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# REMARKS ON THE TERMINOLOGY OF THE MAPPINGS IN FIXED POINT ITERATIVE METHODS IN METRIC SPACES

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**Abstract.** In this paper we present some suggestions for unifying the terminology of the mappings appearing in fixed point iterative methods for the case when the setting is a metric space. We consider the following concepts: contraction type mapping, contractive type mapping and nonexpansive type mapping, for which some problems are formulated.
**Key Words and Phrases**: Metric spaces, contraction mapping, contractive mapping, nonexpansive mapping, quasicontraction, quasicontractive, quasinonexpansive, cyclic contraction, cyclic contractive mapping, cyclic nonexpansive mapping, graphic contraction, graphic contractive mapping, graphic nonexpansive mapping, Caristi mapping, Caristi-Browder mapping, Caristi-Kirk mapping, weakly Picard mapping, Picard mapping, well-posedness of fixed point problem, Ostrowski property, data dependence, open problem.
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## 1. Introduction

In the metric fixed point theory, very often, some basic conditions on mappings appear under several names (see [49], [38], [29], [39], [80], [83], [66], [67], [73], [9], [21], [27], [37], [43], [2], [31], [78], [14], [15], [63], [60], ...)

Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. We denote by $F_T$ the fixed point set of $T$, i.e., $F_T := \{x \in X : x = T(x)\}$. We start by presenting the following four examples:

(1) for the condition: there exists $0 < l < 1$ such that

$$d(T(x), T(y)) \leq ld(x, y), \text{ for all } x, y \in X,$$

we meet the following names: *contraction*, *strong contraction*, *strict contraction*, *Banach contraction*, *Banach-Caccioppoli contraction*, *Picard-Banach-Caccioppoli contraction*, *contractive*, *strongly nonexpansive*,...

(2) for the condition:

$$d(T(x), T(y)) \leq d(x, y), \text{ for all } x, y \in X,$$

we meet the following names: *nonexpansive*, *contraction*, *mapping which do not increase distance*,...

(3) for the condition: there exists $0 < l < 1$ such that

$$d(T^2(x), T(x)) \leq ld(x, T(x)), \text{ for all } x \in X,$$

we meet the following names: *graphic contraction*, *iterative contraction*, *weakly contraction*, *Banach mapping*,...

(4) for the condition: $F_T \neq \emptyset$ and there exists $c > 0$ such that

$$d(x, F_T) \leq cd(x, T(x)), \text{ for all } x \in X$$

we can find the following names: *condition I*, *condition II*, *condition D*, *$F_T$-displacement condition*,...

On the other hand, the ambient space $X$ that appears in fixed point theorems has various structures:

- complete metric space
- compact metric space
- closed subset of a Banach space
- closed convex and bounded subset of a Hilbert space
- compact convex of a Banach space
- ordered metric space
- metric space with graph
- metric space with convex structure
- ...

So, the problem must be considered in the following three cases:

- $X$ is a metric space;
- $X$ is a Hilbert space;
- $X$ is a Banach space.

In this paper, we consider the first case, i.e., the case of a metric space $X$, and we cover the following aspects/sections:

2. Orbits and sequences of successive approximations
3. Metric conditions on $Y \times Y$ for $T : Y \to X$ with $Y \subset X$
4. Metric conditions on $Y \times F_T$ for $T : Y \to X$ with $F_T \neq \emptyset$: quasi-conditions
5. Metric conditions on $G \subset X \times X$ for $T : X \to X$: G-conditions (graphic conditions, cyclic conditions,...)
6. Conditions in terms of the displacement (Caristi condition, retraction-displacement condition, $F_T$-displacement condition...)
7. The case of multivalued mappings
8. Problems
   8.1 Mappings of contraction type

## 2. Orbits and sequences of successive approximations

Let $(X,d)$ be a metric space and $T : X \to X$ be a mapping. In terms of iterates of $T$, we have the following notions and notations (see [7], [9], [10], [14], [15], [73], [71], [78], ...):

