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INEXACT SIMULTANEOUS PROJECTION METHOD FOR SOLVING BILEVEL EQUILIBRIUM PROBLEMS

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Abstract. In this work we consider a bilevel equilibrium problem defined over the intersection of the fixed points set of demicontractive mappings. An inexact simultaneous projection method for solving the problem is introduced and its strong convergence is established under mild and standard conditions. Primary numerical illustrations in finite and infinite dimensional spaces with comparisons to related results in the literature, demonstrate the algorithm performances and emphasize its computational and theoretical advantages.
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1. Introduction

Let $I = \{1, ..., r\}, J = \{1, ..., m\}, \mathcal{H}$ be a real Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. Let $f : C \times C \to \mathcal{R} \cup \{+\infty\}, g_j : C \times C \to \mathcal{R} \cup \{+\infty\}$ be bifunctions such that $f(x, x) = 0, g_j(x, x) = 0$ for all $x \in C, j \in J$, which usually are called *cost bifunctions*. The equilibrium problem [9] for f onto C is to find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solutions set of the problem (1.1) is denoted by $S(C, f)$. Let mappings $S_i : C \to C (i \in I)$ be demicontractive. In this paper, we consider the following bilevel equilibrium problem including demicontractive mappings:

$$\text{Find } x^* \in \Omega \ \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in \Omega, \tag{1.2}$$

where $Fix(S_i)$ is the fixed points set of S_i, and $\Omega = \cap_{i \in I} Fix(S_i) \cap S(C, g_j)_{j \in J}$. Problem (1.2) is a special form of bilevel problems. Note that if g_j is pseudomonotone on C, then the constraint Ω is convex and not given in explicit form. It includes, as special cases, many other problems, such as the follows.

Equibrium problem: Setting $S_i = 0$ and $g_j = 0$ for all $i \in I, j \in J$, it is easy to see that the problem (1.2) coincides with the form (1.1). In recent years, this problem has received a lot of attention by many authors who improved it via various ways in both Euclidean spaces and infinite dimensional Hilbert spaces; see for instance [16, 17, 22, 26, 25, 27, 10, 11] and the references therein.

Bilevel variational inequality problem: Let two mappings $F : C \to \mathcal{H}$ and $G : C \to \mathcal{H}$. The following problem is called the bilevel variational inequality problem [7]:

$$\text{Find } x^* \in VI(C,G) \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in VI(C,G), \tag{1.3}$$

where $VI(C,G) := \{\bar{x} \in C : \langle G(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall y \in C\}$. By choosing $f(x,y) := \langle F(x), y - x \rangle, g_j(x,y) = \langle G(x), y - x \rangle$ and $S_i = 0$ for all $i \in I, j \in J, (x,y) \in C \times C$, we can easily see that (1.3) is equivalent to (1.2). The simplest approach for solving this problem is projection method in which only two projections on the feasible set C is performed per each iteration such as the projection method of Xu et al. in [32] for neural network models, the extragradient methods in [12, 13, 29, 1, 7, 21].

Minimum-norm problem: Let $a \in \mathcal{H}$ and a mapping $F : C \to \mathcal{H}$. The minimum-norm problem [35, 30, 23] is formulated in the following:

$$\min \left\{ \|a - y\|^2 : y \in VI(C,F) \right\}.$$

Taking $f(x,y) = \|y - a\|^2 - \|x - a\|^2, g_j(x,y) = \langle F(x), y - x \rangle$ and $S_i = 0$ for all $(x,y) \in C \times C, i \in I, j \in J$, we can see that the problem (1.2) collapses into the minimum-norm problem. A typical example is the problem of the least-squares solution to the constrained linear inverse problem.

Equilibrium problem over the fixed point sets: Let $f : C \times C \to \mathcal{R} \cup \{+\infty\}$ be a bifunction such that $f(x,x) = 0$ for all $x, y \in C$ and mappings $S_i : C \to C (i \in I)$. The problem is to find $x^* \in \cap Fix(S_i)_{i \in I}$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in \cap Fix(S_i)_{i \in I}. \tag{1.4}$$

By choosing $g_j = 0$ for all $j \in J$, the problem (1.4) may be written in the form of (1.2). In recent years, it is attractive by many researchers. There are increasing interests in studying solution methods for this problem such as the hybrid steepest descent methods of Yamada [33] where $S_i (i \in I)$ are nonexpansive mappings, subgradient-type method of Iiduka [20] and other [34, 15, 14].

Bilevel equilibrium problem (lexicographic Ky Fan inequality): Let $g : C \times C \to \mathcal{R} \cup \{+\infty\}$ such that $g(x,x) = 0$ for all $x \in C$. Let $S_i (i \in I)$ be the identity mapping. Set $g_j(x,y) = g(x,y)$ for all $j \in J$ and $(x,y) \in C \times C$. Then, the problem (1.2) usually called bilevel equilibrium problem or lexicographic Ky Fan inequality as follows:

$$\text{Find } x \in S(C,g) \text{ such that } f(x,y) \geq 0, \quad \forall y \in S(C,g). \tag{1.5}$$

In [28], Moudafi introduced proximal methods for a class of monotone bilevel equilibrium problems. By using Korpelevich's extragradient method, Strodiot et al. [31] proposed the projected extragradient viscosity method for finding the solution of a variational inequality problem whose constraint set is the common elements of the set of fixed points of a demicontractive mapping and the solutions set of monotone equilibrium problems. Some other methods for solving this problem have been studied extensively (see [3, 19, 36]).

In this paper, motivated by the extragradient method for equilibrium and fixed point problems in [2], parallel techniques in [4] and the subgradient projections in [31, 19], we propose a new iterative algorithm to solve the problem (1.2), replacing the metric projection as usual by inexact projection.

The remainder of this paper is organized as follows. In the next section, we recall some basic definitions, properties of approximation projection and some technical lemmas. Section 3 is devoted to the presentation of our scheme and its convergence results. In Section 5, we particularize the proposed method to the equilibrium problem over a fixed point set, and the bilevel equilibrium problem. Some preliminary computational results and comparisons are presented in the last section.

2. Preliminaries

Let C be a nonempty, closed and convex subset of $\mathcal{H}$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. A mapping $S : C \to C$ is called:
(i) *demicontractive* with constant ξ, shortly $\xi-$demicontractive, if $Fix(S) \neq \emptyset$ and there exists $\xi \in [0,1)$ such that

$$\|S(x) - x^*\|^2 \leq \|x - x^*\|^2 + \xi\|x - S(x)\|^2, \quad \forall x \in C, x^* \in Fix(S);$$

(ii) quasinonexpansive, if $Fix(S) \neq \emptyset$ and

$$\|S(x) - x^*\| \leq \|x - x^*\|, \quad \forall x \in C, x^* \in Fix(S);$$

(iii) *demiclosed* at zero, if for each $\{x^k\} \subset C$, then

$$\left\{x^k \rightharpoonup \hat{x}, \|S(x^k) - x^k\| \to 0\right\} \Rightarrow S(\hat{x}) = \hat{x}.$$

For each $x \in \mathcal{H}$, there exists a unique point in C, denoted by $Pr_C(x)$ satisfying

$$\|x - Pr_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

As usual, Pr_C is called *metric projection* of $\mathcal{H}$ on C. Then, a point $\bar{x} = Pr_C(x)$ if and only if $\bar{x} \in C$ which satisfies

$$\langle x - \bar{x}, \bar{x} - y\rangle \geq 0, \quad \forall y \in C.$$

Let $\epsilon > 0$. We may define that a point $w_x \in C$ is called a $\epsilon-$*metric projection* of $x \in \mathcal{H}$ onto C, if

$$\langle x - w_x, w_x - y\rangle \geq -\frac{\epsilon^2}{4}, \quad \forall y \in C. \tag{2.1}$$

The set of all $\epsilon-$metric projections of x onto C is denoted $Pr^{\epsilon}_C(x)$.

