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AN ACCELERATED EXTRAGRADIENT ALGORITHM FOR FIXED POINT, PSEUDOMONOTONE EQUILIBRIUM AND SPLIT NULL POINT PROBLEMS

YASIR ARFAT*, POOM KUMAM*,**,***, MUHAMMAD AQEEL AHMAD KHAN****
AND PARINYA SA NGIAMSUNTHORN*****

*KMUTT Fixed Point Research Laboratory,
KMUTT-Fixed Point Theory and Applications Research Group,
Department of Mathematics, Faculty of Science,
King Mongkut's University of Technology Thonburi (KMUTT),
126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand

**Center of Excellence in Theoretical and Computational Science (TaCS-CoE),
SCL 802 Fixed Point Laboratory, Science Laboratory Building,
King Mongkut's University of Technology Thonburi (KMUTT),
126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand

***Research Center for Interneural Computing, China Medical University Hospital,
China Medical University, Taichung, 40402, Taiwan

****Department of Mathematics, The Islamia University of Bahawalpur,
Bahawalpur 63100, Pakistan

*****Department of Mathematics, Faculty of Science,
King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road,
Bang Mod, Thung Khru, Bangkok 10140, Thailand

E-mail: yasir.arfat@mail.kmutt.ac.th (Y. Arfat), poom.kum@kmutt.ac.th (P. Kumam),
itsakb@hotmail.com (M.A.A.Khan) and parinya.san@kmutt.ac.th. (P.S.Ngiamsunthorn)

Abstract. This paper provides iterative construction of a common solution associated with the fixed point problem of an infinite family of $\Bbbk$-demicontractive mappings, pseudomonotone equilibrium problem satisfying Lipschitz-type continuity and the split common null point problem. We propose an iterative algorithm that combines the classical extragradient method with the inertial extrapolation technique. The analysis of the proposed algorithm is two-fold: firstly, we establish strong convergence results under suitable set of constraints and secondly we verify the viability of the proposed algorithm via numerical experiment with applications.

Key Words and Phrases: Monotone Înclusion, inertial extrapolation technique, pseudomonotone equilibrium problem, fixed point problem, null point problem.
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1. Introduction

Let C be a nonempty subset of a real Hilbert space $\mathcal{H}_1$ with the inner product $< \cdot, \cdot >$ and the associated norm $\| \cdot \|$. For an operator $T : C \to C$, we denote by $Fix(T) = \{x \in C \mid x = Tx\}$ the set of all fixed points of the operator T. Recall that the operator T is known as $\Bbbk$-demicontractive [24] if there exists $\Bbbk \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \Bbbk\|x - Tx\|^2, \ \forall x \in C, p \in Fix(T).$$

The class of $\Bbbk$-demicontractive operators has been studied extensively in various instances of fixed point problems in Hilbert spaces. However, we are concerned with the fixed point problem of an infinite family of $\Bbbk$-demicontractive operators in Hilbert spaces via the following construction of S_k:

$$\begin{aligned}
Q_{k,k+1} &= Id, \\
Q_{k,k} &= \beta_k T_k^{'} Q_{k,k+1} + (1 - \beta_k) Id, \\
Q_{k,k-1} &= \beta_{k-1} T_{k-1}^{'} Q_{k,k} + (1 - \beta_{k-1}) Id, \\
&\vdots \\
Q_{k,m} &= \beta_m T_m^{'} Q_{k,m+1} + (1 - \beta_m) Id, \\
&\vdots \\
Q_{k,2} &= \beta_2 T_2^{'} Q_{k,3} + (1 - \beta_2) Id, \\
S_k = Q_{k,1} &= \beta_1 T_1^{'} Q_{k,2} + (1 - \beta_1) Id,
\end{aligned}$$

where Id is an identity operator, $0 \leq \beta_m \leq 1$ and $T_m^{'} = \alpha x + (1 + \alpha) T_m x$ for all $x \in C$ with T_m being $\Bbbk$-demicontractive operator and $\alpha \in [\Bbbk, 1)$. It is well-known in the context of operator S_k that each $T_m^{'}$ is nonexpansive and the limit $\lim_{k\to\infty} Q_{k,m}$ exists. Moreover

$$Sx = \lim_{k\to\infty} S_k x = \lim_{k\to\infty} Q_{k,1} x, \ \text{for all } x \in C.$$

This implies that $Fix(S) = \bigcap_{k=1}^{\infty} Fix(S_k)$ [31, 33].

Besides fixed point problem, an other abstract formulation in nonlinear functional analysis is the classical equilibrium problem [14] with respect to a (monotone) bifunction g defined on a nonempty subset C of a real Hilbert space $\mathcal{H}_1$ which aims to find a point $\bar{x} \in C$ such that

$$g(\bar{x}, \bar{y}) \geq 0, \ \text{for all } \bar{y} \in C. \tag{1.1}$$

The set of equilibrium points or the solutions of problem (1.1) is denoted by $EP(g)$.

Owing to the wide applicability, the problem (1.1) along with the fixed point problem associated with various nonlinear operators has been studied in the current literature. It is remarked that most of the iterative algorithms dealing with the problem (1.1) solve a strongly monotone regularized equilibrium problem. As a matter of fact, these iterative algorithms fail to converge provided that the bifunction g is pseudomonotone. On the other hand, the extragradient iterative algorithm, based on the Korpelevich method [26], and its various modifications proved to be an important tool for solving the pseudomonotone equilibrium problem [2]. In this connection, we use a modified variant of extragradient iterative algorithm in Hilbert spaces.

In 1994, Censor and Elfving [17] investigated an abstract problem under the name of split convex feasibility problems (SCFP) which is a generalization of the convex feasibility problems. This abstract framework found valuable real-world applications in medical image reconstruction problem and the intensity-modulated radiation therapy [18], see also [15, 21, 20, 19, 22] and the references cited therein. One of the important instances of SCFP is the split common null point problem (SCNPP) defined as follows: given two multivalued operators $A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $A_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ the SCNPP problem deals with a model aiming to find a point

$$\hat{x} \in \mathcal{H}_1 \text{ such that } 0 \in A_1(\hat{x}) \text{ and } 0 \in A_2(\hbar\hat{x}), \tag{1.2}$$

where $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. The set of solutions of the SCNPP (1.2) is denoted by $\Omega := \{\hat{x} \in A_1^{-1}0 : \hbar\hat{x} \in A_2^{-1}0\}$, where $(.)^{-1}$ indicates the inverse operator. In 2012, Byrne et al.[15] suggested the following iterative algorithms to solve the SCNPP associated with two maximal monotone operators A_1 and A_2:

$$x_{k+1} = J_m^{A_1}(x_k + \delta\hbar^*(J_m^{A_2} - I)\hbar x_k), \ k \in \mathbb{N}, \tag{1.3}$$

and

$$\begin{cases} x_0, v \in \mathcal{H}_1; \\ x_{k+1} = \beta_k v + (1-\beta_k)J_m^{A_1}(x_k + \delta\hbar^*(J_m^{A_2} - I)\hbar x_k), \ k \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\hbar^*$ denotes the adjoint operator of $\hbar$ and $J_m^{A_1}, \ J_m^{A_2}$ denotes the corresponding resolvents of $A_1, \ A_2$, respectively. It is remarked that the algorithm (1.3) exhibits weak convergence while the algorithm (1.4) exhibits strong convergence under suitable sets of constraints.

The algorithms (1.3) and (1.4) proved to be a major source of inspiration to study the SCNPP in Hilbert spaces. Since then various optimization algorithms have been analyzed for various instances of SCFP in Hilbert spaces [1, 11, 9, 10, 6, 7, 5, 8, 3, 4, 16, 27, 28]. Quite recently, Yasir et al.[10] investigated an accelerated hybrid projection algorithm for the SCNPP and the fixed point problem in Hilbert spaces. Inspired and motivated by the results presented in [10] and [15], we are aiming to analyze a modified version of the extragradient iterative algorithm for computing a common solution of the SCNPP along with the pseudomonotone equilibrium problem and the fixed point problem of $\mathbb{k}$-demicontractive operators in Hilbert spaces.