- $F_T := \{x \in X \mid T(x) = x\}$, the set of fixed points of $T$;
- $O_T(x) := \{T^n(x) \mid n \in \mathbb{N} := 0, 1, \ldots\}$, the orbit of $T$ at $x \in X$;
- $(T^n(x))_{n\in\mathbb{N}}$, the sequence of successive approximations of $T$ at $x \in X$;
- $\omega_T(x) :=$ the cluster point set of $(f^n(x))_{n\in\mathbb{N}}$, i.e., the set $\{y \in X \mid \exists\, n_k \to \infty$ such that $f^{n_k}(x) \to y$ as $n_k \to \infty\}$;
- $T$ is weakly Picard mapping (WPM) iff for each $x \in X$, $f^n(x) \to x^*(x) \in F_T$ as $n \to \infty$;
- if $T$ is WPM with $F_T = \{x^*\}$, then $T$ is by definition, Picard mapping (PM); by $F_T = \{x^*\}$ we understand that $T$ has a unique fixed point and we denote it by $x^*$;
- $T$ is pre-WPM if for each $x \in X$, $(f^n(x))_{n\in\mathbb{N}}$ is convergent;
- $T$ is pre-PM if there exists $y \in X$ such that $f^n(x) \to y$ as $n \to \infty$, for all $x \in X$;
- if $T$ is pre-WPM, then we define the operator $T^\infty : X \to X$ by
$$T^\infty(x) := \lim_{n\to\infty} T^n(x);$$
- $T$ is asymptotically regular iff for each $x \in X$, $d(T^{n+1}(x), T^n(x)) \to 0$ as $n \to \infty$;
- $T$ has diminishing orbital diameters on $X$ iff for each $x \in X$ with $0 < \delta(O_T(x)) < +\infty$ we have that $\lim\limits_{n\to\infty} \delta(O_T(T^n(x)) < \delta(O_T(x))$.

## 3. Metric conditions on $Y \times Y$ for $T : Y \to X$ with $Y \subset X$

Let $(X,d)$ be a metric space, $Y \subset X$ and $T : Y \to X$ a mapping.

We name the following metric conditions on $Y \times Y$ for $T$ as follows (see [73], [67], [38], [66], [74], ...):

(1) *contraction mapping*: there exists $0 < l < 1$ such that
$$d(T(x), T(y)) \le l d(x,y), \text{ for all } x, y \in Y;$$
(2) *contractive mapping*:
$$d(T(x), T(y)) < d(x,y), \text{ for all } x, y \in Y, x \ne y;$$
(3) *nonexpansive mapping*:
$$d(T(x), T(y)) \le d(x,y), \text{ for all } x, y \in Y;$$

(4) *$\varphi$-contraction mapping*: the function $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is increasing such that $\psi : \mathbb{R}_+ \to \mathbb{R}_+,\ \psi(t) = \varphi(t,t,t,t,t)$ is a comparison function and

$$d(T(x),T(y)) \leq \varphi(d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(y,T(x))),$$

for all $x, y \in Y$.

Some examples of $\varphi$-contractions (see [9], [34], [49], [63], [64], [66], [73], ...) are:

- *Kannan mappings*: there exist $l_1, l_2 \in \mathbb{R}_+, l_1 + l_2 < 1$ such that

$$d(T(x),T(y)) \leq l_1 d(x,T(x)) + l_2 d(y,T(y)), \text{ for all } x, y \in Y.$$

- *Ćirić-Reich-Rus mappings*: there exist $l_1, l_2, l_3 \in \mathbb{R}_+, l_1 + l_2 + l_3 < 1$ such that

$$d(T(x),T(y)) \leq l_1 d(x,y) + l_2 d(x,T(x)) + l_3 d(y,T(y)), \text{ for all } x, y \in Y.$$

- *Hardy-Rogers mappings*: there exist $l_1, l_2, l_3, l_4, l_5 \in \mathbb{R}_+,\ l_1+l_2+l_3+l_4+l_5 < 1$ such that

$$d(T(x),T(y)) \leq l_1 d(x,y) + l_2 d(x,T(x)) + l_3 d(y,T(y)) + l_4 d(x,T(y)) + l_5 d(y,T(x)),$$

for all $x, y \in Y$.

- *Ćirić mappings*: there exists $l \in ]0,1[$ s.t.

$$d(T(x),T(y)) \leq l \max\{d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(y,T(x))\},$$

for all $x, y \in Y$.

(5) *$\varphi-$contractive mappings*: the function $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is increasing such that $\varphi(t,t,t,t,t) \leq t,\ t \in \mathbb{R}_+$ and

$$d(T(x),T(y)) < \varphi(d(x,y), d(x,f(x)), d(y,f(y)), d(x,T(y)), d(y,T(x)))),$$

for all $x, y \in Y, x \neq y$.

Some examples of $\varphi$-contractive mappings (see [9], [27], [39], [65], [68], [73], [17], [29], [40], [63], [64], [76], [79], ...) are:

- *Kannan contractive mappings*: there exist $l_1, l_2 \in \mathbb{R}_+, l_1 + l_2 = 1$ such that

$$d(T(x),T(y)) < l_1 d(x,T(x)) + l_2 d(y,T(y)), \text{ for all } x, y \in Y, x \neq y.$$

- *Ćirić-Reich-Rus contractive mappings*: there exist $l_1, l_2, l_3 \in \mathbb{R}_+, l_1+l_2+l_3 = 1$ such that

$$d(T(x),T(y)) < l_1 d(x,y) + l_2 d(x,T(x)) + l_3 d(y,T(y)), \text{ for all } x, y \in Y, x \neq y.$$

- *Hardy-Rogers contractive mappings*: there exist $l_1, l_2, l_3, l_4, l_5 \in \mathbb{R}_+,\ l_1 + l_2 + l_3 + l_4 + l_5 = 1$ such that

$$d(T(x),T(y)) < l_1 d(x,y) + l_2 d(x,T(x)) + l_3 d(y,T(y)) + l_4 d(x,T(y)) + l_5 d(y,T(x)),$$

for all $x, y \in Y, x \neq y$.