Lemma 2.1. ([6, Remark 2.1]) *For each $x, y \in \mathcal{H}$, the following holds*

$$\|w_x - w_y\|^2 \leq \|x - y\|^2 + \epsilon^2, \quad \forall w_x \in Pr^{\epsilon}_C(x), w_y \in Pr^{\epsilon}_C(y). \tag{2.2}$$

From the above definitions, it is clear that Pr_C is an $\epsilon-$metric projection onto C for all $\epsilon > 0$, quasinonexpansive on $\mathcal{H}$ and

$$\|w_x - w_y\| \leq \|x - y\| + \epsilon, \quad \forall x, y \in \mathcal{H}, w_x \in Pr^{\epsilon}_C(x), w_y \in Pr^{\epsilon}_C(y).$$

Let $f : C \times C \to \mathcal{R} \cup \{+\infty\}$ be a bifunction such that $f(x,x) = 0$ for all $x \in C$. Then,

– The $\epsilon-$*diagonal subdifferential* $\partial_2^\epsilon f(x,x)$ at $x \in C$ is given by

$$\begin{aligned}\partial_2^\epsilon f(x,x) =&\{w \in \mathcal{H} : f(x,y) - f(x,x) \geq \langle w, y-x\rangle - \epsilon, \quad \forall y \in C\}\\ =&\{w \in \mathcal{H} : f(x,y) + \epsilon \geq \langle w, y-x\rangle, \quad \forall y \in C\};\end{aligned}$$

– It is said to be *strongly monotone* on C with constant $\beta > 0$ (shortly $\beta-$ strongly monotone), if for each $x, y \in C$,

$$f(x,y) + f(y,x) \leq -\beta\|x-y\|^2.$$

– It is *pseudomonotone*, if for each $x, y \in C$,

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0.$$

– It is *Lipschitz-type* on C with constants $\gamma_1 > 0$ and $\gamma_2 > 0$, if for each $x, y, z \in C$,

$$f(x,y) + f(y,z) \geq f(x,z) - \gamma_1\|x-y\|^2 - \gamma_2\|y-z\|^2.$$

Let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued mapping. As usual, T is said to be
(a) $\epsilon-$*Lipschitz continuous* on C with constant $L > 0$ if

$$\rho(T(x), T(y)) \leq L\|x-y\| + \epsilon, \quad \forall x, y \in C,$$

where ρ denotes the Hausdorff distance. By definition, the Hausdorff distance of two sets A and B is defined as

$$\rho(A,B) := \max\{d(A,B), d(B,A)\},$$

where $d(A,B) := \sup_{a\in A}\inf_{b\in B}\|a-b\|$ and $d(B,A) := \sup_{b\in B}\inf_{a\in A}\|a-b\|$. In the case $L \in [0,1)$, the mapping T is called to be $\epsilon-$*contractive* with constant L on C. Let us note that the $0-$Lipschitz continuous is Lipschitz continuous;
(b) $\beta-$*strongly monotone* on C if

$$\langle w_x - w_y, x-y\rangle \geq \beta\|x-y\|^2, \quad \forall x, y \in C, w_x \in T(x), w_y \in T(y).$$

To investigate the convergence of our iteration schemes, we recall the following technical lemma which will be used in the sequel.

Lemma 2.2. ([8, Lemma 2.1]) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $g : C \times C \to \mathcal{R}$ be a bifunction such that $g(x,x) = 0$ for all $x \in C$, and for each $x \in C$, $g(x,y)$ is lower semicontinuous, convex and subdifferentiable on C respect to y. If g is β-strongly monotone on C and for each $\epsilon \geq 0$, $\partial_2^\epsilon g(x,x)$ is Lipschitz continuous with constant $L > 0$ on C, then the multivalued mapping*

$$S(x) := \{x - \gamma w_x : w_x \in \partial_2^\epsilon g(x,x)\}, \quad \forall x \in C,$$

is $2\sqrt{\gamma\epsilon}-$contractive with constant $\delta = \sqrt{1-\gamma(2\beta-\gamma L^2)}$, where $\gamma \in \left(0, \frac{2\beta}{L^2}\right)$.

Lemma 2.3. ([24, Remark 4.4]) *Let $\{a_k\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let k_0 be an integer such that $a_{k_0} \leq a_{k_0+1}$ and define, for all integers $k \geq k_0$,*

$$\tau(k) = \max\{i \in \mathcal{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}.$$

Then, $0 \leq a_k \leq a_{\tau(k)+1}$ *for all* $k \geq k_0$*. Furthermore, the sequence* $\{\tau(k)\}_{k\geq k_0}$ *is nondecreasing and tends to* $+\infty$ *as* $k \to \infty$*.*

Lemma 2.4. ([24, Remark 4.2]) *Assume that* $S : C \to C$ *is an* $m-$*demicontractive mapping such that* $Fix(S) \neq \emptyset$ *and* $\alpha \in [0, 1-m]$*. Then, the mapping*

$$S_\alpha = (1-\alpha)I + \alpha S$$

is quasinonexpansive on C*. Moreover,*

$$\|S_\alpha(x) - x^*\|^2 \leq \|x - x^*\|^2 - \alpha(1-m-\alpha)\|S(x) - x\|^2, \quad \forall x \in C, x^* \in Fix(S).$$

Lemma 2.5. ([2, Lemma 3.1]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$*, and a bifunction* $h : C \times C \to \mathcal{R} \cup \{+\infty\}$ *satisfies the conditions:*

- $h(x,x) = 0$ *for all* $x \in C$*;*
- *for each* $x \in C$*,* $h(x,\cdot)$ *is convex and subdifferentable on* C*;*
- h *is pseudomonotone on* C*;*
- h *is Lipschitz-type with constants* $\gamma_1 > 0$ *and* $\gamma_2 > 0$*.*

Then, if $\lambda \in \left(0, \min\left\{\frac{1}{2\gamma_1}, \frac{1}{2\gamma_2}\right\}\right)$*, then the mapping* S *is defined in, for each* $x \in C$*,*

$$y_x = \operatorname{argmin}\left\{\lambda h(x,y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\},$$
$$S(x) = \operatorname{argmin}\left\{\lambda h(y_x,y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\},$$

which is quasinonexpansive on C*.*

3. Main results

Now we will discuss the iteration scheme and the convergence of the parallel projection method with computing inexact subgradients and approximate metric projections.

Theorem 3.1. *Let* f *be* $\beta-$*strongly monotone and weakly continuous,* $\partial_2^\epsilon f(x,x)$ *be* $L-$ *Lipschitz continuous on* C*. For each* $i \in I$*, let the mapping* $S_i : C \to C$ *be* β_i-*demicontractive such that* $\Omega \neq \emptyset$*. Let* $g_j (j \in J)$ *be pseudomonotone, weakly continuous and Lipschitz-type with constants* c_{1j} *and* c_{2j}*. Suppose that the sequence* $\{x^k\}$ *is generated by the following scheme:*

$$\begin{cases} x^0 \in C, \\ y_i^k = (1-\alpha_{k,i})x^k + \alpha_{k,i}S_i(x^k), \quad \forall i \in I, \\ y^k := y_{i_0}^k, \ \textit{where } i_0 = \operatorname{argmax}\{\|y_i^k - x^k\| : i \in I\}, \\ z_j^k = \operatorname{argmin}\left\{\rho_{k,j}g_j(y^k,y) + \frac{1}{2}\|y - y^k\|^2 : y \in C\right\}, \\ \bar{z}_j^k = \operatorname{argmin}\left\{\rho_{k,j}g_j(z_j^k,y) + \frac{1}{2}\|y - y^k\|^2 : y \in C\right\}, \\ z^k := \bar{z}_{j_0}^k, \ \textit{where } j_0 = \operatorname{argmax}\{\|\bar{z}_j^k - y^k\| : j \in J\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(z^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(z^k,z^k). \end{cases} \tag{3.1}$$

Let the positive parameter sequences $\{\alpha_{k,i}\}(i \in I), \{\rho_{k,j}\}(j \in J), \{\epsilon_k\}, \{\gamma_k\}$ *and* $\{\tau_k\}$ *be satisfied the restriction set:*

$$\begin{cases} \tau \in (0,\beta), \tau_k \le \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < a \le \alpha_{k,i} \le \min\left\{\frac{1-\beta_i}{2} : i \in I\right\}, \\ 0 < \bar{a} \le \rho_{k,j} \le \bar{b} < \min\left\{\frac{1}{2c_{1j}}, \frac{1}{2c_{2j}} : j \in J\right\}, \\ \epsilon_k \le \gamma_k, \sum_{k=0}^{\infty} \epsilon_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \gamma_k = +\infty, \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \sum_{k=0}^{\infty} \gamma_k \tau_k < +\infty. \end{cases} \tag{3.2}$$

Then, the sequences $\{x^k\}, \{y^k\}$ *and* $\{z^k\}$ *converge strongly to the unique solution* x^* *of the problem* (1.2).

