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SEHGAL CONTRACTIONS ON MENGER SPACES

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. We present some remarks and comments on four proofs of Sehgal & Bharucha-Reid fixed point principle for probabilistic B-contractions.

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1. Introduction

A Sehgal-contraction, or a $B-$contraction, on a probabilistic metric space $(S, \mathcal{F})$ is a self-mapping A of S such that

$$(BC_L) \qquad F_{ApAq}(Lx) \geq F_{pq}(x), \ \forall p, q \in S$$

for some $L \in (0,1)$ and every x. As it is well known [1, 2, 4, 10, 11, 12, 19, 20], every $B-$contraction on a complete Menger space $(S, \mathcal{F}, Min)$ has a unique fixed point, which is globally attractive. Therefore $B-$contractions on complete Menger spaces $(S, \mathcal{F}, Min)$ belong to the class of Picard operators, extensively studied by I. A. Rus (see [15], [16] and [9]). In fact, the following more general result holds:

Theorem 1.1. *Every t-norm of Hadžić-type has the fixed point property for $B-$contractions.*

Indeed, let $(S, \mathcal{F}, T)$ be a complete Menger space such that T is of Hadžić-type and consider a B-contraction $A : S \to S$. Without loss of generality, we can suppose $L \in (0, \frac{1}{2}]$. Let $p_0 \in S$ and $x \in (0, \infty)$ be fixed. If m is a positive integer, then

$$F_{p_0 A^m p_0}(2x) \geq T(F_{p_0 A p_0}(x), F_{A p_0 A^m p_0}(x)) \geq T(F_{p_0 A p_0}(x), F_{p_0 A^{m-1} p_0}(2x))$$

and we can see by induction that

$$F_{p_0 A^m p_0}(2x) \geq T_m(F_{p_0 A p_0}(x)), \forall m \geq 1. \tag{1.1}$$

Thus we have, for all integers n, m:

$$F_{A^n p_0 A^{n+m} p_0}(2x) \geq F_{p_0 A^m p_0}(2x/L^n) \geq T_m(F_{p_0 A p_0}(x/L^n)). \tag{1.2}$$

Since T is of Hadžić-type (that is the iterates $\{T_m\}$ are equicontinuous at $a = 1$) and $F_{p_0 A p_0} \in D^+$ (so that $\lim_{t \to \infty} F_{p_0 A p_0}(t) = 1$), then

$$\lim_{n \longrightarrow \infty} F_{A^n p_0 A^{n+m} p_0}(2x) = 1, \tag{1.3}$$

uniformly in m, for each $x \in (0, \infty)$. By definition, this means that $\{A^n p_0\}$ is $\mathcal{F}$-Cauchy and the conclusion follows.

The proof of the following result of type Sherwood (compare with [20]) is easy to reproduce:

Lemma 1.2. *Let T be an lc-t-norm and fix an F in D^+. Let $S = \{1, 2, ...\}$ and define a probabilistic metric by*

$$\begin{cases} F_{nn+m}(x) = T^m[F(2^n x), F(2^{n+1} x), ..., F(2^{n+m} x)], n, m \in S \\ F_{nn} = \varepsilon_0 \end{cases} \tag{1.4}$$

Then $(S, \mathcal{F}, T)$ is a Menger space and the mapping $n \stackrel{A}{\rightarrow} n+1$ is a $B-$contraction with $L = \frac{1}{2}$.

The following theorem is a partial converse to Theorem 1.1:

Theorem 1.3. *If T is a continuous t-norm with the fixed point property for $B-$contractions, then T is of Hadžić-type.*

Proof. Suppose that T is not of Hadžić-type. Then there exists $a \in (0, 1)$ such that for each $1 > b > a$ there is $m_b \geq 1$ for which $T_{m_b}(b) < a$. Now let $b_n \in (a, 1)$ be increasing to 1. Then there exists $m_n \geq 1$ such that

$$T_{m_n}(b_n) < a, n = 1, 2, ... \tag{1.5}$$

Obviously we can suppose that m_n is increasing on n. Let $F \in D^+$ be defined by

$$F(x) = \begin{cases} 0 \ \ if \ \ \ x \leq 1 \\ b_1 \ \ if \ \ \ x \in (1, 2^{2+m_1}] \\ b_{n+1} \ \ if \ \ \ x \in (2^{2n+m_n}, 2^{2n+m_{n+1}}], \ \ n \geq 1 \end{cases} \tag{1.6}$$