2. Preliminaries

Throughout this section, we assume certain concepts of the monotone operator theory and other related concepts from the celebrated monograph of Bauschke and Combettes [12]. Assume that $P_C^{\mathcal{H}_1}$ is a metric projection operator associated with $C \subset \mathcal{H}_1$ provided that the subset C is nonempty, closed and convex. We also assume that $A_1 \subseteq \mathcal{H}_1 \times \mathcal{H}_1$ is a set-valued operator with the usual definitions of $dom(A_1)$ and $zer(A_1)$ whereas the set $gra(A_1) = \{(x,u) \in \mathcal{H}_1 \times \mathcal{H}_1 | u \in A_1 x\}$ denotes the graph of A_1. The operator A_1^{-1} denotes the inverse of A_1. The operator A_1 is said to be monotone if $\langle x-y, u-v\rangle \geq 0$, for all $(x,u),(y,v) \in gra(A_1)$. A monotone operator A_1 is called as maximal monotone operator if there is no proper monotone extension of A_1, equivalently if $ran(Id + mA_1) = \mathcal{H}_1$ for all $m > 0$, where $ran(A_1)$ denotes the range of the operator A_1. The monotone operator A_1 is also connected with the resolvent operator $J_m^{A_1} = (Id + mA_1)^{-1}$ which is well-defined, single-valued, nonexpansive and satisfies $Fix(J_m^{A_1}) = A_1^{-1}(0)$ for all $m > 0$.

Let $g : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction. Then g is said to be (i) monotone if $g(x, y) + g(y, x) \leq 0$, for all $x, y \in C$; (ii) pseudomonotone if $g(x, y) \geq 0 \Rightarrow g(y, x) \leq 0$, for all $x, y \in C$ and (iii) strongly pseudomonotone if $g(x, y) \geq 0 \Rightarrow g(y, x) \leq -\alpha\|x - y\|^2$, for all $x, y \in C$, where $\alpha > 0$. It is worth mentioning that the monotonicity of a bifunction implies the pseudo-monotonicity, but the converse is not true.

The rest of this section is organized with the celebrated results required in the sequel. We first define certain important assumptions for modeling the pseudomonotone equilibrium problem.

Assumption 2.1. [13] *Let $g : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying the following assumptions:*
(A1): g is pseudomonotone, i.e., $g(x, y) \geq 0 \Rightarrow g(x, y) \leq 0$, for all $x, y \in C$;
(A2): g is Lipschitz-type continuous, i.e., there exist two nonnegative constants d_1, d_2 such that

$$g(x, y) + g(y, z) \geq g(x, z) - d_1\|x - y\|^2 - d_2\|y - z\|^2, \text{ for all } x, y, z \in C;$$

(A3): g is weakly continuous on $C \times C$, imply that, if $x, y \in C$ and $(x_k), (y_k)$ are two sequences in C converge weakly to x and y respectively, then $g(x_k, y_k)$ converges to $g(x, y)$;
(A4): For each fixed $x \in C$, $g(x, \cdot)$ is convex and subdifferentiable on C.

Lemma 2.2. *Let $x, y \in \mathcal{H}_1$ and $\beta \in \mathbb{R}$ then*
1. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
2. $\|x - y\|^2 \leq \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$;
3. $\|\beta x + (1 - \beta)y\|^2 = \beta\|x\|^2 + (1 - \beta)\|y\|^2 - \beta(1 - \beta)\|x - y\|^2$.

Lemma 2.3 ([12])**.** *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}_1$. For every $x, y, z \in \mathcal{H}_1$ and $\gamma \in \mathbb{R}$, the set*

$$D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + \gamma\},$$

is closed and convex.

Lemma 2.4 ([12])**.** *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}_1$. The operator $Id - T$ is said to be demiclosed at the origin, if for any sequence (x_k) in C that converges weakly to some x and if the sequence $((Id - T)x_k)$ converges strongly to 0, then $(Id - T)(x) = 0$.*

Lemma 2.5 ([12])**.** *Let $\mathcal{H}_1$ be a real Hilbert space and let (x_k) be a sequence in $\mathcal{H}_1$. Then the following results hold:*
1. *$\mathcal{H}_1$ has the Kadec-Klee property, i.e., if $x_k \rightharpoonup x$ and $\|x_k\| \to \|x\|$ as $k \to \infty$, then $x_k \to x$ as $k \to \infty$;*
2. *If $x_k \rightharpoonup x$ as $k \to \infty$, then $\|x\| \leq \liminf_{k\to\infty} \|x_k\|$.*

Lemma 2.6 ([32]). *Let C be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}_1$ and let $h : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then, $\bar{x}$ is the solution of convex problem $\min\{h(x) : x \in C\}$, if and only if $0 \in \partial h(\bar{x}) + N_C(\bar{x})$, where $\partial h(\cdot)$ denotes the subdifferential of h and $N_C(\bar{x})$ is the normal cone of C at $\bar{x}$.*

Lemma 2.7 ([33]). *Let C be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}_1$ and let (T'_m) be a sequence of nonexpansive operators such that $\bigcap_{k=1}^{\infty} Fix(T'_k) \neq \emptyset$ and $0 \leq \beta_m \leq b < 1$. Then for a bounded subset K of C, we have*

$$\lim_{k\to\infty} \sup_{x\in K} \|Sx - S_k x\| = 0.$$

3. Algorithm and convergence analysis

In this section, we present the convergence analysis of our proposed iterative algorithm. In order to proceed, first observe that we can take the same Lipschitz coefficients (d_1, d_2) for all bifunctions g_i for all $i \in \{1, 2, \cdots, M\}$. Note that the condition $(A2)$ which implies that

$$g_i(x,z) - g_i(x,y) - g_i(y,z) \leq d_{1,i}\|x-y\|^2 + d_{2,i}\|y-z\|^2 \leq d_1\|x-y\|^2 + d_2\|y-z\|^2,$$

where $d_1 = \max\{d_{1,i} : i = 1,2,3,\cdots,M\}$ and $d_2 = \max\{d_{2,i} : i = 1,2,3,\cdots,M\}$. Therefore, $g_i(x,y) + g_i(y,z) \geq g_i(x,z) - d_1\|x-y\|^2 - d_2\|y-z\|^2$.

Now, we set the following hypotheses required in the sequel:
Let $\mathcal{H}_1$,$\mathcal{H}_2$ be two real Hilbert spaces and let $C \subseteq \mathcal{H}_1$ be a nonempty, closed and convex subset of $\mathcal{H}_1$.

- (H1) Let $A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $A_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two maximal monotone operators with the associated resolvents $J_m^{A_1}$ and $J_m^{A_2}$, respectively;
- (H2) Let $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with the associated adjoint operator $\hbar^*$;
- (H3) For each $i \in \{1, 2, \cdots, M\}$, let $g_i : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a finite family of bifunctions satisfying Assumption 2.1;
- (H4) Let S_k be the S-operator;
- (H5) Assume that $\Gamma := \Omega \cap \bigcap_{i=1}^{M} EP(g_i) \cap Fix(S) \neq \emptyset$.

Theorem 3.1. *If $\Gamma \neq \emptyset$ with hypotheses (H1)-(H5), then the sequence (x_k) generated by the Algorithm 1 converges strongly to an element $\bar{x} \in \Gamma$, provided the following conditions hold:*

- (C1) $\sum_{k=1}^{\infty} \Theta_k \|x_k - x_{k-1}\| < \infty$;
- (C2) $0 < a^* \leq \alpha_k \leq b^* < 1$;
- (C3) $\liminf_{k\to\infty} \beta_k > 0$;
- (C4) $\liminf_{k\to\infty} m_k > 0$.

