + (q-r)t^{q-r-1}\|(u,v)\|_q^q \\ &\quad -\nu(2-\alpha-\beta-r)t^{1-\alpha-\beta-r}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\ &= t^{1-\alpha-\beta-r}[(p-r)t^{p+\alpha+\beta}\|(u,v)\|_p^p + (q-r)t^{q+\alpha+\beta}\|(u,v)\|_q^q \\ &\quad -\nu(2-\alpha-\beta-r)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx]. \end{aligned}$$

Let

$$\begin{aligned} \psi_{u,v}(t) = (p-r)t^{p+\alpha+\beta}\|(u,v)\|_p^p + (q-r)t^{q+\alpha+\beta}\|(u,v)\|_q^q \\ -\nu(2-\alpha-\beta-r)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx. \end{aligned}$$

We also have

$$\lim_{t\to 0^+}\psi_{u,v}(t) = \nu(r+\alpha+\beta-2)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx,$$
$$\lim_{t\to\infty}\psi_{u,v}(t) = -\infty$$

and

$$\begin{aligned} \psi'_{u,v}(t) = (p-r)(p+\alpha+\beta)t^{p+\alpha+\beta-1}\|(u,v)\|_p^p \\ + (q-r)(q+\alpha+\beta)t^{q+\alpha+\beta-1}\|(u,v)\|_q^q < 0. \end{aligned}$$

Thus, for each $(u,v) \in X$ with $\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx > 0$, $F_{u,v}(t)$ attains its maximum at some $t_{max} = t_{max}(u,v)$. This unique t_{max} can be evaluated by solving for t from the equation

$$(r-p)t^{p+\alpha+\beta}\|(u,v)\|_p^p + (r-q)t^{q+\alpha+\beta}\|(u,v)\|_q^q = \nu(r+\alpha+\beta-2)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.$$

A simple calculation yields

$$\begin{aligned} F_{u,v}(t_{max}) = t_{max}^{p-r}\left(1 + \frac{r-p}{r+\alpha+\beta-2}t_{max}^2\right)\|(u,v)\|_p^p \\ + t_{max}^{q-r}\left(1 + \frac{r-q}{r+\alpha+\beta-2}t_{max}^2\right)\|(u,v)\|_q^q > 0. \end{aligned}$$

Thus for $t \in (0, t_{max})$ we have $F'_{u,v}(t) > 0$ and $F'_{u,v}(t) < 0$ for $t \in (t_{max}, \infty)$.
We now have the following lemma as a consequence.

Lemma 2.6. *For every $(u, v) \in X \setminus \{(0, 0)\}$ there exists a unique $0 < t^+ < t_{max}$ such that $(t^+u, t^+v) \in \mathcal{N}^+_{\alpha,\beta}$ and*

$$I_{\alpha,\beta}(t^+u, t^+v) = \inf_{t \geq 0} I_{\alpha,\beta}(tu, tv).$$

Furthermore, if

$$\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx > 0$$

then there exists unique $0 < t^+ < t_{max} < t^-$ such that

$$(t^+u, t^+v) \in \mathcal{N}^+_{\alpha,\beta},\ (t^-u, t^-v) \in \mathcal{N}^-_{\alpha,\beta}$$

and

$$I_{\alpha,\beta}(t^+u, t^+v) = \inf_{0 \leq t \leq t_{max}} I_{\alpha,\beta}(tu, tv),\ I_{\alpha,\beta}(t^-u, t^-v) = \sup_{t \geq 0} I_{\alpha,\beta}(tu, tv).$$

Proof. We only prove the case when

$$\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx > 0.$$

Thus the equation

$$F_{u,v}(t) = \int_\Omega (\lambda f(x)u^r + \beta g(x)v^r)dx$$

has only two solutions namely, $0 < t^+ < t_{max} < t^-$ such that $I'_{\alpha,\beta}(t^+) > 0$ and $I'_{\alpha,\beta}(t^-) < 0$. Since

$$\Phi''(t^+) = (t^+)^{r-1}[F_{u,v}(t^+) - \int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx] > 0$$

and

$$\Phi''(t^-) = (t^-)^{r-1}[F_{u,v}(t^-) - \int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)dx] < 0,$$

therefore $(t^+u, t^+v) \in \mathcal{N}^+_{\alpha,\beta}$ and $(t^-u, t^-v) \in \mathcal{N}^-_{\alpha,\beta}$. Thus $\Phi(t)$ decreases in $(0, t^+)$, increases in (t^+, t^-) and decreases in (t^-, ∞). Hence the lemma. □

We now define the palais-Smale sequence ((PS)-sequence), (PS)-condition and (PS)-value in X for $I_{\alpha,\beta}$ corresponding to the functional $I_{\alpha,\beta}$ which is as follows.

Definition 2.7. Suppose for $c \in \mathbb{R}$, a sequence $\{(u_n, v_n)\} \subset X$ is a $(PS)_c$-sequence for the functional $I_{\alpha,\beta}$ if $I_{\alpha,\beta}(u_n, v_n) \to c$ and $I'_{\alpha,\beta}(u_n, v_n) \to 0$ in X' as $n \to \infty$, then:

(1) $c \in \mathbb{R}$ is a (PS) value in X for the functional $I_{\alpha,\beta}$ if there exists a $(PS)_c$-sequence in X for $I_{\alpha,\beta}$.
(2) The functional $I_{\alpha,\beta}$ satisfies the $(PS)_c$-condition in X for $I_{\alpha,\beta}$ if any $(PS)_c$-sequence admits a strongly convergent subsequence in X.

Remark 2.8. We will sometimes denote $\lim_{n\to\infty} x_n = 0$ as $x_n = o(1)$ for a sequence of real numbers (x_n).

Remark 2.9. X' refers to the dual space of X.

Lemma 2.10. *For any* $0<\alpha,\beta<1$*, the functional* $I_{\alpha,\beta}$ *satisfies the* $(PS)_c$*-condition for* $c\in\left(-\infty,\frac{S_{\alpha,\beta}^{\frac{r}{r-p}}}{\Lambda}-\nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\right)$ *where*

$$\Lambda=2M'(\lambda^{\frac{r}{r-p}}+\mu^{\frac{r}{r-p}})\}^{\frac{p}{r-p}}|\Omega|^{\frac{1}{r}}.$$

Proof. Suppose $\{(u_n,v_n)\}$ is a $(PS)_c$-sequence in X for the functional $I_{\alpha,\beta}$ with

$$c\in\left(-\infty,\frac{S_{\alpha,\beta}^{\frac{r}{r-p}}}{\Lambda}-\nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\right).$$

Then

$$I_{\alpha,\beta}(u_n,v_n)=c+o(1),\ I'_{\alpha,\beta}(u_n,v_n)=o(1). \tag{2.6}$$

We now claim that $\{(u_n,v_n)\}$ is bounded in X. We prove this claim by contradiction, i.e. say $\|(u_n,v_n)\|_p\to\infty$. Let

$$(\tilde{u_n},\tilde{v_n})=\left(\frac{u_n}{\|(u_n,v_n)\|_p},\frac{v_n}{\|(u_n,v_n\|_p)}\right),$$

then $\|(\tilde{u_n},\tilde{v_n})\|_p=1$ which implies that $(\tilde{u_n},\tilde{v_n})$ is bounded in X. Therefore, due to the reflexivity of the space X, we have upto a subsequence

$$(\tilde{u_n},\tilde{v_n})\rightharpoonup(u_n,v_n)$$

as $n\to\infty$ in X. This further implies that

$$\tilde{u_n}\rightharpoonup\tilde{u},\ \tilde{v_n}\rightharpoonup\tilde{v}\text{ in }W_0^{1,p}(\Omega),$$
$$\tilde{u_n}\to\tilde{u},\ \tilde{v_n}\to\tilde{v}\text{ in }L^s(\Omega),\ 1\le s<p^*,$$
$$\int_\Omega\nu h(x)\tilde{u_n}^{1-\alpha}\tilde{v_n}^{1-\beta}dx\to\int_\Omega\nu h(x)u^{1-\alpha}v^{1-\beta}dx.$$

