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SEQUENCES OF OPERATORS AND FIXED POINTS

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Abstract. Let X be a nonempty set and $\mathbb{M}(X)$ be the set of all selfoperators of X. Let $(X, \to)$ and $(\mathbb{M}(X), \rightrightarrows)$ be L-spaces. In this paper we study the following problem:

Let $g, g_n \in \mathbb{M}(X)$, $n \in \mathbb{N}$, be such that

$$g_n \rightrightarrows g \text{ as } n \to \infty.$$

If x_n is a fixed point of g_n, does $(x_n)_{n\in\mathbb{N}}$ or some subsequence of $(x_n)_{n\in\mathbb{N}}$ converge to a fixed point of g?

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1. Introduction

Let X be a set and $\mathbb{M}(X)$ the set of all selfoperators of X. Let $(X, \to)$ and $(\mathbb{M}(X), \rightrightarrows)$ be L-spaces. The aim of this paper is to study the following problem:

Let $g, g_n \in \mathbb{M}(X)$, $n \in \mathbb{N}$, be such that $g_n \rightrightarrows g$ as $n \to \infty$. If x_n is a fixed point of g_n, does $(x_n)_{n\in\mathbb{N}}$ or some subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converge to a fixed point of g?

Diverse aspects of the above problem appear in subjects such as:

• Data dependence of fixed points (Sz. András [1], V. G. Angelov and I. A. Rus [3], V. Berinde [6], S. Czerwik [18], I. Del Prete and C. Esposito [20],

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W. A. Kirk and B. Sims (eds.) [38], T.-C. Lim [40], S. B. Nadler [42], [43], I. A. Rus [56]-[59], [62], I. A. Rus, A. Petruşel and G. Petruşel [66], T. Wang [74],...).

• Iteration methods for operatorial equations (V. Berinde [7], A. Buică [10], T. A. Burton [11], Y.-Z. Chen [12], C. E. Chidume and H. Zegeye [13], Y.-P. Fang, J. K. Kim and N.-J. Huang [25], A. M. Harder and T. L. Hicks [31], L. S. Liu [41], R. D. Nussbaum [45], M. O. Osilike [46], B. E. Rhoades [54],...).

• Approximation scheme theory (W. V. Petryshyn [52], M. A. Krasnoselskii [39], E. De Giorgi [19],...).

• Techniques of proof in fixed point theory (M. Angrisani and M. Clavelli [4], W. G. Dotson [21], [22], R. Fiorenza [26], R. B. Fraser and S. B. Nadler [27], M. Furi and M. Martelli [28], L. Górniewicz [30], G. Isac and Sz. Nemeth [33], W. A. Kirk and B. Sims (eds.) [38], I. A. Rus [56], G. Vidossich [73],...).

• Dynamic aspects of operatorial equations (Y.-Z. Chen [12], D. Chiorean, B. Rus, I. A. Rus and D. Trif [14], J. E. Cohen [15], D. Constantinescu and M. Predoi [17], R. Kempf [36], K. Nakajo and W. Takahashi [44], R. D. Nussbaum [45], A. Petruşel [48],...).

Throughout this paper we follow the terminologies and notations in I. A. Rus [63] and A. Petruşel [50].

2. Convergences on $\mathbb{M}(X)$

Let $(X, \to)$ be an L-space. On $\mathbb{M}(X)$ we consider the following convergences $(g, g_n \in \mathbb{M}(X))$:

• $g_n \stackrel{p}{\to} g$ as $n \to \infty$ stands for pointwise convergence;

• convergence with continuity (Angrisani-Clavelli [4]),

$$g_n \stackrel{c}{\to} g \text{ as } n \to \infty \;\Leftrightarrow\; (x_n \to x^* \text{ as } n \to \infty \;\Rightarrow\; g_n(x_n) \to g(x^*) \text{ as } n \to \infty);$$

• (if (X, U) is an uniform space) uniform convergence,

$$g_n \stackrel{u}{\to} g \text{ as } n \to \infty.$$

Let $\Rrightarrow$ be an L-convergence on $\mathbb{M}(X)$. Then we consider the following convergences:

• iterative convergence

$$g_n \stackrel{i}{\to} g \text{ as } n \to \infty \;\Leftrightarrow\; g_n^m \Rrightarrow g^m \text{ as } n \to \infty,\ \forall\, m \in \mathbb{N};$$

- (if g is WPO) asymptotical convergence (I. A. Rus [59])

$$g_n \xrightarrow{a} g \text{ as } n \to \infty \;\Leftrightarrow\; g_n^m \rightrightarrows g^\infty \text{ as } n, m \to \infty;$$

- (if g_n, g are WPOs) fixed point convergence,

$$g_n \xrightarrow{f_p} g \text{ as } n \to \infty \;\Leftrightarrow\; g_n^\infty \rightrightarrows g^\infty \text{ as } n \to \infty,$$

where $g^\infty(x) := \lim\limits_{n\to\infty} g^n(x)$.

