Fixed Point Theory, Volume 5, No. 2, 2004, 379-391
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

THE DARBOUX-IONESCU PROBLEM FOR THIRD ORDER HYPERBOLIC INCLUSIONS WITH MODIFIED ARGUMENT

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Abstract. In this paper we consider the Darboux-Ionescu Problem for a third order hyperbolic inclusion with modified argument of the form

$$u_{xyz} \in F(x, y, z, u(f(x, y, z), g(x, y, z), h(x, y, z))).$$

An existence theorem for a local solution of this problem is proved and some properties of the set of its solutions are established.
Key Words and Phrases: Multifunction, hyperbolic inclusion, upper-semicontinuity, initial values, absolutely continuous in Carathéodory's sense function.
2000 Mathematics Subject Classification: 35L30, 35R70, 47H10.

1. Introduction

In his PhD Thesis (1927) [17], D.V. Ionescu studied - for the first time in the mathematical literature - boundary value problems of Darboux, Cauchy, Picard and Goursat types for second order partial differential equations with modified argument.

More recently, a series of authors studied the same problems for second order hyperbolic equations of various forms and Darboux Problem for third order hyperbolic equations with modified argument [9] - [11].

This paper was presented at International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from August 24 to August 27, 2004.

In this paper we consider the Darboux-Ionescu Problem for the third order hyperbolic inclusion with modified argument

$$\begin{gathered}\frac{\partial^3 u(x,y,z)}{\partial x\partial y\partial z}\in F\left(x,y,z,u\left(f(x,y,z),g(x,y,z),h(x,y,z)\right)\right),\\(x,y,z)\in D=[0,a]\times[0,b]\times[0,c],\quad u\in\Omega\subset\mathbb{R}^n,\end{gathered}\tag{1.1}$$

with initial values

$$\begin{cases}u(x,y,0)=\varphi(x,y), & (x,y)\in D_1=[0,a]\times[0,b],\\ u(0,y,z)=\psi(y,z), & (y,z)\in D_2=[0,b]\times[0,c],\\ u(x,0,z)=\chi(x,z), & (x,z)\in D_3=[0,a]\times[0,c],\end{cases}\tag{1.2}$$

where φ, ψ, χ are absolutely continuous in Carathéodory's sense [1, §565 - §570], $\varphi\in C^*(D_1;\mathbb{R}^n)$, $\psi\in C^*(D_2;\mathbb{R}^n)$, $\chi\in C^*(D_3;\mathbb{R}^n)$ (see Definition 2.7 for $C^*\left(\Delta;\mathbb{R}^n\right)$, $\Delta\subset\mathbb{R}^2$) and they satisfy the conditions

$$\begin{cases}u(x,0,0)=\varphi(x,0)=\chi(x,0)=v^1(x), & x\in[0,a],\\ u(0,y,0)=\varphi(0,y)=\psi(y,0)=v^2(y), & y\in[0,b],\\ u(0,0,z)=\psi(0,z)=\chi(0,z)=v^3(z), & z\in[0,c],\\ u(0,0,0)=v^1(0)=v^2(0)=v^3(0)=v^0,\end{cases}\tag{1.3}$$

where $F:D\times\Omega\to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex and non-empty values, $\Omega\subset\mathbb{R}^n$ is an open subset and $f\in C(D;[0,a])$, $g\in C(D;[0,b])$, $h\in C(D;[0,c])$.

Under suitable assumptions, we prove an existence theorem for a local solution of the Darboux-Ionescu Problem (1.1)-(1.2) and that the set of its solutions is compact in the Banach space $C(D_0;\mathbb{R}^n)$, $D_0=[0,x_0]\times[0,y_0]\times[0,z_0]\subset D$; moreover, as a function of the initial values, this set defines an upper-semicontinuous multifunction.