If we consider the Menger space as in Lemma 1.2 with this F, then we have successively:

$$F_{n\ n+m_n}(1) \leq F_{n\ n+m_n}(2^n) = T^{m_n}[F(2^{2n}), F(2^{2n+1}), ..., F(2^{2n+m_n})] \leq$$

$$\leq T^{m_n}[F(2^{2n+m_n}), ..., F(2^{2n+m_n})] \leq T_{m_n}(b_n) < a$$

Therefore the sequence $\{A^n 1\}$ is not Cauchy, so that T does not have the fixed point property for $B-$contractions.

It is worth noting that in [1], G. L. Cain & R. Kasriel proved the Sehgal's result (for $T = Min$) by using the *Nishiura pseudo-metrics* d_λ, defined by

$$d_\lambda(p, q) = sup\{x \mid F_{pq}(x) \leq 1 - \lambda\}, \ \lambda \in (0, 1), \ p, q \in S.$$

As a matter of fact, the family $\{d_\lambda\}_{\lambda \in (0,1)}$ generates the $(\varepsilon, \lambda)-$topology on S. Moreover, for every $\lambda \in (0, 1)$,

$$d_\lambda(Ap, Aq) \leq L d_\lambda(p, q), \ \forall p, q \in S.$$

For if $d_\lambda(p, q) < r$ then $F_{pq}(r) > 1 - \lambda$ and the contraction condition (BC_L) implies $F_{ApAq}(Lr) > 1 - \lambda$, which shows that $d_\lambda(Ap, Aq) < Lr$. Hence one can apply the Banach contraction principle in the uniform space $(S, \{d_\lambda\}_{\lambda \in (0,1)})$.

As we will see, the result in (Sehgal &Bharucha-Reid, [19]) is also a consequence of the "fixed point alternative" in generalized (Luxemburg) complete metric spaces (see [6] or [13]). Incidentally, this method offers a sort of converse of the fixed point principle in Menger spaces under the t-norm $T_M = Min$, by giving a suitable family of (generalized) metric topologies on such a space. The method also suggests order theoretic proofs of the contraction principle for probabilistic $B-$contractions.

For terminology and notations, we refer to [2], [4] and [18].

2. Some generalized metrics on Menger spaces

Recall that D^+ denotes the set of the distribution functions of all nonnegative (real) random variables, which then are nondecreasing and left-continuous on $(0,\infty)$ and have limit 1 at ∞. Let we are given a probabilistic metric $\mathcal{F}: S \to D^+$, such that $(S, \mathcal{F}, Min)$ is a Menger space, and consider an element G of D^+ which will be fixed. Recall that $\mathcal{F}(p,q)$ is usually denoted by F_{pq}.

Theorem 2.1. *If we define the two-place function* d_G by

$$d_G(p,q) = inf\{a \mid a > 0 \text{ and } F_{pq}(ax) \geq G(x) \ , \ \forall x \in \mathbb{R}\},$$

then

(i) d_G *is a Luxemburg metric on* S;

(ii) *The* d_G-*topology is stronger than the* $(\varepsilon,\lambda)-$*topology*;

(iii) *If* $(S,\mathcal{F})$ *is complete, then* (S, d_G) *is complete.*

Proof. (i) Clearly d_G is symmetric and $d_G(p,p) = 0$. If $d_G(p,q) = 0$ then, for each $a > 0$, $F_{pq}(ax) \geq G(x)$ for all x. Now, if $y = ax$ is fixed and $x \to +\infty$, then $F_{pq}(y) \geq \lim_{x\to\infty} G(x) = 1$, that is $p = q$.

Suppose that $d_G(p,r) < \infty$ and $d_G(r,q) < \infty$. If $d_G(p,r) < a' < a$ and $d_G(r,q) < b' < b$, then

$$F_{pq}[(a'+b')x)] \geq Min\{F_{pr}(a'x), F_{rq}(b'x)\} \geq G(x)$$

which shows that $d_G(p,q) \leq a' + b' < a + b$. Therefore $d_G(p,q) \leq d_G(p,r) + d_G(r,q)$, and it follows that d_G is a Luxemburg metric.