The last convergence follows from the Egoroff's theorem. From (2.6) we have

$$\begin{aligned}c+o(1)&=\frac{1}{p}\|(u_n,v_n)\|_p^p\|(\tilde{u_n},\tilde{v}_n)\|_p^p+\frac{1}{q}\|(u_n,v_n)\|_q^q\|(\tilde{u_n},\tilde{v}_n)\|_q^q\\&-\frac{1}{r}\|(u_n,v_n)\|_p^r\int_\Omega(\lambda f(x)\tilde{u}_n^r+\mu g(x)\tilde{v}_n^r)dx\\&-\frac{\nu}{2-\alpha-\beta}\|(u_n,v_n)\|_p^{2-\alpha-\beta}\int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta}dx\end{aligned}$$

and

$$\begin{aligned}o(1)&=\|(u_n,v_n)\|_p^p\|(\tilde{u}_n,\tilde{v}_n)\|_p^p+\|(u_n,v_n)\|_q^q\|(\tilde{u}_n,\tilde{v}_n)\|_q^q\\&-\|(u_n,v_n)\|_p^r\int_\Omega(\lambda f(x)\tilde{u}_n^r+\mu g(x)\tilde{v}_n^r)dx\\&-\nu\|(u_n,v_n)\|_p^{2-\alpha-\beta}\int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta}dx.\end{aligned}$$

Now by the assumption we made, i.e. $\|(u_n, v_n)\|_p \to \infty$, we obtain

$$\begin{aligned} o(1) &= \frac{1}{p}\|(\tilde{u_n}, \tilde{v}_n)\|_p^p + \frac{1}{q}\|(u_n, v_n)\|_p^{q-p}\|(\tilde{u_n}, \tilde{v}_n)\|_q^q \\ &-\frac{1}{r}\|(u_n, v_n)\|_p^{r-p}\int_\Omega(\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)dx \\ &-\frac{\nu}{2-\alpha-\beta}\|(u_n, v_n)\|_p^{2-\alpha-\beta-p}\int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta}dx \end{aligned}$$

and

$$\begin{aligned} o(1) &= \|(\tilde{u}_n, \tilde{v}_n)\|_p^p + \|(u_n, v_n)\|_p^{q-p}\|(\tilde{u}_n, \tilde{v}_n)\|_q^q \\ &-\|(u_n, v_n)\|_p^{r-p}\int_\Omega(\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)dx \\ &-\nu\|(u_n, v_n)\|_p^{2-\alpha-\beta-p}\int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta}dx. \end{aligned}$$

On using the above to equalities we get

$$\begin{aligned} o(1) = &\left(1 - \frac{2-\alpha-\beta}{p}\right)\|(\tilde{u}_n, \tilde{v}_n)\|_p^p \\ &+\left(1 - \frac{2-\alpha-\beta}{q}\right)\|(u_n, v_n)\|_p^{q-p}\|(\tilde{u}_n, \tilde{v}_n)\|_q^q \\ &+\left(\frac{2-\alpha-\beta}{r} - 1\right)\|(u_n, v_n)\|_p^{r-p}\int_\Omega(\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)dx \end{aligned}$$

as $n \to \infty$.
Therefore we have

$$\begin{aligned} \|(\tilde{u}_n, \tilde{v}_n)\|_p^p &= \frac{p(p-2+\alpha+\beta)}{q(2-\alpha-\beta-q)}\|(u_n, v_n)\|_p^{q-p}\|(\tilde{u}_n, \tilde{v}_n)\|_q^q \\ &+\nu\frac{p(p-2+\alpha+\beta)}{r(r-2+\alpha+\beta)}\|(u_n, v_n)\|_p^{2-\alpha-\beta-p}\int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta}dx + o(1) \end{aligned}$$

as $n \to \infty$. Thus we have $\|(\tilde{u}_n, \tilde{v}_n)\|_p^p \to \infty$ which is a contradiction to our assumption that

$$\|(\tilde{u}_n, \tilde{v}_n)\|_p = 1.$$

Therefore, the sequence $\{(u_n, v_n)\}$ is bounded in X.
We choose a subsequence to this bounded sequence, still denoted by $\{(u_n, v_n)\}$ such that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } X,$$

$$u_n \to u,\ v_n \to v \text{ in } L^s(\Omega),\ 1 \le s < p^*,$$

$$\int_\Omega(\lambda f(x)u_n^r + \mu g(x)v_n^r)dx \to \int_\Omega(\lambda f(x)u^r + \mu g(x)v^r)dx,$$

$$\nu\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx \to \nu\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx$$

as $n \to \infty$.

By the Brezis-Lieb [19] theorem we get

$$\|(u_n-u,v_n-v)\|_p^p=\|(u_n,v_n)\|_p^p-\|(u,v)\|_p^p+o(1),$$

$$\begin{aligned}\int_\Omega(\lambda f(x)(u_n-u)^r+\mu g(x)(v_n-v)^r)dx&=\int_\Omega(\lambda f(x)u_n^r+\mu g(x)v_n^r)dx\\&\quad-\int_\Omega(\lambda f(x)u^r+\mu g(x)v^r)dx+o(1)\end{aligned}$$

and

$$\begin{aligned}\nu\int_\Omega h(x)(u_n-u)^{1-\alpha}(v_n-v)^{1-\beta}dx&=\nu\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx\\&\quad-\nu\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx+o(1).\end{aligned}$$

Thus for any $(\phi_2,\phi_2)\in X$ the following holds.

$$\lim_{n\to\infty}\langle I'_{\alpha,\beta},(\phi_2,\phi_2)\rangle=\langle I'_{\alpha,\beta}(u,v),(\phi_1,\phi_2)\rangle=0.$$

In other words (u,v) is a critical point of $I_{\alpha,\beta}$. All we now need to show is that $(u_n,v_n)\to(u,v)$ in X. We use (2.6), the Brezis-Lieb lemma [19] and some basic functional analysis to obtain

$$\begin{aligned}c-I_{\alpha,\beta}+o(1)&=\frac{1}{p}\|(u_n-u,v_n-v)\|_p^p+\frac{1}{q}\|(u_n-u,v_n-v)\|_q^q\\&\quad-\frac{1}{r}\int_\Omega(\lambda f(x)(u_n-u)^r+\mu g(x)(v_n-v)^r)dx\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}0&=\langle I'_{\alpha,\beta}(u_n,v_n),(u_n-u,v_n-v)\rangle\\&=\langle I'_{\alpha,\beta}(u_n,v_n)-I'_{\alpha,\beta}(u,v),(u_n-u,v_n-v)\rangle\\&=\|(u_n-u,v_n-v)\|_p^p+\|(u_n-u,v_n-v)\|_q^q\\&\quad-\int_\Omega(\lambda f(x)(u_n-u)^r+\mu g(x)(v_n-v)^r)dx+o(1).\end{aligned}\tag{2.8}$$

Now without loss of generality, we let

$$\|(u_n-u,v_n-v)\|_p^p=c'+o(1),$$

$$\|(u_n-u,v_n-v)\|_q^q=d'+o(1)$$

and therefore

$$\int_\Omega(\lambda f(x)(u_n-u)^r+\mu g(x)(v_n-v)^r)dx=c'+d'+o(1).$$

Now if $c' = 0$ the proof is immediate. On the contrary, we assume that $c' > 0$.

$$\begin{aligned}
\left(\frac{c'}{2}\right)^{\frac{p}{p^*}} &\leq \left(\frac{c'+d'}{2}\right)^{\frac{p}{p^*}} \\
&= \lim_{n\to\infty}\int_\Omega (\lambda f(x)(u_n-u)^r + \mu g(x)(v_n-v)^r)dx \\
&\leq M' \lim_{n\to\infty}\int_\Omega (\lambda|u_n-u|^r + \mu|v_n-v|^r)\,dx \\
&\leq M' \lim_{n\to\infty}|\Omega|^{\frac{1}{2-\alpha-\beta}-\frac{1}{r}} S_{\alpha,\beta}^{-\frac{r}{p}}\|(u_n-u,v_n-v)\|_p^r \\
&= M'|\Omega|^{\frac{1}{p}-\frac{1}{r}} S_{\alpha,\beta}^{-\frac{r}{p}}(\lambda^{\frac{r}{r-p}}+\mu^{\frac{r}{r-p}})c'^{\frac{r}{p}}.
\end{aligned}$$