In [28] we consider the Darboux-Ionescu Problem for second order hyperbolic inclusion with modified argument

$$\begin{gathered}\frac{\partial^2 z\left(x,y\right)}{\partial x\partial y}\in F\left(x,y,z\left(g\left(x,y\right),h\left(x,y\right)\right)\right),\quad (x,y)\in D=[0,a]\times[0,b]\,,\\ z\left(x,0\right)=\sigma\left(x\right),\quad x\in[0,a]\,,\quad z\left(0,y\right)=\tau\left(y\right),\quad y\in[0,b]\,,\end{gathered}$$

where $F:D\times\Omega\to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex and non-empty values, Ω is an open subset of $\mathbb{R}^n$, $g\in C\left(D;[0,a]\right)$, $h\in C\left(D;[0,b]\right)$, $\sigma\in AC\left([0,a]\,;\mathbb{R}^n\right)$, $\tau\in AC\left([0,b]\,;\mathbb{R}^n\right)$, $\sigma\left(0\right)=\tau\left(0\right)$.

In [29] we consider the Darboux Problem for third order inclusion with unmodified argument

$$\frac{\partial^3 u\left(x,y,z\right)}{\partial x \partial y \partial z} \in F\left(x,y,z,u\right), \quad (x,y,z) \in D = [0,a] \times [0,b] \times [0,c]\,,\ u \in \Omega \subset \mathbb{R}^n$$

with initial values (1.2) and the conditions (1.3).

In both papers [28], [29] the existence of local solutions of the problems there considered is proved and some properties of the solution set are established.

This study was suggested by several papers which deal with the Darboux Problem for third order hyperbolic equations [4], [5], [8], [13]-[15], [18], [22], [24], [26], [27] and with the Darboux-Ionescu Problem for third order hyperbolic equations with modified argument [9]-[11] and the Darboux problem for second and third order hyperbolic inclusions [25], [28], [29].

2. Preliminaries

The definitions and Theorem 2.1 in this section are taken from [1], [6]-[8], [19]-[21], [23].

Definition 2.1. Let X and Y be two non-empty sets. *A multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup_{x\in X} \Phi(x)$ is the *range* of Φ.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.
a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup_{x\in A} \Phi(x)$;
b) If $B \subset Y$, the *counterimage* of B by Φ is

$$\Phi^{-}(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The graph of Φ, denoted *graph*Φ, is the set

$$graph\ \Phi = \{(x,y) \in X \times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. Let us now take $\Phi : X \to 2^X$. An element $x \in X$ with the property $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.4. A single-valued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.5. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper-semicontinuous* if, for any closed subset $B \subset Y$, $\Phi^-(B)$ is closed in X.

Definition 2.6. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* if $\Phi^-(B) \in \mathcal{F}$ for every closed subset $B \subset Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.

Theorem 2.1. [23]. Let X and Y be two metric spaces, Y compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:

(i) the multifunction Φ is upper-semicontinuous;
(ii) the graph of Φ is a closed subset of $X \times Y$;
(iii) any would be the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi(x_n)$, $y_n \to y$, it follows $y \in \Phi(x)$.

Definition 2.7. [6], [7]. The function $u : \Delta \to \mathbb{R}^n$, $\Delta \subset \mathbb{R}^2$, is *absolutely continuous in Carathéodory's sense* [1, §565 - §570] if $u(x, y)$ is continuous on Δ, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x, y)$ is (possibly after a suitable definition on a two-dimentional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue integrable on Δ.

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\Delta; \mathbb{R}^n)$ [6], [7].

Definition 2.8. [1], [8]. The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^3$, is *absolutely continuous in Carathéodory's sense* if $u(x, y, z)$ is continuous on D, is absolutely continuous in each variable (for any pair of the other two variables), and similarly - $u_x(x, y, z)$, $u_y(x, y, z)$, $u_z(x, y, z)$, $u_{xy}(x, y, z)$, $u_{yz}(x, y, z)$, $u_{xz}(x, y, z)$, and u_{xyz} is Lebesgue integrable.

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(D; \mathbb{R}^n)$, [8].

3. Results

In a similar way as in [2], [25], [28], [29], we define the notion of a *local solution* for the Darboux-Ionescu Problem (1.1)+(1.2) and we prove an existence theorem for a local solution of this problem, together with some properties of the set of its solutions, namely that this is a compact subset in the Banach space $C(D_0;\mathbb{R}^n)$ and, as a function of initial values, it defines an upper-semicontinuous multifunction.