The following simple examples illustrate the relations between these notions.

Example 2.1. We take $X = \mathbb{R}$, $d(x,y) = |x - y|$ and we consider on $\mathbb{R}$ the L-convergence "$\xrightarrow{d}$". Let $g_n(x) = 0$ for $x \in \mathbb{R}_-$, $g_n(x) = x^n$ for $x \in [0,1]$, $g_n(x) = 1$ for $x \geq 1$ and $g(x) = 0$ for $x < 1$, $g(x) = 1$ for $x \geq 1$. In this case we have

a) $g_n \xrightarrow{p} g$ as $n \to \infty$;
b) $g_n \overset{c}{\not\to} g$ as $n \to \infty$;
c) $g_n \xrightarrow{u} g$ as $n \to \infty$;
d) g_n and g are WPOs and $g_n \xrightarrow{f_p} g$ as $n \to \infty$, in $(\mathbb{M}(\mathbb{R}), \xrightarrow{p})$;
e) $g_n \xrightarrow{i} g$ as $n \to \infty$, in $(\mathbb{M}(\mathbb{R}), \xrightarrow{p})$.

Example 2.2. $X = \mathbb{R}$, $g_n(x) = 2x + \dfrac{1}{n!}$, $g(x) = 2x$, $x \in \mathbb{R}$. In this case:

a) $g_n \xrightarrow{u} g$ as $n \to \infty$;
b) g is'nt PO;
c) $g_n \xrightarrow{i} g$ as $n \to \infty$, in $(\mathbb{M}(\mathbb{R}), \xrightarrow{u})$;
d) $F_{g_n} = \left\{-\dfrac{1}{n!}\right\}$, $F_g = \{0\}$.

Example 2.3. We consider the Banach space $X = (C[0,1], \|\cdot\|_C)$ and $B_n : C[0,1] \to C[0,1]$ are classical Bernstein operators,

$$B_n(x)(t) := \sum_{k=0}^{n} x\left(\frac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k}.$$

In this case (see R. P. Kelisky and T. J. Rivlin [37], H. Gonska and I. Raşa [29], I. A. Rus [64])

a) $B_n,\ n \in \mathbb{N}^*$, are WPOs;
b) $B_n \xrightarrow{u} 1_X$;
c) $B_n \overset{f_p}{\not\to} 1_X$;
d) $B_n \overset{a}{\not\to} 1_X$;
e) $B_n \xrightarrow{i} 1_X$.

Example 2.4. Let $(X, \to)$ be an L-space and $g_n, g \in \mathbb{M}$. If

(i) $g_n \xrightarrow{c} g$ as $n \to \infty$,

then

(a) $g_n \xrightarrow{i} g$ as $n \to \infty$.

Proof. (i) $\Rightarrow$ $g_n(x) \to g(x)$ as $n \to \infty$, $\forall\, x \in X$ $\Rightarrow$ $g_n(g_n(x)) \to g(g(x))$. So, $g_n^2 \xrightarrow{p} g^2$ as $n \to \infty$. By induction we have (a).

Example 2.5. Let (X, d) be a complete metric space and $g, g_n \in \mathbb{M}(X)$. We suppose that

(i) g is an α-contraction;

(ii) $g_n \xrightarrow{u} g$ as $n \to \infty$.

Then, $g_n \xrightarrow{a} g$ as $n \to \infty$, in $(\mathbb{M}(X), \xrightarrow{u})$.