Proof. Let x^* be the unique solution of Problem (1.2). We divide the proof into several steps.

Step 1. The following assertion holds

$$a_{k+1} \le (1-\tau\gamma_k)a_k + \frac{\gamma_k(3+\|w_k^*\|)^2}{\tau} - \alpha_{k,i_0}(1-\alpha_{k,i_0}-\beta_{i_0})(1-\tau\gamma_k)\|x^k - S_{i_0}(x^k)\|^2, \tag{3.3}$$

where $a_k = \|x^k - x^*\|^2$ and w_k^* is the projection of x^k onto $\partial_2^{\tau_k} f(x^*, x^*)$. Moreover, $\lim_{k\to\infty} \|x^{k+1} - z^k\| = 0$ and three sequences $\{x^k\}, \{y^k\}$ and $\{z^k\}$ are bounded.

Proof of Step 1. Set $A_k(x) = x - \gamma_k \partial_2^{\tau_k} g(x,x)$ for all $x \in C$. By Lemma 2.1 and Lemma 2.2, and using the conditions (3.2), we obtain

$$\begin{aligned} \|x^{k+1} - x^*\| &\le \|(z^k - \gamma_k u^k) - x^*\| + \epsilon_k \\ &\le \|(z^k - \gamma_k u^k) - (x^* - \gamma_k w_k^*)\| + \gamma_k \|w_k^*\| + \epsilon_k \\ &\le \rho\left(A_k(z^k), A_k(x^*)\right) + \gamma_k \|w_k^*\| + \epsilon_k \\ &\le \rho_k \|z^k - x^*\| + 2\sqrt{\gamma_k \tau_k} + \gamma_k \|w_k^*\| + \epsilon_k \\ &\le (1-\tau\gamma_k)\|z^k - x^*\| + \gamma_k(2 + \|w_k^*\|) + \epsilon_k \\ &\le (1-\tau\gamma_k)\|z^k - x^*\| + \gamma_k(3 + \|w_k^*\|), \end{aligned} \tag{3.4}$$

where $\rho_k = \sqrt{1 - \gamma_k(2\beta - \gamma_k L^2)}$. This implies that

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\le \left[(1-\tau\gamma_k)\|z^k - x^*\| + \gamma_k(3 + \|w_k^*\|)\right]^2 \\ &= \left[(1-\tau\gamma_k)\|z^k - x^*\| + \tau\gamma_k \frac{3 + \|w_k^*\|}{\tau}\right]^2 \\ &\le (1-\tau\gamma_k)\|z^k - x^*\|^2 + \frac{\gamma_k(3 + \|w_k^*\|)^2}{\tau}. \end{aligned} \tag{3.5}$$

Set

$$z_j^x = \text{argmin}\left\{\rho_{k,j} g_j(x,y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\}$$

and

$$S_{k,j}(x) = \text{argmin}\left\{\rho_{k,j} g_j(z_j^x, y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\}.$$

By Lemma 2.5, note that $x^* \in Fix(S_{k,j})$ for all $k \in \mathcal{N}, j \in J$, for each fixed k, j, the mapping $S_{k,j}$ is quasinonexpansive. Then,

$$\|z^k - x^*\| = \|\bar{z}^k_{j_0} - x^*\| = \|S_{k,j_0}(y^k) - x^*\| \leq \|y^k - x^*\|, \quad \forall k \in \mathcal{N}. \tag{3.6}$$

For each $\alpha \in [0, 1)$, set $T_\alpha = (1 - \alpha)I + \alpha S_{i_0}$. By Lemma 2.4, we have

$$\begin{aligned}\|y^k - x^*\|^2 =&\|y^k_{i_0} - x^*\|^2\\ =&\left\|T_{\alpha_{k,i_0}}(x^k) - x^*\right\|^2\\ \leq&\|x^k - x^*\|^2 - \alpha_{k,i_0}(1 - \alpha_{k,i_0} - \beta_{i_0})\|x^k - S_{i_0}(x^k)\|^2.\end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned}\|x^{k+1} - x^*\|^2 \leq&(1 - \tau\gamma_k)\|z^k - x^*\|^2 + \frac{\gamma_k(3 + \|w^*_k\|)^2}{\tau}\\ \leq&(1 - \tau\gamma_k)\|x^k - x^*\|^2 + \frac{\gamma_k(3 + \|w^*_k\|)^2}{\tau}\\ &- \alpha_{k,i_0}(1 - \alpha_{k,i_0} - \beta_{i_0})(1 - \tau\gamma_k)\|x^k - S_{i_0}(x^k)\|^2\\ \leq&(1 - \tau\gamma_k)\|x^k - x^*\|^2 + \tau\gamma_k\frac{(3 + \|w^*_k\|)^2}{\tau^2}\\ \leq&\max\{\|x^k - x^*\|^2, K\}\\ \leq&...\\ \leq&\max\{\|x^0 - x^*\|^2, K\},\end{aligned} \tag{3.8}$$

where

$$K = \frac{1}{\tau^2}\sup_k\left\{(3 + \|w^*_k\|)^2\right\} < +\infty.$$

Consequently, $\{x^k\}$ is bounded and (3.3) is deduced from (3.8). From (3.7), it follows that $\|y^k - x^*\| \leq \|x^k - x^*\|$ and $\{y^k\}$ is also bounded, and the relation (3.6) implies the boundedness of $\{z^k\}$. Note that $x^{k+1} \in Pr^{\epsilon_k}_C(z^k - \gamma_k u^k), z^k \in Pr^{\epsilon_k}_C(z^k)$. Then,

$$0 \leq \lim_{k\to\infty}\left\|x^{k+1} - z^k\right\| \leq \lim_{k\to\infty}\left[\gamma_k\|u^k\| + \epsilon_k\right] = 0.$$

This completes the proof of Step 1.

Step 2. Let us consider two following cases:

Case 2.1. There exists $k_0 \in \mathcal{N}$ such that $a_{k+1} \leq a_k$ for all $k \geq k_0$ and hence $\lim_{k\to\infty} a_k = A < +\infty$. Combining $\lim_{k\to\infty}\gamma_k = 0$, (3.4), (3.6), (3.7) and the boundedness of $\{x^k\}$, we have

$$\lim_{k\to\infty}\|z^k - x^*\|^2 = \lim_{k\to\infty}\|y^k - x^*\|^2 = A.$$

By the definition of i_0 and $0 < a \le \alpha_{k,i} \le \frac{1-\beta_i}{2}$ for all $i \in I$, we have

$$\begin{aligned}\frac{1-\beta_{i_0}}{2}\|y_i^k - x^k\|^2 \le& \frac{1-\beta_{i_0}}{2}\|y_{i_0}^k - x^k\|^2\\ =&\frac{\alpha_{k,i_0}(1-\beta_{i_0})}{2}\|S_{i_0}(x^k) - x^k\|^2\\ \le&\alpha_{k,i_0}(1-\beta_{i_0}-\alpha_{k,i_0})\|x^k - S_{i_0}(x^k)\|^2\\ \le&\|x^k - x^*\|^2 - \|y^k - x^*\|^2,\end{aligned}$$

where the last relation is deduced from (3.7). Then, we have $\lim_{k\to\infty}\|y_i^k - x^k\|^2 = 0$ and hence

$$0 \le a\lim_{k\to\infty}\|S_i(x^k) - x^k\| \le \lim_{k\to\infty}\alpha_{k,i}\|S_i(x^k) - x^k\| = \lim_{k\to\infty}\|y_i^k - x^k\| = 0, \quad \forall i \in I. \tag{3.9}$$

Applying Lemma 2.5 for

$$x := y^k; \bar{x} := x^*; \lambda := \rho_{k,j_0}; \gamma_1 := c_1; y_x := z_{j_0}^k; \gamma_2 := c_2; S(x) := z^k; h := g_{j_0},$$

we obtain that

$$\begin{aligned}\|z^k - x^*\|^2 =& \|\bar{z}_{j_0}^k - x^*\|^2\\ \le&\|y^k - x^*\|^2 - (1-2\rho_{k,j_0}c_1)\|z_{j_0}^k - y^k\|^2 - (1-2\rho_{k,j_0}c_2)\|z_{j_0}^k - \bar{z}_{j_0}^k\|^2\\ \le&\|y^k - x^*\|^2 - (1-2\bar{b}c_1)\|z_{j_0}^k - y^k\|^2 - (1-2\bar{b}c_2)\|z_{j_0}^k - \bar{z}_{j_0}^k\|^2.\end{aligned}$$