Thus,

$$c' \geq \frac{S_{\alpha,\beta}^{\frac{r}{r-p}}}{\{2M'(\lambda^{\frac{r}{r-p}}+\mu^{\frac{r}{r-p}})\}^{\frac{p}{r-p}}|\Omega|^{\frac{1}{r}}} = \frac{S_{\alpha,\beta}^{\frac{r}{r-p}}}{\Lambda}.$$

Therefore from (2.7), (2.8) and $(u,v) \in \mathcal{N}_{\alpha,\beta}\bigcup\{(0,0)\}$ we have

$$\begin{aligned}
c' &= I_{\alpha,\beta}(u,v) + \frac{c'}{p} + \frac{d'}{q} - \frac{c'+d'}{r} \\
&\geq \frac{S_{\alpha,\beta}^{\frac{r}{r-p}}}{\Lambda} - \nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]
\end{aligned}$$

which contradicts

$$c' < \frac{S_{\alpha,\beta}^{\frac{r}{r-p}}}{\Lambda} - \nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right].$$

Thus $c' = 0$ and hence $(u_n, v_n) \to (u, v)$ in X. □

We will now see the proof of the existence of a local minimizer for $I_{\alpha,\beta}$ in $\mathcal{N}^+_{\alpha,\beta}$.

Lemma 2.11. *There exists* $\Lambda^* > 0$ *such that*

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right] \in (0, \Lambda^*),$$

$I_{\alpha,\beta}$ *has a minimizer* $(u_\nu, v_\nu) \in \mathcal{N}^+_{\alpha,\beta}$ *and it satisfies*

(*i*) $I_{\alpha,\beta}(u_\nu, v_\nu) = i^+$ *is a weak solution to the problem* (1.1)

(*ii*) $I_{\alpha,\beta}(u_\nu, v_\nu) \to 0$ *and* $\|(u_\nu, v_\nu)\|_p \to 0$, $\|(u_\nu, v_\nu)\|_q \to 0$ *as* $\nu \to 0$.

Proof. For the proof of (*i*) we follow Hsu [15], Theorem 4.2.
Since $i^+ = \inf_{(u,v)\in\mathcal{N}_{\alpha,\beta}}\{I_{\alpha,\beta}(u,v)\}$, there exists a sequence $(u_n, v_n) \in \mathcal{N}_{\alpha,\beta}$ such that $I_{\alpha,\beta}(u_n, v_n) \to i^+$ and $I'_{\alpha,\beta}(u_n, v_n) \to 0$ in X^* as $n \to \infty$. Since the functional $I_{\alpha,\beta}$ is coercive and therefore (u_n, v_n) is bounded in X. Thus there exists a subsequence of (u_n, v_n), still denoted as (u_n, v_n), such that $((u_n, v_n)) \rightharpoonup (u, v) \in X$. So we have

$$u_n \rightharpoonup u,\ v_n \rightharpoonup v,$$

$$u_n \to u,\ v_n \to v \text{ a.e.in } \Omega,$$
$$u_n \to u,\ v_n \to v \text{ in } L^s(\Omega) \text{ for } 1 \le s < p^*$$

as $n \to \infty$. This implies

$$\frac{2\nu}{2-\alpha-\beta}\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx \to \frac{2\nu}{2-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.$$

Clearly (u,v) is a weak solution of (1.1). Also since $(u_n,v_n)\in \mathcal{N}_{\alpha,\beta}$ we have

$$\begin{aligned} L^\nu_{\alpha,\beta}(u_n,v_n) &= \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{p}-\frac{1}{r}\right)\|(u_n,v_n)\|_p^p \\ &+ \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{q}-\frac{1}{r}\right)\|(u_n,v_n)\|_q^q \\ &- \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}I_{\alpha,\beta}(u_n,v_n) \end{aligned}$$

where

$$L^\nu_{\alpha,\beta}(u_n,v_n) = \int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx.$$

Also

$$\begin{aligned} L^\nu_{\alpha,\beta}(u_n,v_n) \quad &\ge \quad \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p \\ &\quad + \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{q}-\frac{1}{r}\right)\|(u,v)\|_q^q - \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}i^+ \\ &\ge \quad -\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}i^+ > 0 \end{aligned}$$

where we have used the lower-semicontinuity of $\|\cdot\|_p$, $\|\cdot\|_q$ and $i^+ < 0$. Therefore $(u,v)\neq(0,0)$. Thus we have a nontrivial weak solution.
Claim: We now claim that $(u_n,v_n)\to(u,v)$ in X and $I_{\alpha,\beta}(u,v) = i^+$.
For any $(u_0,v_0)\in\mathcal{N}_{\alpha,\beta}$ we have

$$\begin{aligned} L^\nu_{\alpha,\beta}(u_0,v_0) &= \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{p}-\frac{1}{r}\right)\|(u_0,v_0)\|_p^p \\ &+ \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{q}-\frac{1}{r}\right)\|(u_0,v_0)\|_q^q \\ &- \frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}I_{\alpha,\beta}(u_0,v_0). \end{aligned}$$

Thus

$$\begin{aligned} i^+ \quad &\le \quad I_{\alpha,\beta}(u,v) \\ &\le \quad \lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{r}\right)\|(u_n,v_n)\|_p^p + \left(\frac{1}{q}-\frac{1}{r}\right)\|u_n,v_n\|_q^q \right. \\ &\quad \left. -\frac{2\nu}{2-\alpha-\beta}L^\nu_{\alpha,\beta}(u_n,v_n)\right] \\ &= \quad I_{\alpha,\beta}(u,v) = i^+. \end{aligned}$$

Thus $I_{\alpha,\beta}(u,v) = i^+$. This also implies that $(u_n, v_n) \to (u,v)$ in X.
For the proof of (ii) let $(u_\nu, v_\nu) \in \mathcal{N}^+_{\alpha,\beta}$. From Lemmas 2.1, 2.2 we have that

$$0 > I_{\alpha,\beta}(u_\nu, v_\nu) \geq -\nu A_0 \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right].$$

Therefore it is obvious that as $\nu \to 0$ we have $I_{\alpha,\beta}(u_\nu, v_\nu) \to 0$.
Further we have

$$\begin{aligned} 0 &= \lim_{\nu \to 0} I_{\alpha,\beta}(u_\nu, v_\nu) \\ &= \lim_{\nu \to 0} \left[\left(\frac{1}{p} - \frac{1}{r} \right) \|(u_\nu, v_\nu)\|_p^p + \left(\frac{1}{q} - \frac{1}{r} \right) \|u_\nu, v_\nu\|_q^q \right. \\ &\quad \left. - \frac{2\nu}{2-\alpha-\beta} \int_\Omega h(x) u_\nu^{1-\alpha} v_\nu^{1-\beta} dx \right]. \end{aligned}$$

As seen earlier that the functional $I_{\alpha,\beta}$ is coercive over $\mathcal{N}^+_{\alpha,\beta}$ and therefore (u_ν, v_ν) is bounded. Also using the fact $\lim\limits_{\nu \to 0} \frac{2\nu}{2-\alpha-\beta} \int_\Omega h(x) u_\nu^{1-\alpha} v_\nu^{1-\beta} dx = 0$ we clearly have

$$\lim_{\nu \to 0} \|(u_\nu, v_\nu)\|_p^p = 0 = \lim_{\nu \to 0} \|(u_\nu, v_\nu)\|_q^q. \qquad \square$$