(ii) Now, suppose that $\{p_n\}$ is d_G-convergent to p. Let $\varepsilon > 0$ and $\lambda \in (0,1)$ be given. Since $G \in D^+$, then there exists x_0 such that $G(x_0) > 1 - \lambda$. For $a < \frac{\varepsilon}{x_0}$, we choose n_0 such that $d_G(p_n, p) < a$ for all $n \geq n_0$. Therefore $F_{p_n p}(\varepsilon) \geq F_{p_n p}(ax_0) \geq G(x_0) > 1 - \lambda$, and we see that $\{p_n\}$ is $\mathcal{F}-$convergent to p.

(iii) Suppose that $\{p_n\}$ is d_G-Cauchy and $(S,\mathcal{F})$ is complete. Then, as above, we obtain that $\{p_n\}$ is $\mathcal{F}$-Cauchy and thus there exists $p \in S$ such that $\{p_n\}$ is $\mathcal{F}$-convergent to p. Let $a, \ \delta > 0$ be given. Then there exists n_0 such that $F_{p_n p_{n+m}}(ax) \geq G(x)$ for all $n > n_0$, all $m \geq 1$ and each x. Let $n > n_0$ and $x > 0$ be fixed. Since

$$F_{p_n p}((a+\delta)x) \geq Min\{F_{p_n p_{n+m}}(ax), F_{p_{n+m} p}(\delta x)\} \geq$$

$$\geq Min\{G(x), F_{p_{n+m}p}(\delta x)\},$$

then, by letting $m \to \infty$, $F_{p_n p}((a+\delta)x) \geq Min\{G(x), 1\} = G(x)$. Therefore $d_G(p_n, p) \leq a + \delta$, $\forall n \geq n_0$ and we can see that $\{p_n\}$ is d_G-convergent.

Example 2.2. Since

$$G(x) = \varepsilon_1(x) = \begin{cases} 0, & x \leq 1 \\ 1, & x > 1 \end{cases} \Rightarrow d_G(p,q) = inf\{x \mid F_{pq}(x) = 1\},$$

then d_G needs not be nontrivial: Every metric space (S, d) is a Menger space under $T_M = Min$ if we set $F_{pq}(x) = \frac{x}{x+d(p,q)}$, $\forall x \geq 0$. Clearly, $d_G(p,q) < \infty \Leftrightarrow p = q$. On the other hand, (S, d) can also be regarded as a Menger space under T_M with $F_{pq}(x) = \varepsilon_{d(p,q)}(x) = G(\frac{x}{d(p,q)})$, for $d(p,q) \neq 0$ and it is easily seen that $d_G(p,q) = d(p,q)$ in this case. The situation is reversed for $G(x) = \frac{x}{x+1}$, $\forall x \geq 0$.

Remark 2.3. The condition $\lim_{t\to\infty} G(t) = 1$ is essential for the last conclusions in Theorem 2.1, as shown by the following

Example 2.4. Let $\lambda \in (0,1)$ and $G(x) = \begin{cases} 0, & x \leq 1 \\ 1-\lambda, & x > 1 \end{cases}$. Then

$$d_G(p,q) = inf\{x \mid F_{pq}(x) \geq 1-\lambda\} =_{not} R_\lambda(p,q).$$

Generally, R_λ is a *pseudo-metric* and *the family* $\{R_\lambda\}_{\lambda\in(0,1)}$ does generate the $\mathcal{F}-$uniformity:

$$R_\lambda(p,q) < \varepsilon \Rightarrow F_{pq}(\varepsilon) \geq 1-\lambda \text{ and } F_{pq}(\varepsilon) > 1-\lambda \Rightarrow R_\lambda(p,q) < \varepsilon.$$

Notice also that $S = \mathbb{R}$ is a complete Menger space under $T_P = Prod$, if we set

$$F_{pq}(x) = e^{-\frac{|p-q|}{x}}, \ \forall\ x > 0, p, q \in \mathbb{R}.$$

As it is easily seen, $R_\lambda(p,q) = \frac{|p-q|}{-log(1-\lambda)}$ gives a complete *metric for each* λ, although T_P is *strictly weaker* than $T_M = Min$. Moreover, each R_λ generates the $(\varepsilon, \lambda)-$topology.