The last conclusion is deduced from the assumptions $0 < \rho_{k,j} \le \bar{b} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ for all $j \in J, k \in \mathcal{N}$. This together $j_0 = \text{argmax}\{\|\bar{z}_j^k - y^k\| : j \in J\}$ implies that

$$0 \le \lim_{k\to\infty}\|z_{j_0}^k - y^k\| \le \frac{1}{1-2\bar{b}c_1}\lim_{k\to\infty}\left(\|y^k - x^*\|^2 - \|\|z^k - x^*\|^2\right) = 0.$$

By a similar way, we also have $\lim_{k\to\infty}\|z_{j_0}^k - z^k\| = \lim_{k\to\infty}\|z_{j_0}^k - \bar{z}_{j_0}^k\| = 0$. Then,

$$0 \le \lim_{k\to\infty}[\|z^k - y^k\|] \le \lim_{k\to\infty}\left[\|z_{j_0}^k - y^k\| + \|z_{j_0}^k - z^k\|\right] = 0.$$

By the definition of j_0, we have

$$0 \le \lim_{k\to\infty}\|\bar{z}_j^k - y^k\| \le \lim_{k\to\infty}\|z^k - y^k\| = 0. \tag{3.10}$$

By Step 1, $\{y^k\}$ is bounded, we may assume that $y^{k_s} \rightharpoonup \bar{x} \in C$ and

$$\liminf_{k\to\infty}[-f(y^k, x^*)] = \lim_{s\to\infty}[-f(y^{k_s}, x^*)] = -f(\bar{x}, x^*),$$

where the last equality is deduced from the weak continuity of f. Since (3.9), we have $x^{k_s} \rightharpoonup \bar{x}$. Thus, $\bar{z}_j^{k_s} \rightharpoonup \bar{x}$ as $s \to \infty$ for all $j \in J$. Using the demiclosed property at zero of S_i for all $i \in I$, $x^{k_s} \rightharpoonup \bar{x} \in C$ and (3.9), we obtain $\bar{x} \in \cap_{i\in I}Fix(S_i)$. By a similar way as the proof of Lemma 2.5 that since

$$z_j^k = \text{argmin}\left\{\rho_{k,j}g_j(y^k, y) + \frac{1}{2}\|y - y^k\|^2 : y \in C\right\},$$

we have

$$\rho_{k,j}\left[g_j(y^k,y)-g_j(y^k,z_j^k)\right]\geq\left\langle z_j^k-y^k,z_j^k-y\right\rangle,\quad\forall y\in C,j\in J.$$

Substituting $y=\bar{z}_j^k\in C$ into this inequality, we obtain

$$\rho_{k,j}\left[g_j(y^k,\bar{z}_j^k)-g_j(y^k,z_j^k)\right]\geq\left\langle z_j^k-y^k,z_j^k-\bar{z}_j^k\right\rangle. \tag{3.11}$$

Using the Lipschitz-type property of g_j and the relation (3.11), we have

$$\begin{aligned}\rho_{k,j}g_j(z_j^k,\bar{z}_j^k)\geq&\rho_{k,j}[g_j(y^k,\bar{z}_j^k)-g_j(y^k,z_j^k)]-c_1\rho_{k,j}\|z_j^k-y^k\|^2-c_2\rho_{k,j}\|\bar{z}_j^k-z_j^k\|^2\\ \geq&\left\langle z_j^k-y^k,z_j^k-\bar{z}_j^k\right\rangle-c_1\rho_{k,j}\|z_j^k-y^k\|^2-c_2\rho_{k,j}\|\bar{z}_j^k-z_j^k\|^2.\end{aligned} \tag{3.12}$$

Since

$$\bar{z}_j^k=\operatorname{argmin}\left\{\rho_{k,j}g_j(z_j^k,y)+\frac{1}{2}\|y-y^k\|^2:y\in C\right\},$$

we have

$$\rho_{k,j}\left[g_j(z_j^k,y)-g_j(z_j^k,\bar{z}_j^k)\right]\geq\left\langle\bar{z}_j^k-y^k,\bar{z}_j^k-y\right\rangle,\quad\forall y\in C,j\in J. \tag{3.13}$$

Adding the two inequalities (3.12) and (3.13), we get

$$\begin{aligned}\rho_{k,j}g_j(z_j^k,y)\geq&\left\langle z_j^k-y^k,z_j^k-\bar{z}_j^k\right\rangle-c_1\rho_{k,j}\|z_j^k-y^k\|^2-c_2\rho_{k,j}\|\bar{z}_j^k-z_j^k\|^2\\&+\left\langle\bar{z}_j^k-y^k,\bar{z}_j^k-y\right\rangle,\quad\forall y\in C,j\in J.\end{aligned}$$

Consequently,

$$\begin{aligned}\rho_{k_s,j}g_j(z_j^{k_s},y)\geq&\left\langle z_j^{k_s}-y^{k_s},z_j^{k_s}-\bar{z}_j^{k_s}\right\rangle-c_1\rho_{k_s,j}\|z_j^{k_s}-y^{k_s}\|^2-c_2\rho_{k_s,j}\|\bar{z}_j^{k_s}-z_j^{k_s}\|^2\\&+\left\langle\bar{z}_j^{k_s}-y^{k_s},\bar{z}_j^{k_s}-y\right\rangle,\quad\forall y\in C,j\in J.\end{aligned}$$

Letting $s\to\infty$ in the above inequality and using the weak continuity of g_j, the assumption $0<a\leq\rho_{k,j}\leq b$ and (3.10), we obtain

$$0\leq\limsup_{s\to\infty}\rho_{k_s,j}g_j(z_j^{k_s},y)\leq g_j(\bar{x},y),\quad\forall y\in C.$$

Then, $\bar{x}\in\cap_{j\in J}S(C,g_j)$ and so $\bar{x}\in\cap_{i\in I}Fix(S_i)\cap_{j\in J}S(C,g_j)$.
From $u^k\in\partial_2^{\tau_k}f(z^k,z^k)$ and $f(z^k,z^k)=0$, it follows

$$f(z^k,x^*)=f(z^k,x^*)-f(z^k,z^k)\geq\langle u^k,x^*-z^k\rangle-\tau_k. \tag{3.14}$$

Combining (2.2), (3.6), (3.7) and (3.14), note that $x^{k+1}=Pr_C^{\epsilon_k}(z^k-\gamma_k u^k)$, we get

$$\begin{aligned}\|x^{k+1}-x^*\|^2=&\|Pr_C^{\epsilon_k}(z^k-\gamma_k u^k)-Pr_C^{\epsilon_k}(x^*)\|^2\\ \leq&\|z^k-\gamma_k u^k-x^*\|^2+\epsilon_k^2\\ =&\|z^k-x^*\|^2-2\gamma_k\langle u^k,z^k-x^*\rangle+\gamma_k^2\|u^k\|^2+\epsilon_k^2\\ \leq&\|z^k-x^*\|^2+2\gamma_k[f(z^k,x^*)+\tau_k]+\gamma_k^2\|u^k\|^2+\epsilon_k^2\\ \leq&\|x^k-x^*\|^2+2\gamma_k f(z^k,x^*)+2\gamma_k\tau_k+\gamma_k^2\|u^k\|^2+\epsilon_k^2.\end{aligned} \tag{3.15}$$

This implies

$$2\sum_{i=k_0}^{k}\gamma_i[-f(z^i,x^*)]\leq\|x^{k_0}-x^*\|^2+2\sum_{i=k_0}^{k}\gamma_i\tau_i+M\sum_{i=k_0}^{k}\gamma_i^2+\sum_{i=k_0}^{k}\epsilon_i^2,\quad\forall k\geq k_0, \tag{3.16}$$

where $M = \sup\{\|u^k\|^2 : k \in \mathcal{N}\} < +\infty$. Under the condition set (3.2) and

$$\liminf_{k\to\infty} \left[-f(z^k, x^*)\right] = \lim_{s\to\infty} \left[-f(y^{k_s}, x^*)\right] = \liminf_{k\to\infty} \left[-f(y^k, x^*)\right] = -f(\bar{x}, x^*) \geq 0,$$

we may include that $\liminf_{k\to\infty}[-f(y^k, x^*)] = 0$. On the other hand, the $\beta-$strong monotonicity of f yields

$$\begin{aligned} 0 \leq & f(x^*, \bar{x}) \\ = & \liminf_{k\to\infty} f(x^*, y^k) \\ \leq & \liminf_{k\to\infty}[-\beta\|y^k - x^*\|^2 - f(y^k, x^*)] \\ = & -\beta A + \liminf_{k\to\infty}[-f(y^k, x^*)] \\ = & -\beta A. \end{aligned}$$

Hence $A = 0$, which implies $x^k \to x^*, y^k \to x^*$ and $z^k \to x^*$ as $k \to \infty$.