Remark 2.12. For $\epsilon > 0$ we define

$$u_\epsilon(x) = \frac{\eta(x)}{(\epsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}, \; v_\epsilon(x) = \frac{u_\epsilon(x)}{|u_\epsilon(x)|_{p^*}}$$

where $\eta(x) \in C_0^\infty(\Omega)$ is a radially symmetric function defined by

$$\eta(x) = \begin{cases} 1 & |x| < \rho_0 \\ 0 & |x| > 2\rho_0 \\ 0 \leq \eta(x) \leq 1 & \text{otherwise.} \end{cases}$$

Further let $|\nabla \eta| \leq C$, where ρ_0 is such that $B(0, 2\rho_0) \subset \Omega$. Then $\int_\Omega |u_\epsilon|^{p^*} dx = 1$ and we have the following estimates

$$\int_\Omega |u_\epsilon|^t dx = \begin{cases} C_1 \epsilon^{\frac{N(p-1)-t(N-p)}{p}} + O(1) & t > \frac{N(p-1)}{N-p} \\ C_1 |\ln \epsilon| + O(1) & t = \frac{N(p-1)}{N-p} \\ O(1) \leq \eta(x) \leq 1 & t < \frac{N(p-1)}{N-p}. \end{cases}$$

Therefore in particular we have

$$\int_\Omega |\nabla u_\epsilon|^p dx = K_2 \epsilon^{\frac{p-N}{p}} + O(1)$$

and

$$\left(\int_\Omega |u_\epsilon|^{p^*} dx \right)^{\frac{1}{p^*}} = K_3 \epsilon^{\frac{p-N}{p}} + O(1)$$

where $K_1, K_2, K_3 > 0$ independent of ϵ. Further there exists ϵ_0 such that S, the best sobolev constant, is close to $\frac{K_2}{K_3}$ for every $0 < \epsilon < \epsilon_0$. In other words we will take $S = \frac{K_2}{K_3}$.

We now prove the following lemma which will be used in guaranteeing the multiplicity of solutions.

Lemma 2.13. *There exists ϵ_1, Λ^*, $\sigma(\epsilon) > 0$ such that for $\epsilon \in (0, \epsilon_1)$,*

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right] \in (0, \Lambda^*)$$

and $\sigma \in (0, \sigma(\epsilon))$, we have

$$\sup_{t\geq 0} I_{\alpha,\beta}(t_\epsilon \sqrt[p]{\nu} v_\epsilon, t_\epsilon \sqrt[p]{\nu} v_\epsilon) < c_{\alpha,\beta} - \sigma,$$

where

$$c_{\alpha,\beta} = \frac{r-p}{rp} S^{\frac{r}{r-p}} - \nu A_0 \left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right].$$

Proof. Define

$$\begin{aligned} a(t) &= I_{\alpha,\beta}(t\sqrt[p]{\nu} v_\epsilon, t\sqrt[p]{\nu} v_\epsilon) \\ &= \frac{t^p}{p}\nu \int_\Omega |\nabla v_\epsilon|^p dx + \frac{t^q}{q}(2\nu^{\frac{q}{p}}) \int_\Omega |\nabla v_\epsilon|^q dx \\ &\quad -\frac{1}{r}\int_\Omega (\lambda f(x) + \mu g(x))(t v_\epsilon \nu^{\frac{1}{p}})^r dx - \frac{2\nu^{\frac{p-\alpha-\beta+2}{p}} t^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx. \end{aligned}$$

Clearly $a(0) = 0$, $\lim_{t\to\infty} a(t) = -\infty$. Therefore there exists $t_\epsilon > 0$ such that

$$I_{\alpha,\beta}(t_\epsilon \sqrt[p]{\nu} v_\epsilon, t_\epsilon \sqrt[p]{\nu} v_\epsilon) = \sup_{t\geq 0} I_{\alpha,\beta}(t\sqrt[p]{\nu} v_\epsilon, t\sqrt[p]{\nu} v_\epsilon).$$

This yields that

$$\begin{aligned} (2\nu) t_\epsilon^{p-1} \int_\Omega |\nabla v_\epsilon|^p dx + 2 t_\epsilon^{q-1} \nu^{\frac{q}{p}} \int_\Omega |\nabla v_\epsilon|^q dx &= t_\epsilon^{r-1} \int_\Omega (\lambda f(x) + \mu g(x))(v_\epsilon \nu^{\frac{1}{p}})^r dx \\ &\quad + 2\nu^{\frac{p-\alpha-\beta+2}{p}} t_\epsilon^{1-\alpha-\beta} \int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx. \end{aligned} \tag{2.9}$$

From (2.9) we have the following

$$\begin{aligned} t_\epsilon^{p+\alpha+\beta-2} \int_\Omega |\nabla v_\epsilon|^p dx &\leq t_\epsilon^{r+\alpha+\beta-2} \int_\Omega (\lambda f(x) + \mu g(x))(v_\epsilon \nu^{\frac{1}{p}})^r dx \\ &\quad + 2\nu^{\frac{p-\alpha-\beta+2}{p}} \int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx. \end{aligned} \tag{2.10}$$

and

$$(2\nu) t_\epsilon^{p-q} \int_\Omega |\nabla v_\epsilon|^p dx + 2\nu^{\frac{q}{p}} \int_\Omega |\nabla v_\epsilon|^q dx \geq t_\epsilon^{r-q} \int_\Omega (\lambda f(x) + \mu g(x))(v_\epsilon \nu^{\frac{1}{p}})^r dx. \tag{2.11}$$

From the estimates for u_ϵ obtained in the Remark 2.12, i.e.

$$\int_\Omega |\nabla v_\epsilon|^p dx = S + O\left(\epsilon^{\frac{N-p}{p}}\right),$$

$$\int_\Omega |v_\epsilon|^r dx = O\left(\epsilon^{\frac{r(N-p)}{p^2}}\right), \quad \int_\Omega |v_\epsilon|^{2-\alpha-\beta} dx = O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-p)}{p^2}}\right).$$

From (2.9) it very easily follows now that

$$t_\epsilon^{p+\alpha+\beta-2}\left(S + O\left(\epsilon^{\frac{N-p}{p}}\right)\right) \leq CM' t_\epsilon^{r+\alpha+\beta-2} + 2M\nu^{\frac{p-\alpha-\beta+2}{p}} O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-p)}{p^2}}\right) \tag{2.12}$$

where we have use the estimate

$$\int_\Omega (\lambda f(x) + \mu g(x)) v_\epsilon^r dx \leq CM' \|v_\epsilon\|_{p^*}^r = CM'.$$

Thus, there exists $T_1 > 0$, $\epsilon_1 > 0$ such that for any $\epsilon \in (0, \epsilon_1)$, we have $t_\epsilon \geq T_1$. Likewise we have

$$(2\nu) t_\epsilon^{p-q}(S + O(\epsilon^{\frac{N-p}{p}})) + 2C\nu^{\frac{q}{p}} \quad \geq \quad C t_\epsilon^{2-\alpha-\beta-q}. \tag{2.13}$$

Then, there exists $T_2 > 0$, $\epsilon_2 > 0$ such that for any $\epsilon \in (0, \epsilon_2)$, we have $t_\epsilon \leq T_2$. Let $\tilde{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$. Then for any $\epsilon \in (0, \tilde{\epsilon})$ we have $T_1 \leq t_\epsilon \leq T_2$. Consider

$$b(t) = \frac{t^p}{p}\nu \int_\Omega |\nabla v_\epsilon|^p dx - \frac{1}{r}\int_\Omega (\lambda f(x) + \mu g(x))(t v_\epsilon \nu^{\frac{1}{p}})^r dx.$$

Then a simple calculation gives

$$\sup_{t\geq 0} b(t) \quad = \quad \frac{r-p}{rp} S^{\frac{r}{r-p}} + O\left(\epsilon^{\frac{N-p}{p}}\right).$$