- *Ćirić contractive mappings*:

$$d(T(x),T(y)) < \max\{d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(y,T(x))\},$$

for all $x, y \in Y, x \neq y$.

(6) *$\varphi$-nonexpansive mappings*: the function $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is increasing such that $\varphi(t,t,t,t,t) \le t,\ t \in \mathbb{R}_+$ and

$$d(T(x),T(y)) \le \varphi(d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(Y,T(x))),$$

for all $x, y \in Y$.

Some examples of $\varphi$-nonexpansive mappings (see [49], [63], [65], [67], [73], [76], [64], [78], [79], ...) are:

- *Kannan nonexpansive mappings*: there exist $l_1, l_2 \in \mathbb{R}_+, l_1 + l_2 \le 1$ such that

$$d(T(x),T(y)) \le l_1 d(x,T(x)) + l_2 d(y,T(y)), \text{ for all } x,y \in Y.$$

- *Ćirić-Reich-Rus nonexpansive mappings*: there exist $l_1, l_2, l_3 \in \mathbb{R}_+$, $l_1 + l_2 + l_3 \le 1$ such that

$$d(T(x),T(y)) \le l_1 d(x,y) + l_2 d(x,T(x)) + l_3 d(y,T(y)), \text{ for all } x,y \in Y.$$

- *Hardy-Rogers nonexpansive mappings*: there exist $l_1, l_2, l_3, l_4, l_5 \in \mathbb{R}_+, l_1 + l_2 + l_3 + l_4 + l_5 \le 1$ such that

$$d(T(x),T(y)) \le l_1 d(x,y) + l_2 d(x,T(x)) + l_3 d(y,T(y)) + l_4 d(x,T(y)) + l_5 d(y,T(x)),$$

for all $x, y \in Y$.

- *Ćirić nonexpansive mappings*:

$$d(T(x),T(y)) \le \max(d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(y,T(x))),$$

for all $x, y \in Y$.

## 4. Metric conditions on $Y \times F_T$ for $T : Y \to X$ with $F_T \ne \emptyset$: quasi-conditions

$(X,d)$ a metric space, $Y \subset X$ and $T : Y \to X$ with $F_T \ne \emptyset$

In what follows we present conditions on $Y \times F_T$ as a restriction of conditions on $Y \times Y$ to $Y \times F_T$ (see [18], [49], [60], [69], [70], [71], [79], [9], [12], ...):

(1) *quasicontraction mapping*: there exists $0 < l < 1$ such that

$$d(T(x),x^*) \le l d(x,x^*), \text{ for all } x \in Y,\ x^* \in F_T \ne \emptyset;$$

(2) *quasicontractive mapping*:

$$d(T(x),x^*) < d(x,x^*), \text{ for all } x \in X,\ x^* \in F_T \ne \emptyset,\ x \ne x^*;$$

(3) *quasinonexpansive mapping*:

$$d(T(x),x^*) \le d(x,x^*), \text{ for all } x \in X,\ x^* \in F_T \ne \emptyset;$$

(4) *quasi-$\varphi$-contraction mapping*: the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and

$$d(T(x),x^*) \le \varphi(d(x,x^*)), \text{ for all } x \in X,\ x^* \in F_T \ne \emptyset;$$

(5) *quasi-$\varphi$-contractive mapping*: the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing such that $\varphi(t) \le t$, $\forall\ t \in \mathbb{R}_+$ and

$$d(T(x),x^*) < \varphi(d(x,x^*)), \text{ for all } x \in X,\ x^* \in F_T \ne \emptyset,\ x \ne x^*;$$

(6) *quasi-$\varphi$-nonexpansive mapping*: the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing such that $\varphi(t) \le t$, $\forall\, t \in \mathbb{R}_+$ and

$$d(T(x), x^*) \le \varphi(d(x, x^*)),\ \forall\, x \in X,\ x^* \in F_T \neq \emptyset.$$

In general if $C$ is a condition on $Y \times Y$, then quasi $C$ is the restriction of $C$ to $Y \times F_T$ with $F_T \neq \emptyset$.