Case 2.2. There does not exist $k_1 \in \mathcal{N}$ such that $a_{k+1} \leq a_k$ for all $k \geq k_1$. So there exists a positive integer k_0 such that $a_{k_0} \leq a_{k_0+1}$. By Lemma 2.3, Maingé proposed a subsequence $\{a_{\xi(k)}\}$ of $\{a_k\}$ which is defined as

$$\xi(k) = \max\{i \in \mathcal{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}.$$

Then, he showed that

$$\xi(k) \nearrow +\infty, 0 \leq a_k \leq a_{\xi(k)+1}, a_{\xi(k)} \leq a_{\xi(k)+1}, \quad \forall k \geq k_0. \tag{3.17}$$

By Step 1, $\{z^{\xi(k)}\}$ is bounded, and it has a convergent subsequence. Without general, we may assume that $\lim_{k\to\infty} a_{\xi(k)} = B < +\infty$ and $z^{\xi(k)} \rightharpoonup \bar{x} \in C$. Similar to Case 2.1, we also claim that $\bar{x} \in \Omega$ and $\lim_{k\to\infty} f(x^*, z^{\xi(k)}) = f(x^*, \bar{x}) \geq 0$. Using (3.15) and the $\beta-$strong monotonicity of f, i.e.,

$$f(z^k, x^*) \leq -f(x^*, z^k) - \beta\|z^k - x^*\|^2,$$

we have

$$\begin{aligned} a_{k+1} \leq & a_k + 2\gamma_k f(z^k, x^*) + 2\gamma_k\tau_k + \gamma_k^2\|u^k\|^2 + \epsilon_k^2 \\ \leq & a_k + 2\gamma_k \left[-f(x^*, z^k) - \beta\|z^k - x^*\|^2\right] + 2\gamma_k\tau_k + +\gamma_k^2\|u^k\|^2 + \epsilon_k^2. \end{aligned} \tag{3.18}$$

It results

$$\begin{aligned} 2\gamma_{\xi(k)} f(x^*, z^{\xi(k)}) \leq & \, a_{\xi(k)} - a_{\xi(k)+1} - 2\gamma_{\xi(k)}\beta\|z^{\xi(k)} - x^*\|^2 \\ & + 2\tau_{\xi(k)}\gamma_{\xi(k)} + \gamma_{\xi(k)}^2\|u^{\xi(k)}\|^2 + \epsilon_{\xi(k)}^2 \\ \leq & -2\gamma_{\xi(k)}\beta\|z^{\xi(k)} - x^*\|^2 + 2\tau_{\xi(k)}\gamma_{\xi(k)} \\ & + \gamma_{\xi(k)}^2\|u^{\xi(k)}\|^2 + \epsilon_{\xi(k)}^2 \end{aligned}$$

Since

$$\sum_{k=0}^{\infty} \gamma_k = +\infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \quad \sum_{k=0}^{\infty} \gamma_k\tau_k < +\infty$$

and by Step 1 that $\{u^k\}$ is bounded, we deduce that

$$\limsup_{k\to\infty} f\left(x^*, z^{\xi(k)}\right) = 0 \Rightarrow \lim_{k\to\infty} f\left(x^*, z^{\xi(k)}\right) = 0. \tag{3.19}$$

By (3.7), Remark 2.1 and using the $\beta-$strong monotonicity of f, we have

$$\begin{aligned}
a_{k+1} =& \|Pr_C^{\epsilon_k}(z^k - \gamma_k u^k) - Pr_C^{\epsilon_k}(x^*)\|^2 \\
\leq& \|z^k - \gamma_k u^k - x^*\|^2 + \epsilon_k^2 \\
=& \|z^k - x^*\|^2 + 2\gamma_k\langle u^k, x^* - z^k\rangle + \gamma_k^2\|u^k\|^2 + \epsilon_k^2 \\
\leq& \|z^k - x^*\|^2 + 2\gamma_k f(z^k, x^*) + \gamma_k^2\|u^k\|^2 + \epsilon_k^2 \\
\leq& \|z^k - x^*\|^2 + 2\gamma_k\left[-f(x^*, z^k) - \beta\|z^k - x^*\|^2\right] + \gamma_k^2\|u^k\|^2 + \epsilon_k^2 \\
=& (1 - 2\beta\gamma_k)\|z^k - x^*\|^2 - 2\gamma_k f(x^*, z^k) + \gamma_k^2\|u^k\|^2 + \epsilon_k^2 \\
\leq& (1 - 2\beta\gamma_k)a_k - 2\gamma_k f(x^*, z^k) + \gamma_k^2\|u^k\|^2 + \epsilon_k^2.
\end{aligned}$$

This implies that

$$a_{\xi(k)+1} \leq \left[1 - 2\beta\gamma_{\xi(k)}\right] a_{\xi(k)} - 2\gamma_{\xi(k)} f(x^*, z^{\xi(k)}) + \gamma_{\xi(k)}^2\|u^{\xi(k)}\|^2 + \epsilon_{\xi(k)}^2.$$

Combining this and (3.17), we obtain

$$a_{\xi(k)} \leq \left[1 - 2\beta\gamma_{\xi(k)}\right] a_{\xi(k)} - 2\gamma_{\xi(k)} f(x^*, z^{\xi(k)}) + \gamma_{\xi(k)}^2\|u^{\xi(k)}\|^2 + \epsilon_{\xi(k)}^2.$$

Thus

$$2\beta a_{\xi(k)} \leq -2f(x^*, z^{\xi(k)}) + \gamma_{\xi(k)}\|u^{\xi(k)}\|^2 + \epsilon_{\xi(k)}.$$

Letting the limit yields

$$\lim_{k\to\infty} a_{\xi(k)} = 0,$$

and hence $\lim_{k\to\infty} a_{\xi(k)+1} = 0$. Combining this and (3.17), we have $\lim_{k\to\infty} a_k = 0$. Thus, $\{x^k\}$ and $\{y^k\}$ converge strongly to x^*. This completes the proof. □

4. Applications

In this section, we suppose that $f, S_i(i \in I)$ and $g : C \times C \to \mathcal{R} \cup \{+\infty\}$ satisfy the following assumptions:

(1) The bifunction f is $\beta-$strongly monotone, weakly continuous and $\partial_2^\epsilon f(x, x)$ is Lipschitz continuous on C with constant $L > 0$ for all $\epsilon > 0$;
(2) The mappings $\{S_i : i \in I\}$ are β_i-demicontractive;
(3) The bifunction g is pseudomonotone, weakly continuous, Lipschitz-type with constants $c_1 > 0$ and $c_2 > 0$, $g(x, x) = 0$ for all $x \in C$.

When $S_i(i \in I)$ is the identity mapping and $g_j = g(j \in J)$, we obtain the following corollary as an immediate consequence of Theorem 3.1.

Corollary 4.1. *Let positive parameter sequences* $\{\rho_k\}, \{\epsilon_k\}, \{\gamma_k\}$ *and* $\{\tau_k\}$ *satisfy the restriction set:*

$$\begin{cases} \tau \in (0,\beta), 0 < \tau_k \leq \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < \bar{a} \leq \rho_k \leq \bar{b} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \\ \epsilon_k \leq \gamma_k, \sum_{k=0}^{\infty} \epsilon_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \gamma_k = +\infty, \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \sum_{k=0}^{\infty} \gamma_k\tau_k < +\infty. \end{cases}$$

Then, the sequences $\{x^k\}$ *and* $\{y^k\}$ *defined by the iteration scheme:*

$$\begin{cases} x^0 \in C, \\ y^k = \operatorname{argmin}\left\{\rho_k g(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ z^k = \operatorname{argmin}\left\{\rho_k g(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(z^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(z^k, z^k), \end{cases} \tag{4.1}$$

converge strongly to the unique solution of the bilevel equilibrium problem (1.5).