Let the following hypotheses be satisfied:

(H_1) $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex, non-empty values in $\mathbb{R}^n$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^n$ is an open subset.

(H_2) For any $(x,y,z) \in D$, the mapping $u \to F(x,y,z,u)$ is upper-semicontinuous on Ω.

(H_3) For any $u \in \Omega$ the mapping $(x,y,z) \to F(x,y,z,u)$ is Lebesgue-measurable on Ω.

(H_4) $f \in C(D;[0,a])$, $g \in C(D;[0,b])$, $h \in C(D;[0,c])$, $0 \leq f(x,y,z) \leq x \leq a$, $0 \leq g(x,y,z) \leq y \leq b$, $0 \leq h(x,y,z) \leq z \leq c$.

(H_5) There exists a function $k : D \to \mathbb{R}_+$, $k \in \mathcal{L}^1(D;\mathbb{R}_+)$ such that

$$\|\zeta\| \leq k(x,y,z), \quad \forall \zeta \in F(x,y,z,u), \quad \forall (x,y,z) \in D, \quad \forall u \in \Omega,$$

where $\| \, . \, \|$ is the Euclidian norm on $\mathbb{R}^n$.

(H_6) The functions $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ are absolutely continuous in Carathéodory's sense and satisfy conditions (1.3), $\varphi(x,y) \in \Omega$ for $(x,y) \in D_1$, $\psi(y,z) \in \Omega$ for $(y,z) \in D_2$, $\chi(x,z) \in \Omega$ for $(x,z) \in D_3$.

Remark 1. The function $\alpha : D \to \mathbb{R}^n$, defined by

$$\alpha(x,y,z) = \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \psi(0,z) + \psi(0,0) =$$

$$= \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0, \tag{3.1}$$

is absolutely continuous in Carathéodory's sense on D, $\alpha \in C^*(D;\mathbb{R}^n)$, [1,§565 - §570].

Denote by $M \subset \Omega$ the convex compact set in which the function $\alpha : D \to \mathbb{R}^n$, defined by (3.1), takes its values for all $(x,y,z) \in D_0$, $\alpha\,(D_0) \subseteq M \subset \Omega$.

Remark 2. Since hypothesis (H_5) ensures that the function k is integrable, a point $(x_0,y_0,z_0) \in \,]0,a] \times]0,b] \times]0,c]$ can be found so that

$$\int_0^{x_0}\int_0^{y_0}\int_0^{z_0} k(r,s,t)dr\;ds\;dt < d(M,C_\Omega),$$

where $d(M,C_\Omega)$ is the distance from M to $C_\Omega = \mathbb{R}^n - \Omega$.

Definition 3.1. The *Darboux-Ionescu Problem* for the third order hyperbolic inclusion with modified argument (1.1) consists in determining a *solution* of this inclusion which satisfies the initial conditions (1.2).

Definition 3.2. A *local solution* of Darboux-Ionescu Problem (1.1)+(1.2) is defined as a function $U : D_0 \to \Omega$, $U \in C^*(D_0; \mathbb{R}^n)$, absolutely continuous in Carathéodory's sense [1], which satisfies (1.1) for a.e. $(x, y, z) \in D_0$, and also initial conditions (1.2) for all $(x, y) \in [0, x_0] \times [0, y_0]$, all $(y, z) \in [0, y_0] \times [0, z_0]$, all $(x, z) \in [0, x_0] \times [0, z_0]$.

Theorem 3.1. Let the hypotheses (H_1) - (H_6) be satisfied. Then:

(i) there exists at least a local solution U of the Darboux-Ionescu Problem (1.1)+(1.2);

(ii) the set S_α of the local solutions U is compact in the Banach space $C(D_0; \mathbb{R}^n)$;

(iii) the multifunction $\alpha \to S_\alpha$ is upper-semicontinuous on $C^*(D_0; \mathbb{R}^n)$, taking values in $C(D_0; \mathbb{R}^n)$.