By a conditional quasi $C$ we understand that: if $F_T \neq \emptyset$, then $T$ is quasi $C$. For example, $T$ is conditional quasinonexpansive if $F_T \neq \emptyset$ implies that $T$ is quasinonexpansive.

## 5. Metric conditions on $G \subset X \times X$ for $T : X \to X$: G-conditions (graphic conditions, cyclic conditions,...)

Let $(X, d)$ be a metric space, $T : X \to X$ a mapping and $G \subset X \times X$ a nonempty subset. If $(C)$ is a condition on $T$, on $X \times X$, then by condition G-C we understand the condition which is the restriction of condition $(C)$ to $G$.
For various examples (see also [4], [54], [52], [56], [58], [69], [70], [72], [73], ...):

$(A_1)$ *G-contraction mapping*: there exists $0 < l < 1$ such that

$$d(T(x), T(y)) \le l d(x, y),\ \text{for all } (x, y) \in G;$$

$(A_2)$ *G-contractive mapping*:

$$d(T(x), T(y)) < d(x, y),\ \text{for all } (x, y) \in G,\ x \neq y;$$

$(A_3)$ *G-nonexpansive mapping*:

$$d(T(x), T(y)) \le d(x, y),\ \text{for all } (x, y) \in G.$$

If we take $G := \{(x, T(x)) \mid x \in X\}$, then we have the following conditions (see [56], [57], [59], [74], ...):

$(B_1)$ *graphic contraction mapping*: there exists $0 < l < 1$ such that

$$d(T(x), T^2(x)) \le l d(x, T(x)),\ \text{for all } x \in X;$$

$(B_2)$ *graphic contractive mapping*:

$$d(T(x), T^2(x)) < d(x, T(x)),\ \text{for all } x \in X,\ x \neq T(x);$$

$(B_3)$ *graphic nonexpansive mapping*:

$$d(T(x), T^2(x)) \le d(x, T(x)),\ \text{for all } x \in X.$$

Let $(X, d)$ be a metric space,

$$X = \bigcup_{i=1}^{m} X_i,\ m \ge 2,$$

be a covering of $X$ with $X_i \neq \emptyset$, $i = \overline{1, m}$. Let $T : X \to X$ be such that:

$$T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m,\ T(X_m) \subset X_1.$$

We call a such mapping, a cyclic mapping with respect to this covering.

Let

$$G := \bigcup_{i=1}^{m} (X_i \times X_{i+1}), \text{ where } X_{m+1} = X_1.$$

By definition a G-condition on $T$ is a cyclic condition on $T$.

We present now some examples of cyclic conditions for $T$ a cyclic mapping (see [11], [35], [38], [49], [67], [70], [73], [79], [82], ...):

$(C_1)$ *cyclic contraction mapping*: there exists $0 < l < 1$ such that

$$d(T(x), T(y)) \leq l d(x, y), \text{ for all } x \in X_i,\ y \in X_{i+1},\ i = \overline{1, m};$$

$(C_2)$ *cyclic contractive mapping*:

$$d(T(x), T(y)) < d(x, y), \text{ for all } x \in X_i,\ y \in X_{i+1},\ x \neq y,\ i = \overline{1, m};$$

$(C_3)$ *cyclic nonexpansive mapping*:

$$d(T(x), T(y)) \leq d(x, y), \text{ for all } x \in X_i,\ y \in X_{i+1},\ i = \overline{1, m};$$

..........

## 6. Conditions in terms of the displacement (Caristi condition, retraction-displacement condition, and $F_T$-displacement condition,...)

Let $(X, d)$ be a metric space, $Y \subset X$ a nonempty subset and $T : Y \to X$ a mapping. In terms of displacement of $T$, functional of $T$, $x \mapsto d(x, T(x))$, we have the following conditions:

$(A_1)$ *Caristi mapping*: there exists $\varphi : X \to \mathbb{R}_+$ such that

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \text{ for all } x \in X;$$

$(A_2)$ *Caristi-Kirk mapping*: $T$ is Caristi mapping with $\varphi$ a lower semicontinuous mapping;

$(A_3)$ *Caristi-Browder mapping*: $T$ is Caristi mapping and $T$ is orbitally continuous.

Let $T : Y \to X$ with $F_T \neq \emptyset$. In this case we have:

$(B_1)$ *$F_T$-displacement condition*: there exists $c > 0$ such that

$$d(x, F_T) \leq c d(x, T(x)), \text{ for all } x \in X;$$

$(B_2)$ *retraction-displacement condition*: there exists a retraction $r : X \to F_T$ (i.e., $r\Big|_{F_T} = 1_{F_T}$) such that

$$d(x, r(x)) \leq c d(x, T(x)), \text{ for all } x \in X;$$

$(B_3)$ *c-WPM*: $T$ is a WPM and there exists $c > 0$ such that

$$d(x, T^{\infty}(x)) \leq c d(x, T(x)), \text{ for all } x \in X.$$

## 7. Problems

7.1. **Mappings of contraction type.** In the metrical fixed point theory we meet some variants of the contraction principle (see [49], [73], [9], [50], [38], [45], [48], [84], [70], ...), like the following ones:

($V_1$) (S. Banach – R. Caccioppoli). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction. Then:

(i) $T$ has a unique fixed point in $X$, i.e., $F_T = \{x^*\}$.