Therefore, for any $\epsilon \in (0, \tilde{\epsilon})$, we have

$$\begin{aligned}
a(t_\epsilon) \quad &= \quad b(t_\epsilon) + \frac{t_\epsilon^q}{q}(\nu^{\frac{q}{p}}) \int_\Omega |\nabla v_\epsilon|^q dx \\
&\quad - \frac{\nu^{\frac{p-\alpha-\beta+2}{p}} t_\epsilon^{2-\alpha-\beta}}{2-\alpha-\beta} \int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx \\
&\leq \quad \frac{r-p}{rp} S^{\frac{r}{r-p}} + O\left(\epsilon^{\frac{N-p}{p}}\right) + \frac{t_\epsilon^q}{q}(2\nu^{\frac{q}{p}}) \int_\Omega |\nabla v_\epsilon|^q dx \\
&\quad - \frac{\nu^{\frac{p-\alpha-\beta+2}{p}} t_\epsilon^{2-\alpha-\beta}}{2-\alpha-\beta} \int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx \\
&\leq \quad \frac{r-p}{rp} S^{\frac{r}{r-p}} + O\left(\epsilon^{\frac{N-p}{p}}\right) + \frac{T_2^q}{q}(2\nu^{\frac{q}{p}}) \int_\Omega |\nabla v_\epsilon|^q dx \\
&\quad - \frac{\nu^{\frac{p-\alpha-\beta+2}{p}} T_1^{2-\alpha-\beta}}{2-\alpha-\beta} \int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx \\
&\leq \quad \frac{r-p}{rp} S^{\frac{r}{r-p}} + O\left(\epsilon^{\frac{N-p}{p}}\right) + O\left(\epsilon^{\frac{q(N-p)}{p^2}}\right) - O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-p)}{p^2}}\right).
\end{aligned}$$

From the assumptions in the problem in (1.1) we also have

$$0<\frac{(2-\alpha-\beta)(N-p)}{p^2}<\frac{q(N-p)}{p^2}<\frac{N-p}{p}.$$

Therefore, one can choose $\epsilon_1>0$, sufficiently small, Λ^*, $\sigma(\epsilon)>0$ such that for $\epsilon\in(0,\epsilon_1)$, $\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\in(0,\Lambda^*)$ and $\sigma\in(0,\sigma(\epsilon))$, we have

$$\begin{aligned}&O\left(\epsilon^{\frac{N-p}{p}}\right)+O\left(\epsilon^{\frac{q(N-p)}{p^2}}\right)-O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-p)}{p^2}}\right)\\&<-A_0\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]-\sigma.\end{aligned}$$

□

3. Few useful lemmas

This section is devoted to recall and prove some important lemmas which are crucial to the proof of the main theorem. We first consider a submanifold of $\mathcal{N}^-_{\alpha,\beta}$ defined as follows.

$$\mathcal{N}^-_{\alpha,\beta}(c_{\alpha,\beta})=\{(u,v)\in\mathcal{N}^-_{\alpha,\beta}:I_{\alpha,\beta}(u,v)\le c_{\alpha,\beta}\}.$$

The main result which we will prove in this section is that the problem in (1.1) admits at least cat(Ω) number of solutions in this set.

Definition 3.1. (a) For a topological space X, we say that a non-empty, closed subspace $Y\subset X$ is contractible to a point if and only if there exists a continuous mapping

$$\xi:[0,1]\times Y\to X$$

such that for some $x_0\in X$. there hold

$$\xi(0,x)=x,\ \text{for all } x\in Y$$

and

$$\xi(1,x)=x_0,\ \text{for all } x\in Y.$$

(b) If Y is closed subset of a topological space X, $\text{cat}_X(Y)$ denotes Lusternik-Schnirelman category of Y, i.e., the least number of closed and contractible sets in X which cover Y.

We now state an auxiliary lemma which can be found in the form of Theorem 1 in [1].

Lemma 3.2. *Suppose that X is a $C^{1,1}$ complete Riemanian manifold and $I\in C^1(X,\mathbb{R})$. Assume that for $c_0\in\mathbb{R}$ and $k\in\mathbb{N}$:*

$$\begin{aligned}&(i)\quad I \text{ satisfies the } (PS)_c \text{ condition for } c\le c_0\\&(ii)\quad cat(u\in X:I(u)\le c_0)\ge k.\end{aligned}\tag{3.1}$$

Then I has at least k critical points in $u\in X:I(u)\le c_0$.

The following lemma is a standard one and can be proved if one works in the lines of the argument in [31].

Lemma 3.3. *Let* $\{(u_n, v_n)\} \subset X$ *be a nonnegative sequence of functions with*

$$\int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r) dx = 1 \text{ and } \|(u_n, v_n)\|_p^p \to S_{\alpha,\beta}.$$

Then there exists a sequence $\{(y_n, \theta_n)\} \subset \mathbb{R}^N \times \mathbb{R}^+$ *such that*

$$\omega_n(x) = (\omega_n^1(x), \omega_n^2(x)) = \theta_n^{\frac{N}{r}} (u_n(\theta_n x + y_n), v_n(\theta_n x + y_n))$$

contains a convergent subsequence denoted again by $\{\omega_n\}$ *such that*

$$\omega_n \to \omega \text{ in } W^{1,p}(\mathbb{R}^N \times W^{1,p}(\mathbb{R}^N),$$

where $\omega = (\omega^1, \omega^2) > 0$ *in* $\mathbb{R}^N$. *Moreover, we have* $\theta_n \to 0$ *and* $y_n \to y \in \overline{\Omega}$ *as* $n \to \infty$.

Up to translations, we assume that $0 \in \Omega$. Moreover, we choose $\delta > 0$ small enough such that $B_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, \partial\Omega) < \delta\}$ and the sets

$$\Omega_\delta^+ = \{x \in \mathbb{R}^N : \text{dist}(x, \partial\Omega) < \delta\}, \quad \Omega_\delta^- = \{x \in \mathbb{R}^N : \text{dist}(x, \partial\Omega) > \delta\}$$

are both homotopically equivalent to Ω. By using the idea of [14] or [20] we define a continuous mapping $\tau : \mathcal{N}_{\alpha,\beta}^- \to \mathbb{R}^N$ by setting

$$\tau(u, v) = \frac{\int_\Omega x(\lambda f u^r + \mu g v^r) dx}{\int_\Omega (\lambda f u^r + \mu g v^r) dx}.$$

Remark 3.4. As told before that the functional $I_{\alpha,\beta}$ is not a C^1-functional, we might fail to use some very useful techniques in variational techniques. For this we will define a *cut-off* functional using a subsolution (refer [13] for a definition) to the system in (1.1). Define,

$$\overline{f}(x, t, s) = \begin{cases} f(x, t, s) & \text{if } t > \underline{u}, s > \underline{v} \\ f(x, t, \underline{v}) & \text{if } t > \underline{u}, s \leq \underline{v} \\ f(x, \underline{u}, s) & \text{if } t \leq \underline{u}, s > \underline{v} \\ f(x, \underline{u}, \underline{v}) & \text{if } t \leq \underline{u}, s \leq \underline{v} \end{cases}$$

where

$$f(x,t,s) = \lambda f(x) t^{r-1} + \mu g(x) s^{r-1} + \nu \frac{1-\alpha}{2-\alpha-\beta} h(x) t^{-\alpha} s^{1-\beta} + \nu \frac{1-\beta}{2-\alpha-\beta} h(x) t^{1-\alpha} s^{-\beta}$$

is a subsolution to (1.1) (the existence of such a solution can be guaranteed by the previous sections by taking $\lambda = \mu = 0$ in (1.1)). Let $\overline{F}(x, t, s) = \int_0^t \int_0^s \overline{f}(x, t, s) ds dt$ and $(\underline{u}, \underline{v})$. Define a function $\overline{I} : X \times X \to \mathbb{R}$ as follows.

$$\overline{I}(u, v) = \frac{1}{p} \|(u, v)\|_p^p + \frac{1}{q} \|(u, v)\|_q^q - \int_\Omega \overline{F}(x, u, v) dx. \tag{3.2}$$

The functional is C^1 (the proof follows the arguments of the Lemma 6.4 in the Appendix of [28]) and weakly lower semicontinuous. The way the functional has been defined, it is not difficult to see that the critical points of the fnctional corresponding to the problem (1.1) and that of the cut-off functional are the same.

Remark 3.5. We will continue to name the cut-off functional $\overline{I}$ as $I_{\alpha,\beta}$.