In the case $g_j = 0 (j \in J)$, the problem (1.2) is formulated in the equilibrium problem over the fixed point set of the demicontractive mappings $S_i (i \in I)$. By Theorem 3.1, the iteration scheme for solving the problem (1.4) and its convergence are given as the following results.

Corollary 4.2. *Suppose that the sequences* $\{x^k\}$ *and* $\{z^k\}$ *are generated by the scheme:*

$$\begin{cases} x^0 \in C, \\ y_i^k = (1 - \alpha_{k,i})x^k + \alpha_{k,i} S_i(x^k), \quad \forall i \in I, \\ y^k := y_{i_0}^k, \ \textit{where } i_0 = \operatorname{argmax}\{\|y_i^k - x^k\| : i \in I\}, \\ x^{k+1} \in Pr_C^{\epsilon_k}(y^k - \gamma_k u^k), u^k \in \partial_2^{\tau_k} f(y^k, y^k). \end{cases} \tag{4.2}$$

Let the positive parameter sequences $\{\alpha_{k,i}\}(i \in I), \{\epsilon_k\}, \{\gamma_k\}$ *and* $\{\tau_k\}$ *be satisfied the conditions:*

$$\begin{cases} \tau \in (0,\beta), 0 < \tau_k \leq \gamma_k < \min\left\{\frac{2\beta}{L^2}, \frac{2(\beta-\tau)}{L^2-\tau^2}, \frac{1}{\tau}\right\}, \\ 0 < a \leq \alpha_{k,i} \leq \min\left\{\frac{1-\beta_i}{2} : i \in I\right\}, \\ \epsilon_k \leq \gamma_k, \sum_{k=0}^{\infty} \epsilon_k^2 < +\infty, \\ \sum_{k=0}^{\infty} \gamma_k = +\infty, \sum_{k=0}^{\infty} \gamma_k^2 < +\infty, \sum_{k=0}^{\infty} \gamma_k\tau_k < +\infty. \end{cases}$$

Then, the sequences $\{x^k\}$ *and* $\{y^k\}$ *converge strongly to a unique solution* x^* *of the problem* (1.4).

5. Numerical results

In this section, we give some numerical experiments for the schemes (4.1) and (4.2). All the programming is implemented in MATLAB R2014a running on a PC with Intel(R) Core(TM) i5-7360U CPU @ 2.30GHz 8.00GB Ram. We will compare the convergence of Scheme (4.2) and the subgradient-type method proposed by Iiduka and Yamada in [33, Algorithm 3.2], the scheme (3.1) and the contraction proximal algorithm in [18, Algorithm 4.1], Scheme (4.1) and the approximate subgradient method of Anh in [5, Algorithm 2].

Example 5.1. [7, Example 5.1] Let C be a polyhedral convex set given by

$$C = \begin{cases} x \in \mathcal{R}^5_+, \\ 0.1 \le x_1, 0.1 \le x_5, x_i \le 1, \quad \forall i = 1, ..., 5, \\ x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \le 3. \end{cases}$$

The mapping $S : C \to C$ is defined by

$$S(x) = \left(\frac{1}{3}x_1, \sin x_2, \frac{1}{3}x_3, x_4, \sin^3(x_5) \right).$$

The equilibrium bifunction $f : \mathcal{R}^5 \times \mathcal{R}^5 \to \mathcal{R}$ is given in the form

$$f(x, y) = \langle F(x) + Qy + q, y - x\rangle,$$

where A is a 5×5 matrix, B is a 5×5 skew-symmetric matrix, D is a 5×5 diagonal matrix, the matrix $Q = AA^T + B + D$ generated as suggested in [7], q is a vector in $\mathcal{R}^5$, $\eta > 1 + \|Q\|$ and the mapping F is defined by

$$F(x) = (\eta x_1 + \eta x_2 + \sin(x_1), -\eta x_1 + \eta x_2 + \sin(x_2), (\eta - 1)x_3, (\eta - 1)x_4, (\eta - 1)x_5).$$

Then, f is strongly monotone with constant $\beta = \eta - \|Q\| - 1$ in the case $\eta > 1 + \|Q\|$. For each $x \in \mathcal{R}^5$, the subdifferential is computed by

$$\partial_2^{\epsilon} f(x, x) = \{F(x) + Qx + q\}, \quad \forall \epsilon > 0,$$

which is Lipschitz continuous with $L = \sqrt{2(2\eta^2 + 2\eta + 1)} + \|Q\|$. The scheme (4.1) for solving the problem (1.4) is simply formulated as follows:

$$\begin{cases} x^0 \in C, \\ y_i^k = (1 - \alpha_{k,i})x^k + \alpha_{k,i}S_i(x^k) \quad i = 1, 2, \\ y^k := y^k_{i_0}, \text{ where } i_0 = \text{argmax}\{\|y_i^k - x^k\| : i = 1, 2\}, \\ u^k = F(x^k) + Qx^k + q, x^{k+1} = Pr_C^{\epsilon_k}(y^k - \gamma_k u^k). \end{cases}$$

As usual, the tolerance error is ϵ-solution of the schemes (4.1) and (4.2), if

$$\max\{\|y^k - x^k\|, \|x^{k+1} - x^k\|\} \le \epsilon.$$

We choose in the Matlab that every entry of A and q is randomly and uniformly generated from $(-3, 3)$, every diagonal entry of D is randomly generated from $(0, 1)$ by

$$A = 6 * rand(5, 5) - 3, B = skewdec(5, 1), D = diag(1 : 5).$$

Taking $\eta := 50 + \|Q\| = 104.8319$ and $\beta = \eta + 9.3213$. As usual, if

$$\max\{\|x^{k+1} - y^k\|, \|y^k - x^k\|\} \le \epsilon,$$

then x^k is called $\epsilon-$ solution of the problem (1.4). The other parameters and data processions in each iteration scheme are chosen as follows:

(i) The scheme (4.2): $\alpha_{k,i} := 0.01 + \frac{1}{k+100}$ for all $i = 1, 2, \epsilon_k = \tau_k = 0, \gamma_k = \frac{1}{7k+10}$, for all $k \in \mathcal{N}$, the starting point $x^0 = (0.25, 0.35, 0.0, 0.1, 0.3)^T$.

(ii) The subgradient-type method (STM):

$$x^0 = (1,1,1,1,1)^T,\ \rho_0 = \|x^0\|, \epsilon_k = 0,$$

$$\xi^k = F(y^k) + Q(2x^k - y^k) + q \in \partial_2 f(y^k, x^k).$$

It is clear that $M \leq L + \|Q\|$ and we choose

$$M = 2L + \|Q\|,\ \lambda_k = \frac{1}{M^2} \in \left(0, \frac{2}{M^2}\right).$$

At each iteration step, $y^k \in C_k := \{x \in \mathcal{R}^5 : \|x\| \leq \rho_k + 1\}$ which satisfies $f(y^k, x^k) \geq 0$ and $\max\{f(y, x^k) : y \in C_k\} \leq f(y^k, x^k) + \epsilon_k$. Then, computing y^k is defined in follows

$$y^k = \text{argmin}\left\{-f(y, x^k) : y \in C_k\right\}.$$

The termination criterion is $\|x^{k+1} - y^k\| \leq \epsilon$.

The comparative results are reported in Table 1.

Problem	Scheme (4.2)		STM	
	Iter.	CPU(s)	Iter.	CPU(s)
1	87	11.0156	178	28.9063
2	87	10.1250	174	28.6875
3	88	9.7813	168	27.2813
4	85	8.5469	172	27.5000
5	90	7.4063	177	30.3281
6	89	7.9375	176	28.8594
7	88	7.9375	171	27.1406
8	87	8.2813	175	28.0625
9	86	7.0094	178	27.8438
10	92	8.1031	170	27.2656

TABLE 1. Comparative results of Scheme (4.2) and the subgradient-type algorithm with the tolerance $\epsilon = 10^{-3}$.