(ii) For each $x \in X$, $T^n(x) \to x^*$ as $n \to \infty$.

($V_2$) (R.S. Palais). Let $(X, d)$ be a metric space and $T : X \to X$ be an $l$-contraction. Then:

(i) $d(x, y) \leq \dfrac{1}{1-l}[d(x, T(x)) + d(y, T(y))],\ \forall\ x, y \in X.$

(ii) $T$ has at most one fixed point.

(iii) For any $x$ in $X$, the sequence $T^n(x)$ of iterates of $x$ under $T$ is a Cauchy sequence.

(iv) If $(X, d)$ is a complete metric space, then $T$ has a unique fixed point $p$, and for any $x$ in $X$ the sequence $T^n(x)$ converges to $p$. In fact,

$$d(T^n(x), p) \leq \frac{l^n}{1-l} d(x, T(x)),\ n \in \mathbb{N}.$$

($V_3$) (W.A. Kirk). Let $(X, d)$ be a complete metric space and $T : X \to X$ be an $l$-contraction. Then:

(i) $T$ has a unique fixed point, say $p \in X$.

(ii) For each $x \in X$ the Picard sequence $\{T^n(x)\}$ converges to $p$.

(iii) The convergence is uniform if $X$ is bounded.

(iv) $d(T^{n+1}(x), p) \leq l d(T^n(x), p)$.

($V_4$) (P.R. Meyers). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction. Then there exist a unique point $x^* \in X$ and a neighborhood $U$ of $x^*$ such that:

(i) $T(x^*) = x^*$.

(ii) For each $x \in X$, $T^n(x) \to x^*$ as $n \to \infty$.

(iii) $T^n(U) \to x^*$ as $n \to \infty$.

($V_5$) (V.I. Opoitsev). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction. Then we have the conclusions (i), (ii) and (iii) in ($V_4$) and the following ones:

(iv) The mapping $T$ is continuous.

(v) Stability of the fixed point $x^*$, i.e., for any neighborhood $V$ of $x^*$ we can find a neighborhood $W$ of $x^*$ such that from $x \in W$ it follows that $T^n(x) \in V$, for all $n \in \mathbb{N}$.

($V_6$) (I.A. Rus). Let $(X, d)$ be a complete metric space and $T : X \to X$ an $l$-contraction. Then:

(i) There exists $x^* \in X$ such that $F_T = F_{T^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$.

(ii) For all $x \in X$, $T^n(x) \to x^*$ as $n \to \infty$.

(iii) $d(x, x^*) \leq \psi(d(x, T(x)))$, $\forall\ x \in X$, where $\psi(t) = \dfrac{t}{1-l}$, $t \geq 0$.

(iv) If $(y_n)_{n\in\mathbb{N}}$ is a sequence in $X$ such that

$$d(y_n, T(y_n)) \to 0 \text{ as } n \to \infty,$$

then $y_n \to x^*$ as $n \to \infty$.

(v) If $(y_n)_{n\in\mathbb{N}}$ is a sequence in $X$ such that

$$d(y_{n+1}, T(y_n)) \to 0 \text{ as } n \to \infty,$$

then $y_n \to x^*$ as $n \to \infty$.

(vi) If $Y \subset X$ is a closed subset such that $T(Y) \subset Y$, then $x^* \in Y$. Moreover, if in addition $Y$ is bounded then:

(a) $(T^n(x))_{n\in\mathbb{N}}$ converges uniformly to $x^*$ on $Y$;

(b) $\bigcap_{n\in\mathbb{N}} T^n(Y) = \{x^*\}$.

Here are some remarks and commentaries, from [70], on the variant $(V_6)$:

"Conclusion (i) is a set-theoretical one. If $X$ is a nonempty set and $T : X \to X$ is a mapping such that $F_{T^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$, then by definition we call $T$ a *Bessaga mapping.*

Conclusion (ii) is a topological one. All Picard iterations converge to the unique fixed point of the mapping. If $(X, \to)$ is an $L$-space and $T : X \to X$ is a mapping such that we have (i) and (ii), then by definition $T$ is a Picard mapping.

Conclusion (iii) is a metrical one and is very important in the theory of fixed point equations. We obtain from this estimate, for example, a data dependence of the fixed point under mapping perturbation.