We then have the following result.

Lemma 3.6. *There exists* Λ^* *such that if*

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\in(0,\Lambda^*)$$

and $(u,v)\in\mathcal{N}^-_{\alpha,\beta}(c_{\alpha,\beta})$, *then* $\tau(u,v)\in\Omega^+_\delta$.

Proof. Let us assume that there exists sequences $\nu_n\to 0$ and $\{(u_n,v_n)\}$ such that $\tau(u_n,v_n)\notin\Omega^+_\delta$. By using the tactics in one of the previous lemmas (2.10) we conclude the boundedness of the sequence $\{(u_n,v_n)\}$ in X. Then we have

$$\nu_n\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx\to 0 \text{ as } n\to\infty.$$

Therefore we get

$$I_{\alpha,\beta}(u_n,v_n) = \left(\frac{1}{p}-\frac{1}{r}\right)\|(u_n,v_n)\|_p^p+\left(\frac{1}{q}-\frac{1}{r}\right)\|(u_n,v_n)\|_q^q+o(1)\le c^{\nu_n}_{\alpha,\beta}+o(1)$$

and

$$\left(\frac{1}{p}-\frac{1}{r}\right)\|(u_n,v_n)\|_p^p \le c^{\nu_n}_{\alpha,\beta}+o(1)\le\frac{S^{\frac{r}{r-p}}}{\Lambda}+o(1).$$

This implies that

$$\|(u_n,v_n)\|_p^p \le \frac{rp}{r-p}\frac{S^{\frac{r}{r-p}}}{\Lambda}+o(1). \tag{3.3}$$

Since $\{(u_n,v_n)\}\subset\mathcal{N}^-_{\alpha,\beta}(c^{\nu_n}_{\alpha,\beta})\subset\mathcal{N}^-_{\alpha,\beta}$, we have

$$\|(u_n,v_n)\|_p^p \le \int_\Omega(\lambda f(x)u_n^r+\mu g(x)v_n^r)dx+o(1)\le M'|(u_n,v_n)|^r_{p^*}+o(1). \tag{3.4}$$

By (3.3) and (3.4) we get

$$\begin{aligned} S_{\alpha,\beta} &\le \frac{\|(u_n,v_n)\|_p^p}{\{\int_\Omega(u_n^{p^*}+v_n^{p^*})dx\}^{\frac{p}{p^*}}}\\ &\le C\|(u_n,v_n)\|_p^p\\ &\le S_{\alpha,\beta}+o(1) \end{aligned} \tag{3.5}$$

which implies that $\|(u_n,v_n)\|_p^p\to CS_{\alpha,\beta}^{\frac{p}{r-p}}$ and $\int_\Omega(\lambda f(x)u_n^r+\mu g(x)v_n^r)dx\to C'S_{\alpha,\beta}^{\frac{p}{r-p}}$. Define

$$(\xi_n,\eta_n)=\left(\frac{u_n}{(\int_\Omega(\lambda fu_n^r+\mu gv_n^r)dx)^{1/r}},\frac{v_n}{(\int_\Omega(\lambda fu_n^r+\mu gv_n^r)dx)^{1/r}}\right).$$

Clearly,

$$\int_\Omega (\lambda \xi_n^r + \mu \eta_n^r) dx = 1$$

and

$$\int_\Omega (|\nabla \xi_n|^p + |\eta_n|^p dx) \to S_{\alpha,\beta}^{\frac{p}{r-p}\frac{r-1}{r}}, \text{ as } n \to \infty.$$

From the Lemma 3.3, there exists a sequence $\{(y_n, \theta_n)\} \subset \mathbb{N} \times \mathbb{R}^+$ such that $\theta_n \to 0$, $y_n \to y \in \overline{\Omega}$ and $\omega(x) = (\omega_n^1(x), \omega_n^2(x)) = \theta_n^{\frac{N}{r}}(\xi_n(\theta_n x + y_n), \eta_n(\theta_n x + y_n)) \to (\omega_1, \omega_2)$ with $\omega_1, \omega_2 > 0$ in $\mathbb{R}^N$ as $n \to \infty$.
Let $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi(x) = x$ in Ω. Then we guarantee that

$$\begin{aligned}
\tau(u_n, v_n) &= \frac{\int_\Omega \chi(x)(\lambda f u_n^r + \mu g v_n^r) dx}{\int_\Omega (\lambda f u_n^r + \mu g v_n^r) dx} \\
&= \int_\Omega \theta_n^N \chi(\theta_n x + y_n)(\lambda \xi_n^r + \mu \eta_n^r) dx \\
&= \int_\Omega \chi(\theta_n x_n + y_n)(\lambda(\omega_n(x)^1)^r + \mu(\omega_n(x)^2)^r) dx. \qquad (3.6)
\end{aligned}$$

By the Lebesgue dominated convergence theorem we have

$$\int_\Omega \chi(\theta_n x_n + y_n)(\lambda(\omega_n^1)^r + \mu(\omega_n^2)^r) dx \to y \in \overline{\Omega}$$

as $n \to \infty$. this implies that $\tau(x_n, y_n) \to y \in \overline{\Omega}$ as $n \to \infty$, which leads to a contradiction to our assumption. □

The analysis done till now tells us that $\inf_{M_\delta} u_{\alpha,\beta} > 0$ and $\inf_{M_\delta} v_{\alpha,\beta} > 0$, thanks to the Lemma 2.11 and the definition of Ω_δ^-. Note that $M_\delta = \{x \in \Omega : \text{dist}(x, \Omega_\delta^-) \leq \frac{\delta}{2}\}$ which is a compact set. Thus by the Lemma 2.13 and using the idea of Lemma 3.4 of [14], Lemma 3.3 of [9], we can obtain a $\tilde{t}^- > 0$ such that

$$(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y), \tilde{t} \sqrt[p]{\nu} v_\epsilon(x-y)) \in \mathcal{N}_{\alpha,\beta}(c_{\alpha,\beta} - \sigma)$$

uniformly in $y \in \Omega_\delta^-$. Further, by the lemma 3.6, $\tau(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y), \tilde{t} \sqrt[p]{\nu} v_\epsilon(x-y)) \in \Omega_\delta^-$. Thus we can define a map $\gamma : \Omega_\delta^- \to \mathcal{N}_{\alpha,\beta}(c_{\alpha,\beta} - \sigma)^-$ by

$$\gamma(y) = \begin{cases} (\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y), \tilde{t} \sqrt[p]{\nu} v_\epsilon(x-y)), & \text{if } x \in B_\delta(y) \\ 0, & \text{otherwise.} \end{cases}$$

We will denote by $\tau_{\alpha,\beta}$ the restriction of τ over $\mathcal{N}_{\alpha,\beta}^-(c_{\alpha,\beta} - \sigma)$. Observe that v_ϵ is a radial function, therefore for each $y \in \Omega_\delta^-$, we have

$$\begin{aligned}
(\tau_{\alpha,\beta} \circ \gamma)(y) &= \frac{\int_\Omega x(\lambda f(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r + \mu g(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r) dx}{\int_\Omega (\lambda f(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r + \mu g(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r) dx} \\
&= \frac{\int_\Omega (y+z)(\tilde{t}^-)^r \nu^{\frac{r}{p}} (\lambda f + \mu g) v_\epsilon^r dz}{\int_\Omega (\tilde{t}^-)^r \nu^{\frac{r}{p}} (\lambda f + \mu g) v_\epsilon^r dz} \\
&= y.
\end{aligned}$$

From [14], we define the map $H_{\alpha,\beta} : [0,1] \times \mathcal{N}^{-}_{\alpha,\beta}(c_{\alpha,\beta} - \sigma) \to \mathbb{R}^N$ by

$$H_{\alpha,\beta}(t,z) = t\tau_{\alpha,\beta}(z) + (1-t)\tau_{\alpha,\beta}(z).$$

We then have the following lemma.