Proof. (i) Let $C^*(D_0; \mathbb{R}^n)$ be the set of absolutely continuous functions in Carathéodory's sense defined on D_0 with values in $\mathbb{R}^n$ [1]. We denote by $\mathcal{U}_M$ the set of functions $U : D_0 \to \mathbb{R}^n$, $U \in C^*(D_0; \mathbb{R}^n)$, which satisfy the inequality

$$\left\| \frac{\partial^3 U(x, y, z)}{\partial x \partial y \partial z} \right\| \leq k(x, y, z), \quad \text{for a.e. } (x, y, z) \in D_0, \tag{3.2}$$

where $\| \, . \, \|$ is the Euclidian norm on $\mathbb{R}^n$ and also conditions (1.2). The notation $\mathcal{U}_M$ is suitable because $\alpha(x, y, z) \in M$ for $(x, y, z) \in D_0$. We remark that the absolute continuity in Carathéodory's sense of the function U assures the existence of the derivative $\dfrac{\partial^3 U(x, y, z)}{\partial x \partial y \partial z}$ for a.e. $(x, y, z) \in D_0$, [1, §565 - §570]. We have $\mathcal{U}_M \subset C^*(D_0; \mathbb{R}^n)$.

Lemma. For any $U \in \mathcal{U}_M$, it follows that $U(x, y, z) \in \Omega$.

Indeed, integrating $\dfrac{\partial^3 U(x, y, z)}{\partial x \partial y \partial z}$ on D_{xyz}, where

$$D_{xyz} = \{(r, s, t) \mid 0 \leq r \leq x, \ 0 \leq s \leq y, \ 0 \leq t \leq z\}, \quad (x, y, z) \in D_0,$$

and using conditions (1.2), we obtain

$$\begin{aligned}
&U(x,y,z) = U(x,y,0) + U(x,0,z) - U(x,0,0) + U(0,y,z) -\\
&-U(0,y,0) - U(0,0,z) + U(0,0,0) + \int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt =\\
&= \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \psi(0,z) +\\
&+\psi(0,0) + \int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt = \varphi(x,y) + \psi(y,z) + \chi(x,z) -\\
&-v^1(x) - v^2(y) - v^3(z) + v^0 + \int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt =\\
&= \alpha(x,y,z) + \int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt.
\end{aligned} \tag{3.3}$$

From the Remark 2 it results that

$$\begin{aligned}
&\|U(x,y,z) - \alpha(x,y,z)\| = \left\| \int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt \right\| \le\\
&\le \int_0^x\int_0^y\int_0^z \left\| \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t} \right\| dr\,ds\,dt \le \int_0^x\int_0^y\int_0^z k(r,s,t)\,dr\,ds\,dt \le\\
&\le \int_0^{x_0}\int_0^{y_0}\int_0^{z_0} k(r,s,t)dr\,ds\,dt < d(M, C_\Omega).
\end{aligned} \tag{3.4}$$

Hence, it follows

$$d(U(x,y,z), \alpha(x,y,z)) = \|U(x,y,z) - \alpha(x,y,z)\| < d(M, C_\Omega), \tag{3.5}$$

and, from $\alpha(x,y,z) \in M$ for $(x,y,z) \in D_0$, we conclude that $U(x,y,z) \in \Omega$.

The set of functions $\mathcal{U}_M$ is *convex* and *compact* in $C(D_0;\mathbb{R}^n)$. The convexity of $\mathcal{U}_M$ results by the definition of this set, and its compactness from the Arzelà-Ascoli theorem, using hypothesis (H_6) and Remarks 1-2.

We denote by $\mathcal{G}$ the set of the triples $(\alpha, U, V) \in C^*(D_0;\mathbb{R}^n) \times \mathcal{U}_M \times \mathcal{U}_M$ with the property that U and V satisfy the membership relation

$$\frac{\partial^3 V(x,y,z)}{\partial x\partial y\partial z} \in F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z))) \tag{3.6}$$

for a.e. $(x,y,z) \in D_0$. From the definition of $\mathcal{U}_M$ it follows that $U, V \in \mathcal{U}_M$ implies that U, V satisfy conditions (1.2) for $(x,y,z) \in D_0$.