If in a metric space a mapping $T : X \to X$ satisfies (i), (ii) and (iii), then, by definition, the mapping $T$ is a $\psi$-Picard mapping and the estimation in (iii) is called retraction-displacement estimation. In this definition, $\psi$ is a function, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, increasing and continuous in 0 with $\psi(0) = 0$.

If in a metric space $(X, d)$ a mapping $T : X \to X$ satisfies (i) and (iv) then, by definition, the fixed point problem for $f$ is well posed. We remark that we can present this notion in a linear $L$-space. Let $(X, +, \mathbb{R}, \to)$ be a linear $L$-space and $T : X \to X$ be a mapping. By definition, the fixed point problem for $T$ is well posed if:

(1) $F_T = \{x^*\}$.

(2) If $(y_n)_{n\in\mathbb{N}}$ is a sequence in $X$ such that $y_n - T(y_n) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.

If in a metric space $(X, d)$ a mapping $T : X \to X$ satisfies (i) and (v), then, by definition, the mapping has the Ostrowski property. We notice that we can present this notion in a linear $L$-space, as follows. Let $(X, +, \mathbb{R}, \to)$ be a linear $L$-space and $T : X \to X$ be a mapping. By definition, the mapping $T$ has the Ostrowski property if:

(1) $F_T = \{x^*\}$.

(2) If $(y_n)_{n\in\mathbb{N}}$ is a sequence in $X$ such that $y_{n+1} - T(y_n) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.

First part of conclusion (vi) is useful for the localization of the fixed point. Second part (b) is a set-theoretical one, under metrical conditions. If $X$ is a nonempty set and $T : X \to X$ is a mapping such that

$$\bigcap_{n \in \mathbb{N}} T^n(X) = \{x^*\},$$

then, by definition, $T$ is a *Janos mapping*.

On the other hand, from (vi), (b), we have the following property of contractions:

If $(X, d)$ is a complete metric space and $T : X \to X$ is a contraction with $F_T = \{x^*\}$ and $(y_n)_{n\in\mathbb{N}}$ is a bounded sequence in $X$, then $T^n(y_n) \to x^*$ as $n \to \infty$."

For the notions in the above commentaries see: [11], [29], [49], [20], [61], [62], [69], [71], [75], [73], [25], [41], [88].

With the above variants in mind, in terms of the proposed conclusions, a notion emerges: contraction type mapping.

Instead to give an abstract notion of "contraction type mapping", in what follows we present some problems which typify some classes of contraction type mappings.

Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping.

**Problem** $(CT)_1$**.** Which metric conditions on $T$ imply that $T$ is a $\psi$-PM, for a suitable $\psi$?

**Problem** $(CT)_2$**.** Which metric conditions on $T$ imply the conclusions in $(V_4)$?

**Problem** $(CT)_3$**.** Which metric conditions on $T$ imply that $T$ is a $\psi$-PM and a quasi-$\varphi$-contraction, with $\varphi$ and $\psi$ appropriately chosen.

Commentaries:

(1) If $T$ is a $\psi$-PM then the fixed point problem for $T$ is well posed.

(2) If $T$ is PM and is a quasicontraction then $T$ has the Ostrowski property.

(3) For example, if $T$ is a Kannan mapping then $T$ satisfies all the conclusions in $(V_6)$.

(4) References: [70], [49], [34], [38], [39], [73].

7.2. **Mappings of contractive type.** We are looking for a good definition of "contractive type" mapping. For to do this we start with some aspects of the theory of contractive mappings. The basic variants of contractive mapping principle (see [23], [49], [67], [47], [17])are the following ones:

$(V_1)$ (V.V. Niemytzki – M. Edelstein). Let $(X, d)$ be a compact metric space and $T : X \to X$ be a contractive mapping. Then $T$ is a PM.

$(V_2)$ (V.V. Niemytzki – M. Edelstein). Let $(X, d)$ be a metric space and $T$ be a contractive mapping. If $\overline{T(X)}$ is a compact subset of $X$, then $T$ is a PM.

$(V_3)$ (M. Edelstein). Let $(X, d)$ be a metric space and $T : X \to X$ be a contractive mapping. If for each $x \in X$, $\omega_T(x) \neq 0$, then $T$ is a PM.

From the above results on contractive mappings the concept of "contractive type" emerges as solution of one of the following problems:

**Problem** $(CVT)_1$**.** Let $(X, d)$ be a compact metric space and $T : X \to X$ be orbitally continuous. Which metric conditions on $T$ imply that $T$ is PM?

**Problem** $(CVT)_2$**.** Let $(X, d)$ be a metric space and $T : X \to X$ be orbitally continuous such that $\overline{T(X)}$ is a compact subset of $X$. Which metric conditions on $T$ imply that $T$ is PM?