3. Other two proofs of the fixed point principle

The proof of the following result is easy to reproduce:

Lemma 3.1. *Every Sehgal-contraction on* $(S, \mathcal{F})$ *is a Banach-contraction on* (S, d_G). *Namely, if* $A : S \to S$ *verifies the condition* (BC_L), *then*

$$d_G(Ap, Aq) \leq L d_G(p,q), \forall p, q \in S.$$

By using this lemma and the alternative of fixed point, we can prove the Sehgal & Bharucha-Reid theorem ([19]).

Theorem 3.2. *Let A be a Sehgal-contraction on the complete Menger space $(S, \mathcal{F}, Min)$. Then A has a unique fixed point $p^\star$ and, for each $p \in S$, $p^\star = \lim_{n\to\infty} A^n p$ in the $(\varepsilon, \lambda)-$topology.*

Proof. Let $p \in S$ and $G := F_{pAp}$. By Lemma 3.1, A is d_G-contractive. Moreover, $d_G(p, Ap) = 1 < \infty$. By the fixed point alternative (see [6] or [7]), the sequence $A^n(p)$ converges to a fixed point $p^\star$ of A, in the metric d_G, and so in the $(\varepsilon, \lambda)-$topology. Clearly $p^\star$ is unique.

Remark 3.3. For G as in the above proof, $S_p := \{q,\ d_G(p, q) < \infty\}$ is a complete *metric* space. And $A^n p \in S_p$ for all $n \geq 1$. Therefore A has a unique fixed point in S_p. Since A has a unique fixed point in S, the theorem follows also in this way.

Remark 3.4. Let $(S, \mathcal{F}, Min)$ be a complete Menger space and suppose that, for some $G \in D^+$, the d_G-topology and the $(\varepsilon, \lambda)-$topology are identical. Then, for every Sehgal-contraction A on S, we have:

(i) For every $p \in S$, $A^n p$ is convergent to the unique fixed point of A;

(ii) For each $p \in S$ there exists $n \geq 0$ such that

$$F_{A^n p A^{n+1} p}(x) \geq G(x),\ \forall\ x.$$

Indeed, the first assertion follows from Theorem 3.2. The second assertion follows from the fixed point alternative, since (i) is always true. In fact, this assertions indicate, to a certain extent, the behavior of the values of $\mathcal{F}$:

Example 3.5. For $\beta > 0$, let $G(x) = \begin{cases} 0, & x \leq 1 \\ 1 - \frac{1}{x^\beta}, & x > 1 \end{cases}$. It is easy to see that $d_G(p, q) = sup\ \alpha^{\frac{1}{\beta}} d_\alpha(p, q)$. If d_G induces the (ε, λ)-topology on S, then for each $p \in S$ there exists $m \geq 0$ such that

$$F_{A^n p A^{n+1} p}(x) \geq 1 - \frac{1}{x^\beta}, \quad \forall x \geq 1\ ,\ \forall n \geq m.$$

Generally, from the fixed point alternative we obtain the following.

Theorem 3.6. *If A is a B-contraction on a complete Menger space $(S, \mathcal{F}, Min)$ then, for each $G \in D^+$ and each $p \in S$,* ***either***

(A_1) $A^n p$ is d_G convergent to the unique fixed point of A, **or**

(A_2) *for each* $n \geq 0$ *and each* $a > 0$ *there exists* $x_{n,a} > 0$ *such that*

$$F_{A^n p A^{n+1} p}(a x_{n,a}) < G(x_{n,a}).$$

Remark 3.7. Having in mind the above results as well as the methods in [3], [5], [9] and [21], one can introduce the following relation on $S \times \mathbb{R}$:

$$(p, \lambda) \leq_G (q, \mu) \Leftrightarrow \lambda \leq \mu \text{ and } F_{pq} \geq (\mu - \lambda) \circ G.$$

Recall that $\nu \circ G(x) = G(\frac{x}{\nu})$ for $\nu \neq 0$ and $\nu \circ G = \varepsilon_0 \Leftrightarrow \nu = 0$. Since $(a+b) \circ G = \tau_M(a \circ G, b \circ G)$, $\forall a, b \geq 0$, then $\leq_G$ is a partial order for every Menger space $(S, \mathcal{F}, Min)$ and any $G \in D^+$. Now, the method of DeMarr can be applied to the monotone mapping $B(p, \lambda) := (Ap, L\lambda)$ and we have an alternative proof of Theorem 3.2.

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