We prove that, for each $\alpha \in C^*(D_0;\mathbb{R}^n)$ such that $\alpha(x,y,z) \in M$ for $(x,y,z) \in D_0$, the set of the pairs (U,V) for which $(\alpha,U,V) \in \mathcal{G}$ is non-empty, and that the set $\mathcal{G}$ is closed.

Indeed, let us take $U \in \mathcal{U}_M$. By Theorem 1 of [2], there exists a μ-measurable (under the Lebesgue measure μ) multifunction $\Gamma_U : D_0 \to 2^{\mathbb{R}^n}$ with compact and non-empty values in $\mathbb{R}^n$ such that

$$\Gamma_U(x,y,z) \subset F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z))), \tag{3.7}$$

$\forall (x,y,z) \in D_0$.

Then, by Theorem 2 or Theorem 3 [3], there exists a measurable selection β_U of Γ_U, i.e. a measurable single-valued function $\beta_U : D_0 \to \mathbb{R}^n$ with $\beta_U(x,y,z) \in \Gamma_U(x,y,z)$ for $(x,y,z) \in D_0$.

Let the function $V : D_0 \to \mathbb{R}^n$ be defined by

$$V(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \beta_U(r,s,t)\, dr\, ds\, dt. \tag{3.8}$$

Then, the set of those pairs (U,V) such that $(\alpha,U,V) \in \mathcal{G}$ is *non-empty* because

$$\beta_U(x,y,z) \in \Gamma_U(x,y,z) \subset F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z))), \tag{3.9}$$

for a.e. $(x,y,z) \in D_0$,

$$\begin{aligned} &\frac{\partial^3 V(x,y,z)}{\partial x \partial y \partial z} = \beta_U(x,y,z) \in \Gamma_U(x,y,z) \subset \\ &\subset F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z))), \end{aligned} \tag{3.10}$$

for a.e. $(x,y,z) \in D_0$,

$$\left\| \frac{\partial^3 V(x,y,z)}{\partial x \partial y \partial z} \right\| = \|\beta_U(x,y,z)\| \le k(x,y,z), \ \ (x,y,z) \in D_0, \tag{3.11}$$

by the hypothesis (H_5) for $\zeta = \beta_U(x,y,z)$, and

$$\begin{cases} V(x,y,0) = \varphi(x,y), & (x,y) \in [0,x_0] \times [0,y_0], \\ V(0,y,z) = \psi(y,z), & (y,z) \in [0,y_0] \times [0,z_0], \\ V(x,0,z) = \chi(x,z), & (x,z) \in [0,x_0] \times [0,z_0]. \end{cases} \tag{3.12}$$

For the proof that $\mathcal{G}$ is closed, we consider a sequence $\{(\alpha_n, U_n, V_n)\}_{n \in \mathbb{N}}$ of elements in $\mathcal{G}$, convergent to (α,U,V) in the space $C^*(D_0;\mathbb{R}^n) \times C(D_0;\mathbb{R}^n) \times L^1(D_0;\mathbb{R}^n)$. We must check that $(\alpha,U,V) \in \mathcal{G}$, what implies, by the definition of set $\mathcal{G}$, that conditions (1.2) and (3.10) are satisfied by U and V.

The set $\left\{\dfrac{\partial^3 V_n(x,y,z)}{\partial x\partial y\partial z}\right\}_{n\in\mathbb{N}}$ is *relatively weakly compact* in $L^1(D_0;\mathbb{R}^n)$ by the Dunford-Pettis Criterion [12]. Indeed, the hypotheses of the Criterion are satisfied, because we have:

1) $\displaystyle\iiint\limits_{D_0}\left\|\frac{\partial^3 V_n(r,s,t)}{\partial r\partial s\partial t}\right\| dr\,ds\,dt \le \iiint\limits_{D_0} k(r,s,t)\,dr\,ds\,dt = K$, $K>0$ is a constante,