Remark 3.2. Note that the condition (C1) is easily carried through the numerical computation since the value of $\|x_k - x_{k-1}\|$ is known before choosing Θ_k. Here the parameter Θ_k can be taken as $0 \leq \Theta_k \leq \widehat{\Theta_k}$, with

$$\widehat{\Theta_k} = \begin{cases} \min\{\frac{\nu_k}{\|x_k - x_{k-1}\|}, \Theta\} & if\ x_k \neq x_{k-1}; \\ \Theta & \text{otherwise,} \end{cases}$$

Algorithm 1 An Accelerated Projection Based Extragradient Algorithm (Alg.1)

Initialization: Choose arbitrarily, $x_0, x_1 \in \mathcal{H}_1$ and $C_0 = \mathcal{H}_1$. Set $k \geq 1$ and nonincreasing sequence $\alpha_k, \beta_k \subset (0,1)$, $0 < \gamma < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$, $\Theta_k \subset [0,1)$, $m_k \in (0,\infty)$ and $\delta \in (0, \frac{2}{\|\hbar\|^2})$ such that $\|\hbar\|^2 = L$ is the spectral radius of $\hbar^*\hbar$.
Iterative Steps: Given $x_k \in \mathcal{H}_1$, calculate b_k, v_k, w_k and y_k as follows:
Step 1. Compute

$$\begin{cases} b_k = x_k + \Theta_k(x_k - x_{k-1}); \\ u_k = \arg\min\{\gamma g_i(b_k, u) + \frac{1}{2}\|b_k - u\|^2 : u \in C\}; \\ v_k = \arg\min\{\gamma g_i(u_k, u) + \frac{1}{2}\|b_k - u\|^2 : u \in C\}; \\ w_k = \alpha_k v_k + (1-\alpha_k) S_k v_k; \\ y_k = \beta_k w_k + (1-\beta_k)(J_{m_k}^{A_1}(w_k + \delta\hbar^*(J_{m_k}^{A_2} - Id)\hbar w_k)); \end{cases}$$

If $y_k = w_k = v_k = b_k = x_k$ then stop and x_k is the solution of problem Γ. Otherwise,
Step 2. Compute

$$C_{k+1} = \{z \in C_k : \|y_k - z\|^2 \leq \|x_k - z\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 + 2\Theta_k\langle x_k - z, x_k - x_{k-1}\rangle\},$$
$$x_{k+1} = P_{C_{k+1}} x_1, \ \forall\, k \geq 1,$$

Set $k =: k+1$ and go back to **Step 1.**

where $\{\nu_k\}$ is a positive sequence such that $\sum_{k=1}^{\infty} \nu_k < \infty$ and $\Theta \in [0,1)$.

We also need the following relation to establish the strong convergence results of the Algorithm 1.

Lemma 3.3 ([30])**.** *Suppose that $\bar{x} \in EP(g_i)$, $i \in \{1,2,3,...,M\}$ and x_k, b_k, u_k, w_k are defined in Step 1 of Algorithm 1. Then we have*

$$\|v_k - \bar{x}\|^2 \leq \|b_k - \bar{x}\|^2 - (1 - 2\gamma d_1)\|u_k - b_k\|^2 - (1 - 2\gamma d_2)\|u_k - v_k\|^2.$$

Proof of Theorem 3.1. The proof is divided into the following steps:
Step 1. The Algorithm 1 is well-defined.
We know that Γ is closed and convex. Moreover, from Lemma 2.3 we have that C_{k+1} is closed and convex for each $k \geq 1$. Hence the projection $P_{C_{k+1}} x_1$ is well defined. Now, for any $\bar{x} \in \Gamma$, we have that $J_{m_k}^{A_1}\bar{x} = \bar{x}$ and $J_{m_k}^{A_2}(\hbar\bar{x}) = \hbar\bar{x}$. Therefore, it follows from Algorithm 1 that

$$\begin{aligned} \|b_k - \bar{x}\|^2 &= \|(x_k - \bar{x}) + \Theta_k(x_k - x_{k-1})\|^2 \\ &\leq \|x_k - \bar{x}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 + 2\Theta_k\langle x_k - \bar{x}, x_k - x_{k-1}\rangle. \quad (3.1) \end{aligned}$$

Further

$$\begin{aligned} \|w_k - \bar{x}\|^2 &= \|\alpha_k v_k + (1-\alpha_k) S_k v_k - \bar{x}\|^2 \\ &\leq \alpha_k\|v_k - \bar{x}\|^2 + (1-\alpha_k)\|S_k v_k - \bar{x}\|^2 - \alpha_k(1-\alpha_k)\|(Id - S_k)v_k\|^2 \\ &\leq \|v_k - \bar{x}\|^2 - \alpha_k(1-\alpha_k)\|(Id - S_k)v_k\|^2 \\ &\leq \|x_k - \bar{x}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 + 2\Theta_k\langle x_k - \bar{x}, x_k - x_{k-1}\rangle. \quad (3.2) \end{aligned}$$

Furthermore

$$\begin{aligned}\|y_k-\bar{x}\|^2 &= \|\beta_k(w_k-\bar{x})+(1-\beta_k)(J^{A_1}_{m_k}(w_k+\delta\hbar^*(J^{A_2}_{m_k}-I)\hbar w_k)-\bar{x})\|^2\\ &\leq \beta_k\|w_k-\bar{x}\|^2+(1-\beta_k)\|J^{A_1}_{m_k}(w_k+\delta\hbar^*(J^{A_2}_{m_k}-I)\hbar w_k)-\bar{x}\|^2. \qquad (3.3)\end{aligned}$$

Since $J^{A_1}_{m_k}$ is nonexpansive, therefore the expression $\|J^{A_1}_{m_k}(w_k+\delta\hbar^*(J^{A_2}_{m_k}-I)\hbar w_k)-\bar{x}\|^2$ simplifies as

$$\begin{aligned}&\|J^{A_1}_{m_k}(w_k+\delta\hbar^*(J^{A_2}_{m_k}-I)\hbar w_k)-J^{A_1}_{m_k}\bar{x}\|^2\\ &\quad\leq \|w_k+\delta\hbar^*(J^{A_2}_{m_k}-I)\hbar w_k-\bar{x}\|^2\\ &\quad\leq \|w_k-\bar{x}\|^2+\delta^2\|\hbar^*(J^{A_2}_{m_k}-I)\hbar w_k\|^2+2\delta\langle w_k-\bar{x},\hbar^*(J^{A_2}_{m_k}-I)\hbar w_k\rangle\\ &\quad\leq \|w_k-\bar{x}\|^2+\delta^2\|\hbar\|^2\|(J^{A_2}_{m_k}-I)\hbar w_k\|^2+2\delta\langle \hbar w_k-\hbar\bar{x},(J^{A_2}_{m_k}-I)\hbar w_k)\rangle. \qquad (3.4)\end{aligned}$$

Similarly, utilizing the firmly nonexpansiveness of $J^{A_2}_{m_k}$, the expression

$$\lambda_k=2\delta\langle \hbar w_k-\hbar\bar{x},(J^{A_2}_{m_k}-I)\hbar w_k\rangle$$

simplifies as

$$\begin{aligned}\lambda_k &= 2\delta\langle \hbar w_k-\hbar\bar{x}+(J^{A_2}_{m_k}(\hbar w_k)-\hbar w_k)-(J^{A_2}_{m_k}(\hbar w_k)-\hbar w_k),J^{A_2}_{m_k}(\hbar w_k)-\hbar w_k\rangle\\ &= 2\delta(\langle J^{A_2}_{m_k}(\hbar w_k)-\hbar\bar{x},J^{A_2}_{m_k}(\hbar w_k)-\hbar w_k\rangle-\|(J^{A_2}_{m_k}-I)\hbar w_k\|^2)\\ &\leq -2\delta\|(J^{A_2}_{m_k}-I)\hbar w_k)\|^2. \qquad (3.5)\end{aligned}$$