Lemma 3.7. *To each $\epsilon \in (0, \epsilon_0)$, there exists $\Lambda^* > 0$ such that if*

$$\nu \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right] \in (0, \Lambda^*),$$

we have $H_{\alpha,\beta}([0,1] \times \mathcal{N}^{-}_{\alpha,\beta}(c_{\alpha,\beta} - \sigma)) \subset \Omega^{-}_{\delta}$.

Proof. We prove by contradiction. Let there exists sequences $t_n \in [0,1]$, $\nu_n \to 0$ and $z_n = (u_n, v_n) \in \mathcal{N}^{-}_{\alpha,\beta}(c_{\alpha,\beta} - \sigma)$ such that $H_{\alpha,\beta}(t_n, z_n) \notin \Omega^{+}_{\delta}$ for all n. We can assume that $t_n \to t \in [0,1]$. Thus by Lemma 2.11 (*ii*) and similar argument in the proof of 3.6, we have

$$H_{\alpha,\beta}(t_n, z_n) \to y \in \overline{\Omega} \text{ as } n \to \infty$$

which leads to a contradiction. □

We now prove the main result of this article which roughly states that under certain assumptions on ν the problem in (1.1) admits at least $\text{cat}(\Omega) + 1$ number of solutions.

Lemma 3.8. *If (u,v) is a critical point of $I_{\alpha,\beta}$ on $\mathcal{N}^{-}_{\alpha,\beta}$, then it is also a critical point of $I_{\alpha,\beta}$ in X.*

Proof. We follow the proof of Lemma 4.1 in [14] or of Lemma 4.1 in [35]. Let (u,v) be a crtical point of $I_{\alpha,\beta}$ in $\mathcal{N}^{-}_{\alpha,\beta}$. Then

$$\langle I'_{\alpha,\beta}(u,v), (u,v) \rangle = 0.$$

Define

$$\begin{aligned}
\psi_{\alpha,\beta}(u,v) &= \langle I'_{\alpha,\beta}(u,v), (u,v) \rangle \\
&= \|(u,v)\|_p^p + \|(u,v)\|_q^q \\
&\quad - \int_{\Omega} (\lambda f(x) u^r + \mu g(x) v^r) dx \\
&\quad - \nu \int_{\Omega} h(x) u^{1-\alpha} v^{1-\beta} dx.
\end{aligned}$$

Since we are now looking for minimizing $I_{\alpha,\beta}$ over the entire space X, to which the Lagrange multiplier method comes to our rescue in finding a $\theta (\neq 0) \in \mathbb{R}$ such that

$$I'_{\alpha,\beta}(u,v) = \theta \psi'(u,v) \tag{3.7}$$

where

$$\psi_{\alpha,\beta}(u,v) \quad = \quad \langle I'_{\alpha,\beta}(u,v), (u,v) \rangle.$$

Since, $(u,v) \in \mathcal{N}^{-}_{\alpha,\beta}$, we have from a simple computation that $\psi'_{\alpha,\beta}(u,v) < 0$. Consequently from (3.7) we have $I'_{\alpha,\beta}(u,v) = 0$. □

Lemma 3.9. *There exists* $\Lambda^* > 0$ *such that any sequence* $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\alpha,\beta}$ *with* $I_{\mathcal{N}^-_{\alpha,\beta}}(u_n, v_n) \to c \in (-\infty, c_{\alpha,\beta})$ *and* $I'_{\mathcal{N}^-_{\alpha,\beta}}(u_n, v_n) \to 0$ *contains a convergent subsequence for all* $0 < \nu \left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} \right] < \Lambda^*$.

Proof. From the Lagrange's multiplier method, there exists a sequence $(a_n) \subset \mathbb{R}$ such that

$$\|I'_{\alpha,\beta}(u_n, v_n) - a_n \psi'_{\alpha,\beta}(u_n, v_n)\|_{X'} \to 0$$

as $n \to \infty$. Here

$$\begin{aligned} \psi_{\alpha,\beta}(u_n, v_n) &= \langle I'_{\alpha,\beta}(u_n, v_n), (u_n, v_n) \rangle \\ &= \|(u_n, v_n)\|_p^p + \|(u_n, v_n)\|_q^q - \int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r) dx \\ &\quad -\nu \int_\Omega h(x) u_n^{1-\alpha} v_n^{1-\beta} dx. \end{aligned}$$

Then

$$I'_{\alpha,\beta}(u_n, v_n) = a_n \psi'_{\alpha,\beta}(u_n, v_n) + o(1).$$

Since $(u_n, v_n) \in \mathcal{N}^-_{\alpha,\beta} \subset \mathcal{N}_{\alpha,\beta}$, by a simple computation we have

$$\langle \psi_{\alpha,\beta}(u_n, v_n), (u_n, v_n) \rangle < 0.$$

Now suppose $\langle \psi'_{\alpha,\beta}(u_n, v_n), (u_n, v_n) \rangle \to 0$, then we have

$$\begin{aligned} &(r-p)\|(u_n, v_n)\|_p^p + (r-q)\|(u_n, v_n)\|_q^q \\ &= \nu(1+\alpha+\beta) \int_\Omega h(x) u_n^{1-\alpha} v_n^{1-\beta} dx + o(1) \\ &\leq \nu(1+\alpha+\beta) M \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right. \\ &\left. + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right]^{\frac{p+\alpha+\beta-2}{p}} \|(u_n, v_n)\|_p^{2-\alpha-\beta} + o(1) \end{aligned}$$

and

$$\begin{aligned} &(p+\alpha+\beta-2)\|(u_n, v_n)\|_p^p + (q+\alpha+\beta-2)\|(u_n, v_n)\|_q^q \\ &= (r+\alpha+\beta-2) \int_\Omega (\lambda f(x) u_n^r + \beta g(x) v_n^r) dx + o(1) \leq M' \|(u_n, v_n)\|_p^{p^*} + o(1) \end{aligned}$$

where we have used the Hölder inequlaity and the Sobolev embedding. Then we have

$$\|(u_n, v_n)\|_p \leq (\nu C_1)^{\frac{1}{p}} \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right]^{\frac{1}{p}} + o(1)$$

and

$$\|(u_n, v_n)\|_p \geq C_2^{\frac{1}{p^*-p}} + o(1).$$

Now if we choose Λ^* small enough, this cannot hold. Therefore let us assume that $\langle \psi_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle \to l < 0$, as $n\to\infty$. since $\langle I_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle = 0$, we conlcude that $a_n \to 0$ and therefore $I'_{\alpha,\beta}(u_n,v_n)\to 0$. This gives us that

$$I_{\alpha,\beta}(u_n,v_n) = c < c_{\alpha,\beta} \text{ and } I'_{\alpha,\beta}(u_n,v_n)\to 0.$$

Therefore by the Lemma 2.10 the proof is complete. □

Lemma 3.10. *Suppose that (C) holds and*

$$\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right] \in (0,\Lambda^*).$$

Then $cat(\mathcal{N}^-_{\lambda,\mu}(c_{\lambda,\mu}-\sigma)) \geq cat(\Omega)$.