2) $\displaystyle\iiint\limits_{A}\left\|\frac{\partial^3 V_n(r,s,t)}{\partial r\partial s\partial t}\right\| dr\,ds\,dt \le \iiint\limits_{A} k(r,s,t)\,dr\,ds\,dt < \varepsilon$, if $\mu(A) < \delta(\varepsilon)$, from the absolute continuity of Lebesgue integral,

3) For every $\varepsilon>0$ compact set $C\subset D_0$ exists such that

$$\iiint\limits_{D_0-C}\left\|\frac{\partial^3 V_n(r,s,t)}{\partial r\partial s\partial t}\right\| dr\,ds\,dt \le \varepsilon.$$

It follows that $\left\{\dfrac{\partial^3 V_n(x,y,z)}{\partial x\partial y\partial z}\right\}_{n\in\mathbb{N}}$ is weakly convergent to a function $W\in L^1(D_0;\mathbb{R}^n)$. For each $(x,y,z)\in D_0$, using equalities of the form (3.12) for $V_n(x,y,z)$, equalities that are of type (1.2) for $(x,y,z)\in D_0$, we have

$$\begin{aligned}
V(x,y,z) &= w-\lim_{n\to\infty} V_n(x,y,z) = \\
&= w-\lim_{n\to\infty}\left[\alpha_n(x,y,z)+\int_0^x\int_0^y\int_0^z \frac{\partial^3 V_n(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt\right] = \\
&= \alpha(x,y,z)+\int_0^x\int_0^y\int_0^z W(r,s,t)\,dr\,ds\,dt.
\end{aligned}\tag{3.13}$$

From the weak convergence $\dfrac{\partial^3 V_n(x,y,z)}{\partial x\partial y\partial z} \rightharpoonup W(x,y,z)$, $(x,y,z)\in D_0$, using the Corollary of Mazur's Theorem [16], it follows that there exists a sequence of convex combinations $\{X_n\}_{n\in\mathbb{N}}$ of the set $\left\{\dfrac{\partial^3 U_n}{\partial x\partial y\partial z}, \dfrac{\partial^3 U_{n+1}}{\partial x\partial y\partial z}, \dots\right\}$, strongly convergent to W in $L^1(D_0;\mathbb{R}^n)$. Then, we can extract a subsequence from the sequence $\{X_n\}_{n\in\mathbb{N}}$, which converges a.e. to W : $X_{n_i}\to W$ for a.e. $(x,y,z)\in D_0$.

Since $F(x,y,z,U)$ is convex and compact for all $(x,y,z)\in D$ and for all $U\in\Omega$, we obtain from the previous results and from Lemma 2 [2] that

$$\begin{aligned}
&W(x,y,z)\in\bigcap_{l=1}^{\infty}\overline{conv\left(\bigcup_{n=l}^{\infty}\frac{\partial^3 V_n(x,y,z)}{\partial x\partial y\partial z}\right)}\subset\\
&\subset\bigcap_{l=1}^{\infty}\overline{conv\left(\bigcup_{n=l}^{\infty}F(x,y,z,U_n(f(x,y,z),g(x,y,z),h(x,y,z)))\right)}\subset\\
&\subset F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z))),
\end{aligned}\tag{3.14}$$

for a.e. $(x,y,z)\in D_0$, from which it follows that $\mathcal{G}$ *is closed*.

Indeed, (3.14) shows that

$$W(x,y,z)\in F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z)))$$

for a.e. $(x,y,z)\in D_0$, and we obtain $\dfrac{\partial^3 V(x,y,z)}{\partial x\partial y\partial z}=W(x,y,z)$ from (3.13); then, using (3.3) and (3.14) we have

$$W(x,y,z)=\frac{\partial^3 V(x,y,z)}{\partial x\partial y\partial z}\in F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z)))\tag{3.15}$$

for a.e. $(x,y,z)\in D_0$, and also (3.12), hence V satisfies the initial conditions (1.2) for $(x,y,z)\in D_0$.