Utilizing (3.4), (3.5) and Lemma 3.3, we then obtain from (3.3) that

$$\begin{aligned}\|y_k-\bar{x}\|^2 &\leq \beta_k\|w_k-\bar{x}\|^2+(1-\beta_k)(\|w_k-\bar{x}\|^2+\delta^2\|\hbar\|^2\|(J^{A_2}_{m_k}-I)\hbar w_k)\|^2\\ &\quad -2\delta\|(J^{A_2}_{m_k}-I)\hbar w_k\|^2),\\ &\leq \beta_k\|w_k-\bar{x}\|^2+(1-\beta_k)(\|w_k-\bar{x}\|^2-\delta(2-\delta\|\hbar\|^2)\|(J^{A_2}_{m_k}-I)\hbar w_k\|^2)\\ &\leq \beta_k\|w_k-\bar{x}\|^2+(1-\beta_k)\|w_k-\bar{x}\|^2\\ &\leq \|x_k-\bar{x}\|^2+\Theta_k^2\|x_k-x_{k-1}\|^2+2\Theta_k\langle x_k-\bar{x},x_k-x_{k-1}\rangle. \qquad (3.6)\end{aligned}$$

It follows from (3.6) that

$$\|y_k-\bar{x}\|^2\leq \|x_k-\bar{x}\|^2+\Theta_k^2\|x_k-x_{k-1}\|^2+2\Theta_k\|x_k-\bar{x}\|\|x_k-x_{k-1}\|. \qquad (3.7)$$

It follows from the above estimate that $\Gamma\subset C_{k+1}$. Summing up these facts, we conclude that the Algorithm 1 is well-defined.

Step 2. The limit of the sequence $(\|x_k-x_1\|)$ exists.

Since Γ is nonempty closed and convex subset of $\mathcal{H}_1$, there exists a unique $x^*\in\Gamma$ such that $x^*=P_\Gamma x_1$. From $x_{k+1}=P_{C_{k+1}}x_1$, we have $\|x_{k+1}-x_1\|\leq\|x^*-x_1\|$, for all $x^*\in\Gamma\subset C_{k+1}$. In particular $\|x_{k+1}-x_1\|\leq\|P_\Gamma x_1-x_1\|$. This proves that the sequence (x_k) is bounded. On the other hand, from $x_k=P_{C_k}x_1$ and $x_{k+1}=P_{C_{k+1}}x_1\in C_{k+1}$, we have that

$$\|x_k-x_1\|\leq\|x_{k+1}-x_1\|.$$

This implies that (x_k) is nondecreasing and hence

$$\lim_{k\to\infty}\|x_k-x_1\| \text{ exists.} \qquad (3.8)$$

Step 3. Show that $\bar{x_*} \in \Gamma$.
In order to proceed, we first calculate the following estimate which is required in the sequel:

$$\begin{aligned} \|x_{k+1} - x_k\|^2 &= \|x_{k+1} - x_1 + x_1 - x_k\|^2 \\ &= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_1\rangle \\ &= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_k + x_k - x_1\rangle \\ &= \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_k\rangle \\ &\leq \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2 . \end{aligned}$$

Taking lim sup on both sides of the above estimate and utilizing (3.8), we have

$$\limsup_{k\to\infty} \|x_{k+1} - x_k\|^2 = 0.$$

That is

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.9}$$

From (b_k) and (C1), we have

$$\lim_{k\to\infty} \|b_k - x_k\| = \lim_{k\to\infty} \Theta_k \|x_k - x_{k-1}\| = 0. \tag{3.10}$$

Utilizing (3.10) and the following triangle inequality, we have

$$\|b_k - x_{k+1}\| \leq \|b_k - x_k\| + \|x_k - x_{k+1}\|.$$

From (3.9) and (3.10), we have

$$\lim_{k\to\infty} \|b_k - x_{k+1}\| = 0. \tag{3.11}$$

Since $x_{k+1} \in C_{k+1}$, therefore we have

$$\|y_k - x_{k+1}\| \leq \|x_k - x_{k+1}\| + 2\Theta_k \|x_k - x_{k-1}\| + 2\langle x_k - x_{k+1}, x_k - x_{k-1}\rangle.$$

Utilizing (3.11) and (C1), the above estimate implies that

$$\lim_{k\to\infty} \|y_k - x_{k+1}\| = 0. \tag{3.12}$$

From (3.9), (3.12) and the following triangular inequality:

$$\|y_k - x_k\| \leq \|y_k - x_{k+1}\| + \|x_{k+1} - x_k\|,$$

we get

$$\lim_{k\to\infty} \|y_k - x_k\| = 0. \tag{3.13}$$

Consider the following re-arranged variant of the estimate (3.7) by applying Lemma 3.3:

$$\begin{aligned} &(1 - 2\gamma d_1)\|u_k - b_k\|^2 - (1 - 2\gamma d_2)\|u_k - v_k\|^2 \\ &\quad \leq (\|x_k - \bar{x}\| + \|y_k - \bar{x}\|)\|x_k - y_k\| + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \|x_k - \bar{x}\|\|x_k - x_{k-1}\|. \end{aligned}$$

Letting $k \to \infty$, using (C1) and (3.13), we have

$$(1 - 2\gamma d_1) \lim_{k\to\infty} \|u_k - b_k\|^2 - (1 - 2\gamma d_2) \lim_{k\to\infty} \|u_k - v_k\|^2 = 0. \tag{3.14}$$

This implies that

$$\lim_{k\to\infty}\|u_k-b_k\|^2=\lim_{k\to\infty}\|u_k-v_k\|^2=0. \tag{3.15}$$

Again, consider the following re-arranged variant of the estimate (3.7) by applying Lemma 3.3:

$$\begin{aligned} a^*(1-b^*)\|(Id-S_k)v_k\|^2 \quad \leq \quad & (\|x_k-\bar{x}\|+\|y_k-\bar{x}\|)\|x_k-y_k\|+\Theta_k^2\|x_k-x_{k-1}\|^2 \\ & +2\Theta_k\|x_k-\bar{x}\|\|x_k-x_{k-1}\|. \end{aligned}$$

Letting $k\to\infty$ and utilizing (C1),(C2) and (3.13), we have

$$\lim_{k\to\infty}\|(Id-S_k)v_k\|=0. \tag{3.16}$$

This implies that

$$\lim_{k\to\infty}\|w_k-v_k\|=\lim_{k\to\infty}a^*\|(Id-S_k)v_k\|=0. \tag{3.17}$$

Utilizing (3.10), (3.15) and (3.17) and the following triangle inequalities, we have

- $\|v_k-b_k\|\leq\|v_k-u_k\|+\|u_k-b_k\|\to 0$;
- $\|v_k-x_k\|\leq\|v_k-b_k\|+\|b_k-x_k\|\to 0$;
- $\|w_k-b_k\|\leq\|w_k-v_k\|+\|v_k-b_k\|\to 0$.
- $\|w_k-x_k\|\leq\|w_k-b_k\|+\|b_k-x_k\|\to 0$;

From (3.4), (3.5) and Lemma 2.2, we have

$$\begin{aligned} \|y_k-\bar{x}\|^2 &= \|\beta_k w_k+(1-\beta_k)(J_{m_k}^{A_1}(w_k+\delta\hbar^*(J_{m_k}^{A_2}-Id)\hbar w_k))-\bar{x})\|^2 \\ &\leq \beta_k\|w_k-\bar{x}\|^2+(1-\beta_k)(\|w_k-\bar{x}\|^2-\delta(2-\delta\|\hbar\|^2)\|(J_{m_k}^{A_2}-Id)\hbar w_k\|^2) \\ &\leq \|w_k-\bar{x}\|^2-(1-\beta_k)\delta(2-\delta\|\hbar\|^2)\|(J_{m_k}^{A_2}-Id)\hbar w_k\|^2 \\ &\leq \|x_k-\bar{x}\|^2+2\Theta_k\langle x_k-x_{k-1},b_k-\bar{x}\rangle \\ &\quad -(1-\beta_k)\delta(2-\delta\|\hbar\|^2)\|(J_{m_k}^{A_2}-Id)\hbar w_k\|^2. \end{aligned} \tag{3.18}$$