Proof. Let $\text{cat}(\mathcal{N}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma)) = n$. Then, by the definition 3.1 of the category of a set in the sense of Lusternik-Schnirelman, we suppose that

$$\mathcal{N}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma) = A_1\cup A_2\cup ...\cup A_n$$

where A_j, $j=1,2,...,n$ are closed and contractible in $\mathcal{N}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma)$, i.e., there exists $h_j \in C([0,1]\times A_j, \mathcal{N}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma))$ such that

$$h_j(0,z) = z,\ h_j(1,z) = \Theta, \text{ for all } z\in A_j,$$

where $\Theta \in A_j$ is fixed. Consider $B_j = \gamma^{-1}(A_j)$, $j=1,2,...,n$. Then the sets B_j are closed

$$\Omega^-_\delta = B_1\cup B_2\cup ...\cup B_n.$$

We now define the deformation $g_j : [0,1]\times B_j \to \Omega^+_\delta$ by setting

$$g_j(t,y) = H_{\alpha,\beta}(t, h_j(t,\gamma(y))).$$

for $\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p+\alpha+\beta-2}{p}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p+\alpha+\beta-2}{p}}\right] \in (0,\Lambda^*)$. Notice that

$$g_j(0,y) = H_{\alpha,\beta}(0,h_j(0,\gamma(y))) = (\tau_{\alpha,\beta}\circ\gamma)(y) = y, \text{ for all } y\in B_j$$

and

$$g_j(1,y) = H_{\alpha,\beta}(0,h_j(1,\gamma(y))) = \tau_{\alpha,\beta}(\Theta)\in\Omega^+_\delta, \text{ for all } y\in B_j.$$

Thus the sets B_j, $j=1,2,...,n$ are contractible in Ω^+_δ.
Therefore $\text{cat}(\mathcal{N}^-_{\alpha,\beta}-\sigma) \geq \text{cat}_{\Omega^+_\delta}(\Omega^-_\delta) = \text{cat}(\Omega)$. □

Proof of Theorem 1.1. By Lemmas 2.10 and 3.9, the functional $I_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for $c\in(-\infty, c_{\alpha,\beta})$. Then, by Lemma 3.2 and 3.10, we have $I_{\alpha,\beta}$ has at least $\text{cat}(\Omega)$ number of critical points in $\mathcal{N}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma)$. By Lemma 3.8, we have $I_{\alpha,\beta}$ has at least $\text{cat}(\Omega)$ number of critical points in $\mathcal{N}^-_{\alpha,\beta}$. Further, since $\mathcal{N}^+_{\alpha,\beta}\cap\mathcal{N}^-_{\alpha,\beta} = \phi$, the proof is now complete.

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References

[1] C.O. Alves, D.C. de Morais Filno, M.A. Souto, *On systems of elliptic equations involving subcritical or critical Sobolev exponents*, Nonlinear Anal., **42**(2000), 771-787.

[2] C.O. Alves, J.M. do Ó, O.H. Miyagaki, *On perturbations of a class of periodic m-laplacian equations with critical growth*, Nonlinear Anal., **45**(2001), 849-863.

[3] A. Ambrosetti, H. Brezis, G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal., **122**(1994), no. 3, 519-543.

[4] J.G. Azvrero, I.P. Aloson, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term,* Trans. Amer. Math. Soc., **323**(1992), 977-895.

[5] V. Benci, G. Cerami, *The effects of the domain topology on the number of positive solutions of nonlinear elliptic problems,* Arch. Ration. Mech. Anal., **114**(1991), 79-93.

[6] V. Benci, A.M. Micheletti, D. Visetti, *An eigenvalue problem for a quasilinear elliptic field equation,* J. Differential Equations, **184**(2002), no. 2, 299-320.

[7] S. Benmouloud, R. Echarghaoni, S.M. Sbaï, *Multiplicity of positive solutions for a critical quasilinear elliptic system with concave and convex noninearities,* J. Math. Anal. Appl., **396**(2012), 375-385.

[8] H. Brézis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent,* Comm. Pure Appl. Math., **36**(1983), 437-477.

[9] C.Y. Chen, T.F. Wu, *The Nehari manifold for indefinite semilinear elliptic systems involving critical exponent,* Appl. Math. Comput., **218**(2012), 10817-10828.

[10] D. Choudhuri, A. Soni, *Existence of multiple solutions to a partial differential equation involving the fractional p-Laplacian*, J. Anal., **23**(2015), 33-46.

[11] L. Ding, S. Xiao, *Multiple positive solutions for a critical quasilinear elliptic systems,* Nonlinear Anal., **72**(2010), 2592-2607.

[12] P. Drabek, Y. Huang, *Multiplicity of positive solutions for some quaslinear elliptic equation in $\mathbb{R}^N$ with critical Sobolev exponent,* J. Differential Equations, **110**(1997), 106-132.

[13] L.C. Evans, *Partial Differential Equations,* Graduate Studies in Mathematics, AMS, **19**, 1997.

[14] H. Fan, *Multiple positive solutions for a critical elliptic system with concave and convex nonlinearities,* Nonlinear Anal. Real World Appl., **18**(2014), 14-22.

[15] T.S. Hsu, *Multiple positive solutions for a critical quasilinear elliptic system with concave convex nonlinearities,* Nonlinear Anal., **71**(2009), 2688-2698.

[16] G. Li, *The existence of nontrivial solution to the $p-q$ Laplacian problem with nonlinearity asymptotic to u^{p-1} at infinity in $\mathbb{R}^N$,* Nonlinear Anal., **68**(2008), 1100-1119.

[17] G. Li, X. Liang, *The existence of nontrivial solutions to nonlinear elliptic equation of $p-q$-Laplacian type on $\mathbb{R}^N$,* Nonlinear Anal., **71**(2009), 2316-2334.

[18] Q. Li, Z. Yang, *Multiplicity of positive solutions for a $p-q$-Laplacian system withconcave and critical nonlinearities,* J. Math. Anal. Appl., **423**(2015), 660-680.

[19] Q. Li, Z. Yang, *Multiple positive solutions for quasilinear elliptic systems,* Electron. J. Differential Equations, **15**(2013), 1-14.

[20] Q. Li, Z.D. Yang, *Multiple positive solutions for quasilinear elliptic systems with critical exponent and sign-changing weight,* Comput. Math. Appl., **67**(2014), 1848-1863.

[21] Q. Li, Z.D. Yang, *Multiple positive solutions for $p-q$-Laplacian problems with critical exponent,* Acta Math. Sci., **29**(2009), 903-918.

[22] W. Liu, G. Dai, N.S. Papageorgiou, P. Winkert, *Existence of solutions for singular double phase problems via the Nehari manifold method*, arXiv:2101.00593.

[23] N.S. Papageorgiou, D.D. Repovš, C. Vetro, *Positive solutions for singular double phase problems,* J. Math. Anal. Appl., (2020), doi.org/10.1016/j.jmaa.2020.123896.

[24] N.S. Papageorgiou, P. Winkert, *Positive solutions for weighted singular p-Laplace equations via Nehari manifolds*, Appl. Anal., https://doi.org/10.1080/00036811.2019.1688791.

[25] P.H. Rabinowitz, Q. Li, Z.D. Yang, *Methods in Critical Point Theory with Applications to Differential Equations,* CBMS Res. Conf. Ser. Math., **65**, Amer. Math. Soc., Providence, RI, 1986.

[26] O. Rey, *A multiplicity results for a variationalproblem with lack of compactness,* Nonlinear Anal., **13**(1989), 1241-1249.
[27] K. Saoudi, *A singular system involving the fractional p-Laplacian operator via the Nehari manifold approach,* Complex Anal. Oper. Theory, **13**(2019), 801-818.
[28] K. Saoudi, S. Ghosh, D. Choudhuri, *Multiplicity and Hölder regularity of solutions for a nonlocal elliptic PDE involving singularity,* J. Math. Phys., **60**(2019), 1-28.
[29] N.E. Sidiripoulos, *Existence of solutions to indefinite quasilinear elliptic problems of $p-q$-Laplacian type,* Electron. J. Differential Equations, **162**(2010), 1-23.
[30] A. Szulkin, T. Weth, *The Method of Nehari Manifold, Handbook of Nonconvex Analysis and Applications*, Int. Press, Somerville, MA, 2010, 597-632.
[31] W. Willem, *Minimax Theorems,* Birkhäuser, Boston, 1996.
[32] M.Z. Wu, Z.D. Yang, *A classof $p-q$-Laplacian system with critical nonlinearities,* Bound. Value Probl., Art. ID 185319, (2009), 1-19.
[33] H.H. Yin, *Existence of multiple positive solutions for a $p-q$-Laplacian system with critical nonlinearities,* J. Math. Anal. Appl., **403**(2013), 200-214.
[34] H.H. Yin, Z.D. Yang, *Multiplicity results for a class of concave-convex elliptic systems involving sign-changing weight functions,* Ann. Polon. Math., **102**(2011), 51-71.
[35] H.H. Yin, Z.D. Yang, *Multiplicity of positive solutions to a $p-q$-laplacian equation involving critical nonlinearity,* Nonlinear Anal., **75**(2012), 3021-3035.

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