**Problem** $(CVT)_3$**.** Let $(X,d)$ be a metric space and $T : X \to X$ be orbitally continuous such that $\omega_T(x) \neq \emptyset$ for all $x \in X$. Which metric conditions imply that $T$ is PM ?

Commentaries:

(1) If a metric condition on $T$ implies that card $F_T \leq 1$ and $T$ is a graphic contractive, then $T$ is a solution of the above problems. For example, Kannan contractive condition and Ćirić-Reich-Rus contractive condition, Hardy-Rogers contractive condition are such conditions.

(2) References: [9], [29], [40], [49], [63], [65], [75], [76], [79].

7.3. **Mapping of nonexpansive type.** A consistent fixed point theory for nonexpansive mappings is in the case when we have self mappings of a subset (closed convex, bounded closed convex,...) of a Hilbert space and of a Banach space. For a better understanding of the impact of the metric structure in this theory, the study of the problem in an abstract metric space is required.

In what follows we consider nonexpansive mappings on a metric space and we are looking for a good definition of the concept "nonexpansive type mapping".

First, we start with the following result on some classes of mappings in the case of nonexpansive mappings (see [8]).

**Equivalence Theorem.** *Let $(X,d)$ be a compact metric space and $T : X \to X$ a nonexpansive mapping. Then the following statements are equivalent:*

*(i) $T$ has diminishing orbital diameters on $X$.*

*(ii) For each $x \in X$, $T$ is not an isometry on $O_T(x)$ if $\delta(O_T(x)) > 0$.*

*(iii) $T$ is asymptotically regular on $X$.*

As basic variants of the nonexpansive mapping principle in a metric space we present the following results, see [7], [8], [14], [15], [27], [32], [49], [58]. ([7], [8], [14], [15], [25], [27], [38], [46]):

$(V_1)$ Let $(X,d)$ be a metric space and $T : X \to X$ be a nonexpansive mapping. If $\omega_T(x) \cap F_T \neq \emptyset$, $\forall\, x \in X$, then $T$ is WPM.

$(V_2)$ (L.P. Belluce – W.A. Kirk). Let $(X,d)$ be a metric space and $T : X \to X$ be a nonexpansive mapping with diminishing orbital diameters. If $\omega_T(x) \neq \emptyset$, $\forall\, x \in X$, then $T$ is WPM.

$(V_3)$ (L.P. Belluce – W.A. Kirk). Let $(X,d)$ be a compact metric space and $T : X \to X$ be a nonexpansive mapping with diminishing orbital diameters on $X$. Then $T$ is WPM.

$(V_4)$ Let $(X,d)$ be a metric space and $T : X \to X$ be a nonexpansive mapping with diminishing orbital diameters on $X$ and with $\overline{T(X)}$ a compact subset of $X$. Then $T$ is WPM.

Now we have the possibility to define the concept of nonexpansive type mapping as a solution of each of the following problems:

**Problem** $(NT)_1$**.** Let $(X,d)$ be a metric space and $T : X \to X$ be a mapping such that $\omega_T(x) \cap F_T \neq \emptyset$, $\forall\, x \in X$. Which metric conditions on $T$ imply that $T$ is WPM?

**Problem** $(NT)_2$**.** Let $(X,d)$ be a metric space and $T : X \to X$ be orbitally continuous with diminishing orbital diameters on $X$ and with $\omega_T(x) \neq \emptyset$, $\forall\, x \in X$. Which metric conditions on $T$ imply that $T$ is WPM?

Commentaries:

(1) If a metric condition on $T$ implies that $T$ is quasinonexpansive, then $T$ is a solution of the above problems. For example, Kannan nonexpansive condition is such a condition.

(2) References: [27], [78], [79], [38], [4], [36], [9], [67], [60],...

7.4. **Conditions in sinergistic relation in a fixed point theorem.** From the theorems in Sections 7.1, 7.2 and 7.3, and from other results in fixed point theory the following problems emerge:

Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping.

**Problem** $(ST)_1$**.** Which conditions imply that $\omega_T(x) \neq \emptyset$, $\forall\ x \in X$ and which ones imply that $\omega_T(X) \cap F_T \neq \emptyset$, $\forall\ x \in X$?

**Problem** $(ST)_2$**.** Which metric conditions imply that $T$ is asymptotically regular?

Commentaries:

(1) If $T$ is orbitally continuous and $T$ is asymptotically regular, then $\omega_T(x) = F_T$. If in addition $\overline{T(X)}$ is compact, then $\omega_T(x) \neq \emptyset$, $\forall\ x \in X$.