Rearranging the above estimate, we have

$$\begin{aligned} &(1-\beta_k)\delta(2-\delta\|\hbar\|^2)\|(J_{m_k}^{A_2}-Id)\hbar w_k\|^2 \\ &\quad\leq \|x_k-\bar{x}\|^2-\|y_k-\bar{x}\|^2+2\Theta_k\langle x_k-x_{k-1},b_k-\bar{x}\rangle \\ &\quad\leq \|x_k-\bar{x}\|^2-\|b_k-\bar{x}\|^2+2\Theta_k\langle x_k-x_{k-1},b_k-\bar{x}\rangle \\ &\quad\leq (\|x_k-\bar{x}\|+\|b_k-\bar{x}\|)\,\|x_k-b_k\|+2\Theta_k\langle x_k-x_{k-1},b_k-\bar{x}\rangle. \end{aligned} \tag{3.19}$$

By using (C1), (C3), (3.10) and $\delta\in(0,\frac{2}{\|\hbar\|^2})$, the estimate (3.19) implies that

$$\lim_{k\to\infty}\|(J_{m_k}^{A_2}-Id)\hbar w_k\|=0. \tag{3.20}$$

Note that $J_{m_k}^{A_1}$ is firmly nonexpansive, it follows that

$$\begin{aligned} \|y_k-\bar{x}\|^2 &= \|\beta_k w_k+(1-\beta_k)(J_{m_k}^{A_1}(w_k+\delta\hbar^*(J_{m_k}^{A_2}-Id)\hbar w_k))-\bar{x}\|^2 \\ &= \|\beta_k(w_k-\bar{x})+(1-\beta_k)(J_{m_k}^{A_1}(w_k+\delta\hbar^*(J_{m_k}^{A_2}-Id)\hbar w_k)-\bar{x})\|^2 \\ &\leq \beta_k\|w_k-\bar{x}\|^2+(1-\beta_k)\|J_{m_k}^{A_1}(w_k+\delta\hbar^*(J_{m_k}^{A_2}-Id)\hbar w_k)-\bar{x}\|^2. \end{aligned} \tag{3.21}$$

Utilizing (3.4) and (3.5), the expression $J^{A_1}_{m_k}(w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k)$ simplifies as

$$\begin{aligned}\|J^{A_1}_{m_k}(w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k) - J^{A_1}_{m_k}\bar{x}\|^2 &\leq \|w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k - \bar{x}\|^2\\ &\leq \|w_k - \bar{x}\|^2. \qquad (3.22)\end{aligned}$$

Setting $\xi_k = J^{A_1}_{m_k}(w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k)$ and (3.21), it follows that

$$\begin{aligned}\|\xi_k - \bar{x}\|^2 &= \|J^{A_1}_{m_k} w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k) - J^{A_1}_{m_k}\bar{x}\|^2\\ &\leq \langle J^{A_1}_{m_k}(w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k) - J^{A_1}_{m_k}\bar{x}, w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k - \bar{x}\rangle\\ &= \langle \xi_k - \bar{x}, w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k - \bar{x}\rangle\\ &= \frac{1}{2}(\|\xi_k - \bar{x}\|^2 + \|w_k + \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k - \bar{x}\|^2\\ &\quad -\|\xi_k - w_k - \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|^2)\\ &\leq \frac{1}{2}(\|\xi_k - \bar{x}\|^2 + \|w_k - \bar{x}\|^2 - \|\xi_k - w_k - \delta\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|^2)\\ &= \frac{1}{2}(\|\xi_k - \bar{x}\|^2 + \|w_k - \bar{x}\|^2 - \|\xi_k - w_k\|^2 - \delta^2\|\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|^2\\ &\quad +2\delta\langle \xi_k - w_k, \hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\rangle)\\ &\leq \frac{1}{2}(\|\xi_k - \bar{x}\|^2 + \|w_k - \bar{x}\|^2 - \|\xi_k - w_k\|^2 - \delta^2\|\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|^2\\ &\quad +2\delta\|\xi_k - w_k\|\|\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|). \qquad (3.23)\end{aligned}$$

That is

$$\|\xi_k - \bar{x}\|^2 \leq \|w_k - \bar{x}\|^2 - \|\xi_k - w_k\|^2 + 2\delta\|\xi_k - w_k\|\|\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|. \qquad (3.24)$$

So, we have

$$\begin{aligned}\|y_k - \bar{x}\|^2 &\leq \beta_k\|w_k - \bar{x}\|^2 + (1 - \beta_k)\|\xi_k - \bar{x}\|^2\\ &\leq \beta_k\|w_k - \bar{x}\|^2 + (1 - \beta_k)(\|w_k - \bar{x}\|^2 - \|\xi_k - w_k\|^2\\ &\quad +2\delta\|\xi_k - w_k\|\|\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|). \qquad (3.25)\end{aligned}$$

After simplification, we have

$$\begin{aligned}(1 - \beta_k)\|\xi_k - w_k\|^2 &\leq \|w_k - \bar{x}\|^2 - \|y_k - \bar{x}\|^2\\ &\quad -2(1 - \beta_k)\delta\|\xi_k - w_k\|\|\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|)\\ &\leq (\|w_k - \bar{x}\| + \|y_k - \bar{x}\|)\|w_k - y_k\|\\ &\quad -2(1 - \beta_k)\delta\|\xi_k - w_k\|\|\hbar^*(J^{A_2}_{m_k} - Id)\hbar w_k\|). \qquad (3.26)\end{aligned}$$

Making use of (3.24), (3.20) and (C3), we have the following estimate:

$$\lim_{k\to\infty} \|\xi_k - w_k\| = 0. \qquad (3.27)$$

This implies that

$$\lim_{k\to\infty} \|\xi_k - v_k\| = 0. \qquad (3.28)$$

Reasoning as above, we get from (b_k), (C1) and (3.28) that

$$\lim_{k\to\infty} \|\xi_k - x_k\| = 0. \qquad (3.29)$$

Since (x_k) is bounded, then there exists a subsequence (x_{k_t}) of (x_k) such that

$$x_{k_t} \rightharpoonup \bar{x_*} \in \mathcal{H}_1 \text{ as } t \to \infty.$$

So, therefore $\xi_{k_t} \rightharpoonup \bar{x_*}$ and $\bar{w}_{k_t} \rightharpoonup \bar{x_*}$ as $t \to \infty$. In order to show that $\bar{x_*} \in \Omega$, we assume that $(r, s) \in gra(A_1)$. Since $\xi_{k_t} = J^{A_1}_{m_{k_t}}(w_{k_t} + \delta\hbar^*(J^{A_2}_{m_{k_t}} - Id)\hbar w_{k_t})$, therefore, we have

$$w_{k_t} + \delta\hbar^*(J^{A_2}_{m_{k_t}} - Id)\hbar w_{k_t} \in \xi_{k_t} + m_{k_t} A_1(\xi_{k_t}).$$

This implies that

$$\frac{1}{m_{k_t}}(w_{k_t} - \xi_{k_t}) + \frac{1}{m_{k_t}}\delta\hbar^*(J^{A_2}_{m_{k_t}} - Id)\hbar w_{k_t} \in A_1(\xi_{k_t}).$$

From the monotonicity of A_1, we have

$$\langle r - \xi_{k_t}, s - (\frac{1}{m_{k_t}}(w_{k_t} - \xi_{k_t}) + \frac{1}{m_{k_t}}(\delta\hbar^*(J^{A_2}_{m_{k_t}} - Id)\hbar w_{k_t}))\rangle \geq 0.$$

From the above estimate, we also have

$$\begin{aligned} \langle r - \xi_{k_t}, s\rangle &\geq \langle r - \xi_{k_t}, \frac{1}{m_{k_t}}(w_{k_t} - \xi_{k_t}) + \frac{1}{m_{k_t}}(\delta\hbar^*(J^{A_2}_{m_{k_t}} - Id)\hbar w_{k_t})\rangle \\ &= \langle r - \xi_{k_t}, \frac{1}{m_{k_t}}(w_{k_t} - \xi_{k_t})\rangle \\ &\quad + \langle r - \xi_{k_t}, \frac{1}{m_{k_t}}(\delta\hbar^*(J^{A_2}_{m_{k_t}} - Id)\hbar w_{k_t})\rangle. \end{aligned} \tag{3.30}$$

Since $\xi_{k_t} \rightharpoonup \bar{x_*}$, we obtain

$$\lim_{t\to\infty} \langle r - \xi_{k_t}, v\rangle = \langle r - \bar{x_*}, s\rangle.$$

By making the use of (3.27), (3.28) and (3.30), it follows that

$$\langle r - \bar{x_*}, s\rangle \geq 0.$$

This implies that $0 \in A_1\bar{x_*}$. Since $\hbar$ is a bounded linear operator, we have $\hbar w_{k_t} \rightharpoonup \hbar\bar{x_*}$ as $t \to \infty$. Moreover from (3.20), it then follows from the demiclosed principle that $0 \in A_2(\hbar\bar{x_*})$ and hence $\bar{x_*} \in \Omega$.