Example 5.2. In this example [19, Example 5.1], we compare the performance of the subgradient extragradient method (4.2) ($SubExtr$) with the exact version and the contraction proximal algorithm 4.2 ($ContrPA$) in [18]. Set

$$C = \{x \in \mathcal{R}^3 : \|x\| \leq 3\},$$

$$f(x,y) = \langle Ax + By + d, Py - Px\rangle,$$

where

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 6 & 1 \\ 1 & 2 & 7 \end{pmatrix},\ B = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix},\ P = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 6 \end{pmatrix},\ d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Let

$$D_1 = \{x \in \mathcal{R}^3 : x_1 + 2x_2 + 3x_3 - 46 \geq 0\},$$
$$D_2 = \{x \in \mathcal{R}^3 : 2x_1 - 2x_2 + x_3 - 5 \geq 0\}$$

and

$$D_3 = \{x \in \mathcal{R}^3 : -2x_1 + x_2 + x_3 - 6 \geq 0\}.$$

Then, the mapping

$$T(x) = Pr_C\left\{\frac{1}{3}x + \frac{2}{3}\left[\frac{1}{6}Pr_{D_1}(x) + \frac{1}{3}Pr_{D_2}(x) + \frac{1}{2}Pr_{D_3}(x)\right]\right\}, \quad \forall x \in \mathcal{R}^3$$

is nonexpansive.

Let $H = \{x \in \mathcal{H} : \langle a, x\rangle \leq b, a \neq 0\}$. The projection of $x \in \mathcal{H}$ onto H is defined in the form

$$Pr_H(x) = x - \frac{\langle a, x\rangle - b}{\|a\|^2}a.$$

Therefore, $Pr_{D_k}(x)$ $(k = 1, 2, 3)$ is given in the explicit form. Then, f is strongly monotone with constant

$$\beta = \|P^T(A - B)\|,$$

Lipschitz-type with constant

$$c_1 = c_2 = \frac{1}{2}\|P^T(A - B)\|.$$

Now, consider the problem (1.2) with $f_j = 0 (j \in J)$ and $S_i = T (i \in I)$. In both schemes, we use the same stopping criteria $\|x^{k+1} - x^k\| \leq 10^{-3}$. Parameters and $\partial_2 f$ are chosen as follows:

- Scheme $SubExtr$: $\beta_i = 0$, $\alpha_{k,i} = 0.0001 + \frac{1}{5k+1}$, $\gamma_k = \frac{1}{100k+55}$, $\partial_2 f(y^k, y^k) = \{P^T[(A + B)y^k + d]\}$.
- Algorithm $ContrPA$: $\lambda_k = \frac{0.001}{k^{0.9}}$ for all $k \geq 1$.

The performance of two results are tabulated in Table 2.

St. point x^0	$SubExtr$ Iter.	$SubExtr$ CPU(s)	$ContrPA$ Iter.	$ContrPA$ CPU(s)
$(0, 0, 0)$	30	2.0469	28	2.7406
$(0, 1, 0)$	50	3.8281	29	2.9375
$(0, 1, 1)$	34	2.5313	39	3.2969
$(1, 1, 0)$	53	4.0000	42	3.0313
$(1, 0, 0)$	39	2.9219	41	3.2188
$(1, 0, 1)$	35	2.6250	40	3.4688
$(2, 0, 0)$	53	3.9844	41	3.0469
$(2, 1, 0)$	60	4.2344	44	3.2344
$(2, 0, 1)$	45	3.3281	42	3.2656
$(1, 1, 2)$	33	2.3750	47	4.6927

TABLE 2. Comparison of algorithms in Example 5.2.

Example 5.3. Let $\mathcal{H} = \mathcal{R}^n$. The equilibrium bifunction f is given in Example 5.1. Let the matrices $P, \bar{P}$ be chosen such that $\bar{P}$ is symmetric positive semidefinite and $\bar{P} - P$ is negative semidefinite. The second cost bifunction g and the domain C are given in

$$g(x,y) = \left\langle Px + \bar{P}y + p, y - x \right\rangle, C = \{x \in \mathcal{R}^n : -1 \le x_i \le 1, \quad \forall i = 1, ..., 5\}. \quad (5.1)$$

Then, g is monotone and continuous, for each $x \in \mathcal{R}^n$, $g(x, \cdot)$ is differentiable convex on $\mathcal{R}^n$, Lipschitz-type with constants $c_1 = c_2 = \frac{1}{2}\|P - \bar{P}\|$. Every entry of A, B, P, p and q is randomly and uniformly generated from $(-3, 3)$, $\xi = 58$, every diagonal entry of D is randomly generated from $(0, 1), P = 3\bar{P} - I$ where I is the identity matrix.

The parameters and data processions in each algorithm are chosen as follows:

- Scheme (4.1):

$$\eta = \xi - 1 - \|Q\|, \ \rho_k = \frac{1}{3c_1 + 150 + k}, \ \gamma_k = \frac{1}{100k + 1},$$

  and the stopping rule $\max\{\|y^k - x^k\|, \|x^{k+1} - y^k\|\} \le \epsilon$;
- Approximate subgradient method (ASM): $\eta := 5 + \|Q\|$, $\xi_k := \frac{1}{k^2+10}$, $\lambda_k = 200$, $\beta_k = \frac{1}{7k+1}$ for all $k \in \mathcal{N}$. Then,

$$u^k \in \partial_2 f(y^k, y^k) = \{F(y^k) + Qy^k + q\},$$

$$w^k \in \partial_2 g(x^k, x^k) = \{(P + \bar{P})x^k + p\},$$

  and the stopping rule $\|x^{k+1} - x^k\| \le \epsilon$.

		Scheme (4.1)		ASM	
Test prob.	Dim.No.	Iter.	CPU(s)	Iter.	CPU(s)
1	$n = 5$	7	1.1406	14	1.4688
2	$n = 5$	12	2.8438	12	1.3906
3	$n = 10$	12	2.8281	148	20.2969
4	$n = 10$	10	2.3750	36	3.7969
5	$n = 15$	14	3.7500	76	9.1875
6	$n = 15$	16	4.8125	113	17.2188
7	$n = 20$	19	5.7188	360	56.4844
8	$n = 20$	18	5.0469	281	43.9688
9	$n = 25$	17	5.5156	126	20.3281
10	$n = 25$	16	4.9844	409	54.6601

TABLE 3. Comparative results of Scheme (4.1) and Method (ASM) with the tolerance $\epsilon = 10^{-3}$.

Example 5.4. Let $\mathcal{H}$ be the infinite dimensional Hilbert space $\mathcal{H} := L^2([0,1])$ with the inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt$$

for all $x, y \in \mathcal{H}$ and the norm associated with this inner product

$$\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}.$$

The feasible set $C := \{x \in \mathcal{H} : \|x\| \leq 1\}$. The cost bifunction $g_j : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is of the form

$$g_j(x(t), y(t)) = \langle \max\{0, x(t)\}, y(t) - x(t)\rangle$$

for all $t \in [0,1]$, $j \in J$ and $x \in \mathcal{H}$ and the Lyapunov-type functional

$$f(x, y) := \|x - y\|^2 + \langle \eta x + \xi y + q, y - x\rangle,$$

where $\eta > \xi + 2$. Then, we have

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) =& \|x-y\|^2 + \langle \eta x + \xi y + q, y - x\rangle + \|y - z\|^2 \\ &+ \langle \eta y + \xi z + q, z - y\rangle - \|x - z\|^2 - \langle \eta x + \xi z + q, z - x\rangle \\ =& (\eta - \xi - 2)\langle y - x, z - y\rangle \\ \geq& (\eta - \xi - 2)\|y - x\|\|z - y\| \\ \geq& \frac{\eta - \xi - 2}{2}\|y - x\|^2 + \frac{\eta - \xi - 2}{2}\|z - y\|^2. \end{aligned}$$

Thus, f is Lipschitz-type with constants $c_1 = c_2 = \frac{\eta - \xi - 2}{2}$. For each $x, y \in \mathcal{H}$, we have

$$\begin{aligned} f(x,y) + f(y,x) =& \|x - y\|^2 + \langle \eta x + \xi y + q, y - x\rangle + \|x - y\|^2 + \langle \eta y + \xi x + q, x - y\rangle \\ =& -(\eta - \xi - 2)\|x - y\|^2. \end{aligned}$$