Let us take $\alpha\in C^*(D_0;\mathbb{R}^n)$ with $\alpha(x,y,z)\in M$ for $(x,y,z)\in D_0$. To each $U\in\mathcal{U}_M$ we associate the set $\Phi(U)\subset\mathcal{U}_M$ as follows:

$$\begin{aligned}
&V\in\Phi(U)\Leftrightarrow V\in\mathcal{U}_M,\ \frac{\partial^3 V(x,y,z)}{\partial x\partial y\partial z}\in\\
&\in F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z)))
\end{aligned}\tag{3.16}$$

for a.e. $(x,y,z)\in D_0$.

We thus define a multifunction $\Phi:\mathcal{U}_M\to 2^{\mathcal{U}_M}$. The set $\Phi(U)$ is *convex*, *compact* and *non-empty*. It can be seen that $\Phi(U)$ is convex since $F(x,y,z,U(f(x,y,z),g(x,y,z),h(x,y,z)))$ is convex by the hypothesis (H_1). We have $\Phi(U)\subset\mathcal{U}_M$ but $\mathcal{U}_M$ is compact. The multifunction Φ has a closed graph, because *graph* Φ is the set $\mathcal{G}$ for each fixed α and $\mathcal{G}$ is closed. It follows that $\Phi(U)$ is compact in $C(D_0;\mathbb{R}^n)$ since it is a closed subset of the compact set $\mathcal{U}_M$. The set $\Phi(U)$ is non-empty since there exists V, defined by (3.8), with the property $V\in\Phi(U)$.

The multifunction $\Phi : \mathcal{U}_M \to 2^{\mathcal{U}_M}$, having a closed graph, is upper-semicontinuous by Theorem 2.1. $\Phi : \mathcal{U}_M \to 2^{\mathcal{U}_M}$ is defined on $\mathcal{U}_M$ which is a convex, compact and non-empty set; it is also upper-semicontinuous and its set-values $\Phi(U)$ are convex, closed and non-empty in $\mathcal{U}_M$. From Kakutani-Ky Fan fixed point Theorem [12], [23] it follows that the multifunction Φ has at least a fixed point, that is at least an element $U \in \mathcal{U}_M$ there exists such that $U \in \Phi(U)$. But, in view of definition of $\Phi\,(U)$ given in (3.16), the membership (3.6) holds what implies the expresion (3.8) of function V. It follows from $U, V \in \mathcal{U}_M$ that U and V satisfy (1.2). Since $U \in \Phi\,(U)$ we have $V = U$; but V is of the form (3.8), therefore this fixed point U is a solution of the Darboux-Ionescu Problem (1.1)+(1.2).

(ii) We denote by S_α the set of solutions to problem (1.1)+(1.2), a notation showing that any solution U depends on the function α defined by (3.1). The set S_α contains at least an element. The set S_α is *compact*, *non-empty* in the Banach space $C(D_0; \mathbb{R}^n)$, being the set of the fixed points of multifunction Φ.

iii) The graph $\mathcal{H}$ of the multifunction $\alpha \to S_\alpha$, defined on $C^*(D_0; \mathbb{R}^n)$ with values in $2^{\mathcal{U}_M}$, $S_\alpha \subset \Phi(\mathcal{U}_M) \subset 2^{\mathcal{U}_M}$, is closed in $C^*(D_0; \mathbb{R}^n) \times \mathcal{U}_M$ since $\mathcal{H}$ is the image of the compact set $\mathcal{H}_1$ of the triples $(\alpha, U, V) \in \mathcal{G}$ with $U = V$ through the projection mapping $(\alpha, U, V) \to (\alpha, U)$. The mapping S_α is $-$ in general $-$ a multifunction because several solutions of the problem (1.1)+(1.2) can exist, which are fixed points of mapping Φ corresponding to the same function α. Because the mapping $\alpha \to S_\alpha$ has a closed graph $\mathcal{H}$ by Theorem 2.1, it follows that $\alpha \to S_\alpha$ is upper-semicontinuous on $C^*(D_0; \mathbb{R}^n)$, what completes the proof.

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