Step 4. Show that $\bar{x_*} \in \bigcap_{i=1}^{M} EP(g_i)$.

Note that

$$u_k = \arg\min\{\gamma g_i(b_k, y) + \frac{1}{2}\|b_k - y\|^2 : y \in C\}.$$

By employing Lemma 2.6, we get

$$0 \in \partial_2\{\gamma g_i(b_k, y) + \frac{1}{2}\|b_k - y\|^2\}(u_k) + N_C(u_k).$$

Then, there exist $p \in \partial_2 g_i(b_k, u_k)$ and $\bar{p} \in N_C(u_k)$ such that

$$\gamma p + b_k - u_k + \bar{p}. \tag{3.31}$$

Since $\bar{p} \in N_C(u_k)$ and $\langle \bar{p}, y - u_k\rangle \leq 0$ for all $y \in C$. So, by using (3.31), we have

$$\gamma\langle p, y - u_k\rangle \geq \langle u_k - b_k, y - u_k\rangle, \quad \text{for all } y \in C. \tag{3.32}$$

Since $p \in \partial_2 g_i(b_k, u_k)$,

$$g_i(b_k, y) - g_i(b_k, u_k) \geq \langle p, y - u_k \rangle, \quad \text{for all } y \in C. \tag{3.33}$$

Utilizing, (3.30) and (3.33), we obtain

$$\gamma(g_i(b_k, y) - g_i(b_k, u_k)) \geq \langle u_k - b_k, y - u_k \rangle, \quad \text{for all } y \in C. \tag{3.34}$$

Since $b_k \rightharpoonup \bar{x_*}$ and $\|b_k - u_k\| \to 0$ as $k \to \infty$, this imply $u_k \rightharpoonup \bar{x_*}$. By using (A3) and from (3.34), letting $k \to \infty$, we deduce that $g_i(\bar{x_*}, y) \geq 0$ for all $y \in C$, $i \in \{1, 2, 3, \cdots, M\}$. Therefore, $\bar{x_*} \in \bigcap_{i=1}^{M} EP(g_i)$.

Step 6. Show that $\bar{x_*} \in Fix(S)$.

Observe that

$$\begin{aligned} \|v_k - Sv_k\| &\leq \|v_k - S_k v_k\| + \|S_k v_k - S v_k\| \\ &\leq \|v_k - S_k v_k\| + \sup_{x \in K} \|S_k v_k - S v_k\|. \end{aligned}$$

Utilizing (3.16) and Lemma 2.7, the above estimate implies that

$$\lim_{k \to \infty} \|v_k - Sv_k\| = 0.$$

This together with the fact that $v_{k_t} \rightharpoonup \bar{x_*}$ implies, with the help of Lemma 2.4, that $\bar{x_*} \in Fix(S) = \bigcap_{k=1}^{\infty} Fix(T_k)$ and hence $\bar{x_*} \in \Gamma$.

Step 7. Show that $x_k \to \bar{x} = P_\Gamma x_1$.

Since $\bar{x} = P_\Gamma x_1$ and $\bar{x_*} \in \Gamma$. Therefore, we have

$$\|\bar{x} - x_1\| \leq \|\bar{x_*} - x_1\| \leq \liminf_{k \to \infty} \|x_k - x_1\| \leq \limsup_{k \to \infty} \|x_k - x_1\| \leq \|\bar{x} - x_1\|.$$

From the uniqueness of the nearest point $\bar{x}$, we get that $\bar{x} = \bar{x_*}$. On the other hand, from the estimate $\|x_{k_t} - x_1\| \leq \|\bar{x} - x_1\|$ and Lemma 2.5, we get that $x_{k_t} \to \bar{x}$ as $t \to \infty$. Again, utilizing the uniqueness of $\bar{x}$, we deduce that $x_k \to \bar{x}$ as $k \to \infty$. This completes the proof. □

If we take $A_2 = 0$ then we have the following results:

Corollary 3.4. *Let $\mathcal{H}_1$, A_1, g_i and S be the same as in Theorem 3.1 with*

$$\Gamma := \{x \in A_1^{-1}(0) \cap \left(\bigcap_{i=1}^{M} EP(g_i)\right) \cap Fix(S)\} \neq \emptyset.$$

Then the following sequence:

$$\begin{cases} b_k = x_k + \Theta_k(x_k - x_{k-1}); \\ u_k = \arg\min\{\gamma g_i(b_k, u) + \frac{1}{2}\|b_k - u\|^2 : u \in C\}; \\ v_k = \arg\min\{\gamma g_i(u_k, u) + \frac{1}{2}\|b_k - u\|^2 : u \in C\}; \\ w_k = \alpha_k v_k + (1 - \alpha_k) S_j v_k; \\ y_k = \beta_k w_k + (1 - \beta_k) J_{m_k}^{A_1} \bar{w}_k; \\ C_{k+1} = \{z \in C_k : \|y_k - z\|^2 \leq \|x_k - z\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 \\ \qquad +2\Theta_k \langle x_k - z, x_k - x_{k-1} \rangle\}; \\ x_{k+1} = P_{C_{k+1}} x_1, \forall\ k \geq 1, \end{cases} \tag{3.35}$$

converges strongly to an element $\bar{x} = P_\Gamma x_1$ provided that the conditions (C1)-(C4) hold.

Replacing Mann's iteration in Step 1 of Algorithm 1 by Halpern [23] type algorithm, we have the following algorithm.

Algorithm 2 An Accelerated Projection Based Halpern's-Extragradient Algorithm (Alg.2)

Initialization: Choose arbitrarily, $r, x_0, x_1 \in \mathcal{H}_1$ and $C_0 = \mathcal{H}_1$. Set $k \geq 1$ and nonincreasing sequence $\alpha_k, \beta_k \subset (0,1)$, $0 < \gamma < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$, $\Theta_k \subset [0,1)$, $m_k \in (0,\infty)$ and $\delta \in (0, \frac{2}{\|\hbar\|^2})$ such that $\|\hbar\|^2 = L$ is the spectral radius of $\hbar^*\hbar$.

Iterative Steps: Given $x_k \in \mathcal{H}_1$, calculate b_k, v_k, w_k and y_k as follows:

Step 1. Compute

$$\begin{cases} b_k = x_k + \Theta_k(x_k - x_{k-1}); \\ u_k = \arg\min\{\gamma g_i(b_k, u) + \frac{1}{2}\|b_k - u\|^2 : u \in C\}; \\ v_k = \arg\min\{\gamma g_i(u_k, u) + \frac{1}{2}\|b_k - u\|^2 : u \in C\}; \\ w_k = \alpha_k r + (1-\alpha_k) S_j v_k; \\ y_k = \beta_k w_k + (1-\beta_k)(J_{m_k}^{A_1}(w_k + \delta\hbar^*(J_{m_k}^{A_2} - Id)\hbar w_k)); \end{cases}$$

If $y_k = w_k = v_k = b_k = x_k$ then stop and x_k is the solution of problem Γ. Otherwise,

Step 2. Compute

$$\begin{aligned} C_{k+1} = \{z \in C_k : \|y_k - z\|^2 \leq \alpha_k\|r - z\|^2 + (1-\alpha_k)(\|x_k - x_{k+1}\|^2 + \Theta_k^2\|x_k - x_{k-1}\|^2 \\ + 2\Theta_k\langle x_k - x_{k+1}, x_k - x_{k-1}\rangle)\}, \end{aligned}$$

$$x_{k+1} = P_{C_{k+1}} x_1, \ \forall\ k \geq 1,$$

Set $k =: k+1$ and go back to **Step 1.**

Remark 3.5. In order to obtain the desired convergence result for the Algorithm 2, we have to assume a stopping criteria as $k > k_{max}$ for some chosen sufficiently large number k_{max}.