So, f is $(\eta - \xi - 2)-$strongly monotone, $\partial_2^\nu f(x,x) = \{(\eta + \xi)x + q\}$ is Lipschitz continuous with constant $L := \eta + \xi$ for all $\nu > 0$. It is clearly that g_j is pseudomonotone and Lipschitz-type with constant $c_{1j} = c_{2j} = \frac{1}{2}$. Let also $H_i (i \in I := \{1,2,3\})$ be half-spaces defined by $H_i := \{x \in \mathcal{H} : \langle a_i, x\rangle \leq b_i\}$, where $a_i, b_i \in \mathcal{H}$. Furthermore, we assume that for each $i \in I$, the mapping $S_i = Pr_C Pr_{H_i}$ is demicontractive with constant $\beta_i = 0$. Note that for each $x \in \mathcal{H}$, the projection of x onto H_i is defined as follows:

$$Pr_{H_i}(x) = \begin{cases} x - \frac{\langle a_i, x\rangle - b_i}{\|a_i\|^2} a_i & \text{if } x \notin H_i, \\ x & \text{if } x \in H_i. \end{cases}$$

We take

$$a_i(t) = (2i + 1)t + 3, b_i(t) = 2t^2 + (4i - 5)t + i, \quad \forall i \in I, t \in \mathcal{R}.$$

We compute that

$$z_j^k = Pr_C[y^k - \rho_{k,j} \max\{0, y^k\}],$$

$$\bar{z}_j^k = Pr_C[y^k - \rho_{k,j} \max\{0, z_j^k\}],$$
$$u^k = (\eta + \xi)z^k + q,$$

and

$$x^{k+1} = Pr_C[z^k - \gamma_k u^k].$$

The parameters of the scheme (3.1) are chosen as follows

$$\alpha_{k,i} = 0.0001 + \frac{1}{5k+1}, \rho_{k,j} = 0.5, \gamma_k = \frac{1}{100k+55}.$$

The stopping criterion is $\|x^{k+1} - x^k\| \le \epsilon$ where $\epsilon = 10^{-3}$. Computational results are reported in Figure 1 for different starting points $x^0(t)$.

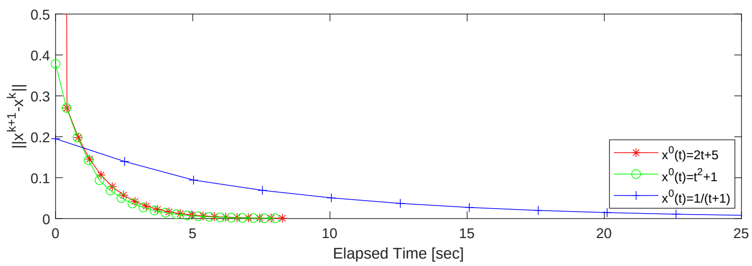

FIGURE 1. Example 4 for different starting points $x^0(t)$.

Denote that

- Test prob.: The tested problem;
- Iter.: The number of iteration loops;
- CPU(s): The averaged CPU-computation times (in second);
- Dim. No: The number of dimension n;
- St. point: The starting point $x^0 \in C$.

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REFERENCES

[1] P.N. Anh, *A new extragradient iteration algorithm for bilevel variational inequalities*, Acta Math. Vietnam., **37**(2012), 95-107.
[2] P.N. Anh, *A hybrid extragradient method extended to fixed point problems and equilibrium problems*, Optimiz., **62(2)**(2013), 271-283.
[3] P.N. Anh, T.T.H. Anh, T. Kuno, *Strong convergence theorems for variational inequalities on the solution set of Ky Fan inequalities*, Acta Math. Vietnamica, **42**(2017), 761-773.
[4] P.K. Anh, C.V. Chung, *Parallel hybrid methods for a finite family of relatively nonexpansive mappings*, Numer. Funct. Anal. Optim., **35**(2014), no. 6, 649-664.
[5] P.N. Anh, L.T. Hoai An, *New subgradient extragradient methods for solving monotone bilevel equilibrium problems*, Optim., **68(11)**(2019), 2097-2122.
[6] P.N. Anh, N.V. Hong, *New projection methods for equilibrium problems over fixed point sets*, Accepted by Optim. Lett., (2020).

[7] P.N. Anh, J.K. Kim, L.D. Muu, *An extragradient algorithm for solving bilevel pseudomonotone variational inequalities*, J. Glob. Optim., **52**(2012), 627-639.
[8] P.N. Anh, H.P. Tu, *Subgradient projection methods extended to monotone bilevel equilibrium problems in Hilbert spaces*, Num. Alg., (2020), DOI: 10.1007/s11075-020-00878-w.
[9] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63**(1994), 123-145.
[10] L.C. Ceng, A. Latif, Q.H. Ansari, *Hybrid extragradient method for hierarchical variational inequalities*, Fixed Point Theory Appl., **2014**(222)(2014), 1-35.
[11] L.C. Ceng, Y.C. Liou, C.F. Wen, *Hybrid extragradient viscosity method for general system of variational inequalities*, J. Ineq. Appl., **2015**(150)(2015), 1-43.
[12] L.C. Ceng, Y.C. Liou, C.F. Wen, *A hybrid extragradientmethod for bilevel pseudomonotone variational inequalities withmultiple solutions*, J. Nonl. Sc. Appl., **9**(2016), 4052-4069.
[13] L.C. Ceng, A. Petruşel, Y.C. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19**(2018), no. 2, 487-501.
[14] Q.L. Dong, X.H. Li, D. Kitkuan, Y.J. Cho, P. Kumam, *Some algorithms for classes of split feasibility problems involving paramonotone equilibria and convex optimization*, J. Inequal. Appl., **77**(2019), 1-23.
[15] P.M. Duc, L.D. Muu, *A splitting algorithm for a class of bilevel equilibrium problems involving nonexpansive mappings*, Optim., **65**(2016), 1855-1866.
[16] F. Facchinei, J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementary Problems*, Springer Verlag, New York, 2003.
[17] F. Giannessi, A. Maugeri, P.M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer, 2004.
[18] T.N. Hai, *Contraction of the proximal mapping and applications to the equilibrium problem*, Optim., **66**(2017), no. 3, 381-396.
[19] T.N. Hai, L.Q. Thuy, *A projected subgradient algorithm for bilevel equilibrium problems and applications*, J. Optim. Theory Appl., **175**(2017), no. 2, 411-431.
[20] H. Iiduka, I. Yamada, *A Subgradient-type method for the equilibrium problem over the fixed point set and its applications*, Optim., **58**(2009), 251-261.
[21] V.V. Kalashnikov, N.I. Kalashnikova, *Solving two-level variational inequality*, J. Glob. Optim., **8**(1996), 289-294.
[22] I.V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer, Berlin, 2000.
[23] I.V. Konnov, S. Kum, *Descent methods for mixed variational inequalities in a Hilbert space*, Nonlinear Anal. Theory Methods Appl., **47**(2001), 561-572.
[24] P.E. Maingé, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim., **47**(2008), 1499-1515.
[25] P.E. Maingé, A. Moudafi, *Coupling viscosity methods with the extragradient algorithm for solving equilibrium problems*, J. Nonl. Conv. Anal., **9**(2008), 283-294.
[26] Yu.V. Malitsky, *Projected reflected gradient methods for variational inequalities*, SIAM J. Optim., **25**(2015), 502-520.
[27] G. Mastroeni, *Gap function for equilibrium problems*, J. Glob. Optim., **27**(2004), 411-426.
[28] A. Moudafi, *Proximal methods for a class of bilevel monotone equilibrium problems*, J. Glob. Optim., **47**(2010), 287-292.
[29] M. Solodov, *An explicit descent method for bilevel convex optimization*, J. Conv. Anal., **14**(2007), 227-237.
[30] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2000), 431-446.
[31] P.T. Vuong, J.J. Strodiot, V.H. Nguyen, *Projected viscosity subgradient methods for variational inequalities with equilibrium problem constraints in Hilbert spaces*, J. Glob. Optim., **59**(2014), no. 1, 173-190.
[32] M.H. Xu, M. Li, C.C. Yang, *Neural networks for a class of bi-level variational inequalities*, J. Glob. Optim., **44**(2009), 535-552.

[33] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, Stud. Comput. Math., **8**(2001), 473-504.
[34] I. Yamada, N. Ogura, *Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim., **25**(2004) 619-655.
[35] Y. Yao, G. Marino, L. Muglia, *A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality*, Optim., **63**(2014), 559-569.
[36] T. Yuying, B.V. Dinh, D.S. Kim, S. Plubtieng, *Extragradient subgradient methods for solving bilevel equilibrium problems*, J. Ineq. Appl., **327**(2018), 1-21.

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