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REMARKS ON A LASALLE CONJECTURE ON GLOBAL ASYMPTOTIC STABILITY

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Abstract. In this paper we present some remarks on the following problem: Let X be a (real or complex) Banach space, $\Omega \subset X$ be an open convex subset and $f : \Omega \to \Omega$ be an operator. We suppose that: (i) $f \in C^1(X, X)$; (ii) the differential of f at x, $df(x) : X \to X$ is a Picard operator for all $x \in \Omega$; (iii) the fixed point set of f, $F_f \neq \emptyset$. The problem is in which conditions f is a Picard operator? In the case $X := \mathbb{R}^m$ or $X := \mathbb{C}^m$, this problem is in connection with a LaSalle Conjecture (J.P. LaSalle, *The Stability of Dynamical Systems*, SIAM, No. 25, 1976) and with the Belitskii-Lyubich Conjecture (G.R. Belitskii and Yu.I. Lyubich, *Matrix Norms and their Applications*, Birkhäuser, 1988).

We also formulate the following conjecture:

Let X be a Banach space, $\Omega \subset X$ be an open convex subset and $f : \Omega \to \Omega$ be an operator. We suppose that: (i) $f \in C^1(\Omega, X)$; (ii) $df^k(x)$ is a Picard operator, $\forall\, x \in \Omega$, $\forall\, k \in \mathbb{N}^*$; (iii) $F_f \neq \emptyset$. Then f is a Picard operator.

Some research directions are also presented.

Key Words and Phrases: Banach space, differentiable nonlinear operator, fixed point, iterate, spectral radius, global asymptotic stability, Picard operator, LaSalle Conjecture, Belitskii-Lyubich Conjecture, discrete Markus-Yamabe Conjecture, Ostrowski property, stability under operator perturbation.

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