Theorem 3.6. *If $\Gamma \neq \emptyset$ with hypotheses (H1)-(H5), then the sequence (x_k) generated by the Algorithm 2 converges strongly to an element $\bar{x} \in \Gamma$, provided the following conditions hold:*

(C1) $\sum_{k=1}^{\infty} \Theta_k\|x_k - x_{k-1}\| < \infty$;
(C2) $0 < a^* \leq \alpha_k \leq b^* < 1$ *and* $\lim_{k\to\infty} \alpha_k = 0$;
(C3) $\liminf_{k\to\infty} \beta_k > 0$;
(C4) $\liminf_{k\to\infty} m_k > 0$.

Proof. Arguing similarly as in the proof of Theorem 3.1 Steps 1-2, the set

$$\begin{aligned} C_{k+1} = \{z \in C_k : \|y_k - z\|^2 \leq \alpha_k\|r - z\|^2 + (1-\alpha_k)(\|x_k - x_{k+1}\|^2 + \xi_k^2\|x_k - x_{k-1}\|^2 \\ + 2\xi_k\langle x_k - x_{k+1}, x_k - x_{k-1}\rangle)\}, \end{aligned}$$

is closed and convex as well as $\Gamma \subset C_{k+1}$ for all $k \geq 0$. Moreover, the sequence (x_k) is bounded and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.36}$$

Since $x_{k+1} = P_{C_k}(r) \in C_k$ and by definition of C_k, we have

$$\begin{aligned} \|y_k - x_{k+1}\|^2 \quad &\leq \quad \alpha_k \|r - x_{k+1}\|^2 + (1-\alpha_k)(\|x_k - x_{k+1}\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 \\ &\quad +2\Theta_k \langle x_k - x_{k+1}, x_k - x_{k-1}\rangle). \end{aligned}$$

Letting $k \to \infty$, from (3.36), (C1), (C2) and the boundedness of (x_k), we obtain

$$\lim_{k\to\infty} \|y_k - x_{k+1}\| = 0.$$

Similarly from (3.12) and (3.36), we get

$$\lim_{k\to\infty} \|y_k - x_k\| = 0.$$

Consequently, we have the following estimates:

$$\begin{aligned} \lim_{k\to\infty} \|u_k - x_k\| &= \lim_{k\to\infty} \|v_k - x_k\| \\ &= \lim_{k\to\infty} \|w_k - x_k\| \\ &= \lim_{k\to\infty} \|w_k - v_k\| = \ 0. \end{aligned}$$

Using $w_k = \alpha_k r + (1-\alpha_k) S_j v_k$, we obtain

$$\|S_k v_k - v_k\| \leq \frac{1}{(1-\alpha_k)} \|w_k - v_k\| + \frac{\alpha_k}{(1-\alpha_k)} \|r - v_k\|.$$

In view of (C2), the above estimate implies that

$$\lim_{k\to\infty} \|S_k v_k - v_k\| = 0.$$

The rest of the proof of Theorem 3.6 follows from the proof of Theorem 3.1. □

4. Numerical experiment and results

This section is devoted to analyze the computation performance of the Algorithm 1 by the following suitable example.

Example 4.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. Define three operators $\hbar, A_1, A_2 : \mathbb{R} \to \mathbb{R}$ as

$$\hbar(x) = 3x \text{ and } A_1 x = 2x \text{ and } A_2 x = 3x$$

for all $x \in \mathbb{R}$. It is evident from the definitions that A_1 and A_2 are maximal monotone operators with

$$\Omega := \{\hat{x} \in A_1^{-1}(0) : \hbar\hat{x} \in A_2^{-1}(0)\} = \{0\}.$$

For the bounded linear operator $\hbar$ and the associated adjoint operator $\hbar^*$, we have

$$\|\hbar\| = \|\hbar^*\| = 3.$$

For each $i \in \{1, 2, 3, \cdots, M\}$, let g_i be a finite family of bifunctions defined by

$$g_i(x, y) = J_i(x)(y - x),$$

where

$$\begin{cases} J_i(x) = 0, \text{ if } 0 \le x \le \mu_i, \\ \text{and} \\ J_i(x) = \sin(x - \mu_i) + \exp(x - \mu_i) - 1, \text{ if } \mu_i \le x \le 1. \end{cases}$$

Note that the family of bifunctions g_i is pseudomonotone and satisfies Assumption 2.1. Moreover, $J_i(x)$ is 4-Lipschitz continuous and

$$\bigcap_{i=1}^{M} EP(g_i) = [0, \mu_1].$$

Let

$$S_k(x) = \begin{cases} -\frac{x}{k}, \ x \in [0, \infty); \\ x, \ x \in (-\infty, 0); \end{cases}$$

be an infinite family of $\mathbb{k}$-demicontractive operators with $\bigcap_{k=1}^{\infty} Fix(S_k) = \{0\}$. Hence

$$\Gamma = \Omega \cap \left(\bigcap_{i=1}^{M} EP(g_i) \right) \cap \left(\bigcap_{k=1}^{\infty} Fix(S_k) \right) = 0.$$

From the Algorithm 1, we have

$$u_k = \arg\min\{\gamma J_i(b_k)(y - b_k) + \frac{1}{2}(y - b_k)^2, \ \forall \ y \in [0, 1]\}.$$

For the sake of clarity, we reformulate the above equation, which is equivalent to

$$u_k = b_k - \gamma J_i(b_k), \quad \text{for all } i \in \{1, 2, \cdots, M\}.$$

Similarly, we get

$$w_k = b_k - \gamma J_i(u_k), \quad \text{for all } i \in \{1, 2, \cdots, M\}.$$

Also, choose $\Theta = 0.5$, $\gamma = \frac{1}{8}$, $\alpha_k = \frac{1}{100k+1}$, $\beta_k = \frac{1}{100k+1}$, $\delta = \frac{1}{9}$, $L = 9$ and $m = 0.01$. Since

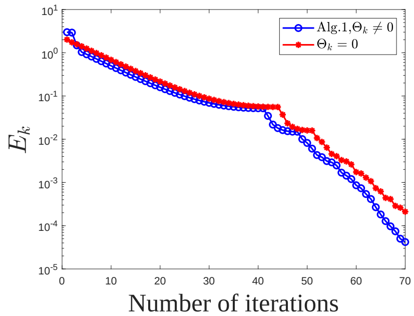
$$\begin{cases} \min\{\frac{1}{k^2\|x_k - x_{k-1}\|}, 0.5\} & \text{if } x_k \neq x_{k-1}; \\ 0.5 & \text{otherwise.} \end{cases}$$

Now, we provide a numerical test for a comparison between our accelerated based projection splitting algorithm defined in Algorithm 1 (i.e $\Theta_k \neq 0$) and the non-inertial variant of the projection splitting algorithm (i.e $\Theta_k = 0$). The stopping criteria is defined as Error=$E_k = \|x_k - x_{k-1}\| < 10^{-6}$. The different choices of x_0 and x_1 are giving as follows:

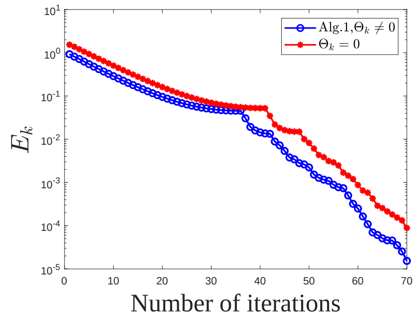
TABLE 1. Numerical results for Example 4.1

	No. of Iter. $\Theta_k = 0$	**Alg.1,** $\Theta_k \neq 0$	**CPU(Sec)** $\Theta_k = 0$	**Alg.1,** $\Theta_k \neq 0$
Choice 1. $x_0 = (5)$, $x_1 = (4)$	85	80	0.079451	0.070391
Choice 2. $x_0 = (4.7)$, $x_1 = (1.7)$	86	77	0.083978	0.075550
Choice 3. $x_0 = (-7)$, $x_1 = (-4)$	91	81	0.084529	0.077949

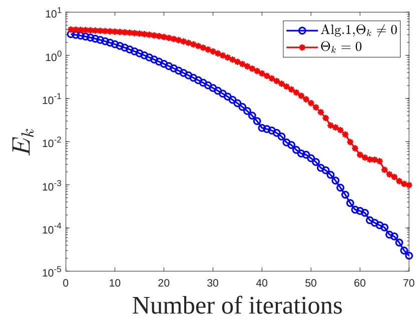
The error plotting E_k and (x_k) against Alg.1, $\Theta_k \neq 0$ and $\Theta_k = 0$ for each choices in Table 1 has shown in Figure 1.