(2) The following result is given in [71]:

Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : X \to \mathbb{R}_+$ such that:

(i) $t_n \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\alpha(t_n) \to 0$ as $n \to \infty \Rightarrow t_n \to 0$ as $n \to \infty$;

(ii) $\alpha(d(x, T(x))) \leq \beta(x) - \beta(T(x))$, $\forall\ x \in X$.

Then, the mapping $T$ is asymptotically regular.

(3) References: [44], [30], [24], [71], [12].

## 8. The case of multivalued mappings

Let us consider now the following sets of subsets of a metric space $(X, d)$:

$$P(X) = \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\};\ P_b(X) = \{Y \in P(X) |\ \text{ Y is bounded }\};$$

$$P_{cl}(X) = \{Y \in P(X) |\ Y \text{ is closed}\};\ P_{cp}(X) = \{Y \in P(X) |\ Y \text{ is compact}\};$$

If $X$ is a normed space, then we denote:

$$P_{cv}(X) = \{Y \in P(X) |\ Y \text{ convex}\};\ P_{cp,cv}(X) = P_{cp}(X) \cap P_{cv}(X).$$

Let us define the following generalized functionals:

(1) $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A, B) := \inf\{d(a, b) |\ a \in A,\ b \in B\}.$$

$D$ is called the gap functional between $A$ and $B$.

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point $x_0$ to the set $B$.

(2) $e : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$e(A, B) := \sup\{D(a, B) |\ a \in A\}.$$

$e$ is called the excess functional of $A$ over $B$.

(3) $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A, B) := \max\{e(A, B), e(B, A)\}.$$

$H$ is called the generalized Pompeiu-Hausdorff functional of $A$ and $B$.

Let $(X,d)$ be a metric space and $T : X \to P(X)$ be a multivalued operator. Then, we denote $F_T := \{x \in X : x \in T(x)\}$, $(SF)_T := \{x \in X : T(x) = \{x\}\}$. The graph of the mapping $T$ is denoted by $Graph(T) := \{(x,y) \in X \times X : y \in T(x)\}$.

Using the above notations, we have in the literature the following metric conditions on a multivalued mapping.

The multivalued operator $T : X \to P(X)$ is said to be:

1) $\alpha$-Lipschitz if $\alpha > 0$ and $H(T(x_1), T(x_2)) \leq \alpha \cdot d(x_1, x_2)$, for each $x_1, x_2 \in X$.

2) $\alpha$-contraction if it is $\alpha$-Lipschitz with $\alpha < 1$.

3) contractive if $H(T(x_1), T(x_2)) < d(x_1, x_2)$, for each $x_1, x_2 \in X$, with $x_1 \neq x_2$.

4) nonexpansive if $H(T(x_1), T(x_2)) \leq d(x_1, x_2)$, for each $x_1, x_2 \in X$.

Let $(X,d)$ be a metric space. If $T : X \to P(X)$ is a multivalued operator and $x_0 \in X$ is an arbitrary point, then the sequence $(x_n)_{n\in\mathbb{N}}$ is, by definition, the successive approximations sequence of $T$ starting from $(x_0, x_1)$ if and only if $x_k \in T(x_{k-1})$, for all $k \in \mathbb{N}^*$.

The following result is the basic theorem for multivalued contraction mappings. It is known in the literature as the Multivalued Contraction Principle of Nadler.

(Nadler [46], Covitz-Nadler [19]) *Let $(X,d)$ be a complete metric space and $x_0 \in X$ be arbitrary. If $T : X \to P_{cl}(X)$ is a multivalued $\alpha$-contraction, then there exists a sequence of successive approximations of $T$ starting from $x_0$ which converges to a fixed point of $T$.*

If the multivalued operator is contractive, then we need a stronger assumption on the space $X$. The following result is the main fixed point result for multivalued contractive mappings.

(Smithson [81]) *Let $(X,d)$ be a compact metric space and $T : X \to P_{cl}(X)$ be a contractive multivalued operator. Then $F_T \neq \emptyset$.*

The main result for nonexpansive mappings was given by T.C. Lim.

(T.C. Lim [42]) *Let $X$ be an uniformly convex Banach space, $Y \in P_{b,cl,cv}(X)$ and $F : Y \to P_{cp}(Y)$ be nonexpansive. Then $F_T \neq \emptyset$.*

As in the singlevalued case, there are many extensions and generalizations of these principles. By similar considerations, we can classify multivalued mappings "of contraction type", "of contractive type" and of "of nonexpansive type". Having in mind the previous main theorems we can formulate, as in the singlevalued case, problems which generate the notions of multivalued mappings of contraction type, multivalued mappings of contractive type and multivalued mappings of nonexpansive type. For related concepts and results see [2], [3], [5], [6], [13], [19], [22], [28], [29], [31], [38], [37], [46], [51], [55], [58], [62], [68], [73], [81], [86], [87].

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