(A) Choice 1



(B) Choice 2



(C) Choice 3

Figure 1. Comparison of Alg.1, $\Theta_k \neq 0$ and $\Theta_k = 0$

We can see from Table 1 and Figure 1 that Alg.1 ($\Theta_k \neq 0$) exhibits better computational performance as compared with Alg.1 ($\Theta_k = 0$).

5. Conclusions

In this paper, we have devised an accelerated based parallel hybrid projection algorithm for computing a common solution associated with the fixed point problem of an infinite family of $\mathbb{k}$-demicontractive mappings, pseudomonotone equilibrium bifunction satisfying Lipschitz-type continuity and the SCNPP in Hilbert spaces. The strong convergence of the algorithm and its variant is established under suitable set of constraints. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. We would like to emphasize that the above mentioned problems occur naturally in many applications, therefore, iterative algorithms are inevitable in this field of investigation. As a consequence, our theoretical framework constitutes an important topic of future research.

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References

[1] F. Alvarez, H. Attouch, *An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping*, Set Valued Anal., **9**(2001), 3-11.

[2] P.N. Anh, *A hybrid extragradient method for pseudomonotne equilibrium problems and fixed point problems*, Bull. Malays. Math. Sci. Sc., **36**(2013), 107-116.

[3] Y. Arfat, O.S. Iyiola, M.A.A. Khan, P. Kumam, W. Kumam, K. Sitthithakerngkiet, *Convergence analysis of the shrinking approximants for fixed point problem and generalized split common null point problem*, J. Inequal. Appl., **67**(2022), 1–21.

[4] Y. Arfat, M.A.A. Khan, P. Kumam, W. Kumam, K. Sitthithakerngkiet, *Iterative solutions via some variants of extragradient approximants in Hilbert spaces*, AIMS Math., **7(8)**(2022), 13910–13926.

[5] Y. Arfat, P. Kumam, M.A.A. Khan, O.S. Iyiola, *Multi-inertial parallel hybrid projection algorithm for generalized split null point problems*, J. Appl. Math. Comput., (2021), https://doi.org/10.1007/s12190-021-01660-4.

[6] Y. Arfat, P. Kumam, M.A.A. Khan, P.S. Ngiamsunthorn, *Parallel shrinking inertial extragradient approximants for pseudomonotone equilibrium, fixed point and generalized split null point problem*, Ricerche Mat., (2021), https://doi.org/10.1007/s11587-021-00647-4.

[7] Y. Arfat, P. Kumam, M.A.A. Khan, P.S. Ngiamsunthorn, *Shrinking approximants for fixed point problem and generalized split null point problem in Hilbert spaces*, Optim. Lett., **16**(2022), 1895-1913.

[8] Y. Arfat, P. Kumam, M.A.A. Khan, P.S. Ngiamsunthorn, *An accelerated Visco-Cesaro means Tseng type splitting method for fixed point and monotone inclusion problems*, Carpathian J. Math., **38**(2022), no. 2, 281-297.

[9] Y. Arfat, P. Kumam, M.A.A. Khan, P.S. Ngiamsunthorn, A. Kaewkhao, *An inertially constructed forward-backward splitting algorithm in Hilbert spaces*, Adv. Differ. Equ., **124**(2021).

[10] Y. Arfat, P. Kumam, P.S. Ngiamsunthorn, M.A.A. Khan, *An accelerated projection based parallel hybrid algorithm for fixed point and split null point problems in Hilbert spaces*, Math. Meth. Appl. Sci., (2021), 1-19.

[11] Y. Arfat, P. Kumam, P.S. Ngiamsunthorn, M.A.A. Khan, H. Sarwar, H.F. Din, *Approximation results for split equilibrium problems and fixed point problems of nonexpansive semigroup in Hilbert spaces*, Adv. Differ. Equ., **512**(2020).

[12] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operators Theory in Hilbert Spaces*, CMS Books in Mathematics, Springer, New York, 2011.

[13] M. Bianchi, S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory. Appl., **90**(1996), 31-43.

[14] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Stud., **63**(1994), 123-145.

[15] C. Byrne, Y. Censor, A. Gibali, S. Reich, *The split common null point problem,* J. Nonlinear Convex Anal., **13**(2012), 759-775.

[16] L.C. Ceng, *Two inertial linesearch extragradient algorithms for the bilevel split pseudomonotone variational inequality with constraints*, J. Appl. Numer. Optim., **2**(2020), 213-233.

[17] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, **8**(1994), 221-239.

[18] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity modulated radiation therapy*, Phys. Med. Biol., **51**(2006), 2353-2365.

[19] Y. Censor, A. Gibali, S. Reich, *Algorithms for the split variational inequality problem,* Numer. Algorithms, **59**(2012), 301-323.
[20] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Probl., **21**(2005), 2071-2084.
[21] Y. Censor, A. Segal, *The split common fixed point problem for directed operators*, J. Nonlinear Convex Anal., **16**(2009), 587-600.
[22] A. Gibali, *A new split inverse problem and an application to least intensity feasible solutions*, Pure. Appl. Func. Anal., **2**(2017), 243-258.
[23] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc., **73**(1967), 957-961.
[24] T.L. Hicks, J.D. Kubicek, *On the Mann iteration process in a Hilbert space*, J. Math. Anal. Appl., **59**(1977), 498-504.
[25] M.A.A. Khan, Y. Arfat, A.R. Butt, *A shrinking projection approach to solve split equilibrium problems and fixed point problems in Hilbert spaces*, UPB, Sci. Bull. Series A, **80**(2018), no. 1, 33-46.
[26] G.M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekon. Mat. Metody, **12**(1976), 747-756.
[27] L. Liya, S.Y. Cho, J.C. Yao, *Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications*, J. Nonlinear Var. Anal., **5**(2021), no. 4, 627-644.
[28] F.U. Ogbuisi, O.S. Iyiola, J.M.T. Ngnotchouye, T.M.M. Shumba, *On inertial type self-adaptive iterative algorithms for solving pseudomonotone equilibrium problems and fixed point problems*, J. Nonlinear Func. Anal., 2021, Article ID **4**(2021), 1-18.
[29] B.T. Polyak, *Some methods of speeding up the convergence of iteration methods*, U.S.S.R. Comput. Math. Math. Phys., **4**(1964), no. 5, 1-17.
[30] T.D. Quoc, L.D. Muu, N.V. Hien, *Extragradient algorithms extended to equilibrium problems,* Optimization, **57**(2008), 749-776.
[31] W. Takahashi, K. Shimoji, *Convergence theorems for nonexpansive mappings and feasibility problems*, Math. Comput. Modelling, **32**(2000), 1463-1471.
[32] J. Tiel, *Convex Analysis. An Introductory Text*, Wiley, 1984.
[33] S. Wang, *A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces*, Appl. Math. Lett., **24**(2011), 901-907.

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