Proof. From $g_n \xrightarrow{u} g$ as $n \to \infty$, we have that there exist $\eta_n > 0$, $\eta_n \to 0$ as $n \to 0$, such that

$$d(g_n(x), g(x)) \leq \eta_n, \ \forall\, x \in X \text{ and } n \in \mathbb{N}.$$

Hence we have

$$d(g_n^m(x), g^m(x)) \leq \frac{\eta_n}{1-\alpha}, \ \forall\, m, n \in \mathbb{N}, \ \ \forall\, x \in X.$$

3. Problem 1

We begin our study with the following question:

Problem 1. Let $(X, \to)$ and $(M(X), \Rrightarrow)$ (where $M(X) \subset \mathbb{M}(X)$) be L-spaces. Let $g, g_n \in M(X)$. We suppose that

(i) $g_n \Rrightarrow g$ as $n \to \infty$;

(ii) $x_n \in F_{g_n}$, $n \in \mathbb{N}$ and $F_g = \{x^*\}$.

In which conditions we have that

(iii) $x_n \to x^*$ as $n \to \infty$?

For a better understanding of Problem 1, we consider the following aspects of this problem.

Problem 1_a. In which conditions on g we have (iii)?

Problem 1_b. For which generalized contractions g, we have (iii)?

Problem 1_c. For which Picard operators g, we have (iii)?

Problem 1_d. For which $M(X) \subset \mathbb{M}(X)$ we have (iii)?

Problem 1_e. For which convergence "$\Rrightarrow$" on $M(X)$ we have (iii)?

The following results are partial answers to these questions:

Theorem 3.1. (Bonsall [9]). *Let (X,d) be a complete metric space and $g, g_n : X \to X$, $n \in \mathbb{N}$, α-contractions. If $g_n \xrightarrow{p} g$ as $n \to \infty$, then we have (iii) in $(X, \xrightarrow{d})$.*

Theorem 3.2. (Nadler [42]). *Let (X,d) be a complete metric space and $g, g_n : X \to X$. If g is a contraction and $x_n \in F_{g_n}$, $n \in \mathbb{N}$, then $g_n \xrightarrow{u} g$ as $n \to \infty \ \Rightarrow\ x_n \to x^*$ as $n \to \infty$.*

Theorem 3.3. (Rus [59]). *Let (X,d) be a metric space, $g : X \to X$ a Picard operator ($F_g = \{x^*\}$) and $g_n : X \to X$ such that $g_n \xrightarrow{a} g$ in $(\mathbb{M}(X), \xrightarrow{u})$. Then $x_n \in F_{g_n}$, $n \in \mathbb{N}$ imply $x_n \xrightarrow{d} x^*$ as $n \to \infty$.*

Remark 3.1. From Theorem 3.3 we have Theorem 3.2.

Theorem 3.4. *Let (X,d) (where $d(x,y) \in \mathbb{R}^m_+$) be a complete generalized metric space and $g, g_n : X \to X$. We suppose that:*

(1) $g_n \xrightarrow{p} g$ as $n \to \infty$;

(2) there exists a matrix $S \in M_{mm}(\mathbb{R}_+)$ such that g, g_n are S-contractions for all $n \in \mathbb{N}$.

Then, $F_g = \{x^*\}$, $F_{g_n} = \{x_n\}$ and $x_n \xrightarrow{d} x^*$ as $n \to \infty$.

Proof. From the definition of an S-contraction it follows that $S^n \to 0$ as $n \to \infty$ (see [53]). From Perov's theorem it follows that

$$F_{g_n} = \{x_n\},\ F_g = \{x^*\}.$$

We have

$$\begin{aligned} d(x_n, x^*) &= d(g_n(x_n), g(x^*)) \\ &\le d(g_n(x_n), g_n(x^*)) + d(g_n(x^*), g(x^*)) \\ &\le S d(x_n, x^*) + d(g_n(x^*), g(x^*)). \end{aligned}$$

Hence

$$d(x_n, x^*) \le (I - S)^{-1} d(g_n(x^*), g(x^*)) \to 0 \text{ as } n \to \infty.$$

Theorem 3.5. *Let (X,d) (where $d(x,y) \in \mathbb{R}^m_+$) be a complete generalized metric space and $g, g_n : X \to X$. We suppose that:*

(1) $g_n \xrightarrow{c} g$ as $n \to \infty$;

(2) there exist $n_0 \in \mathbb{N}^*$ and $S \in M_{mm}(\mathbb{R}_+)$ such that $g^{n_0}, g_n^{n_0}$, $n \in \mathbb{N}$, are S-contractions.

Then, $F_g = \{x^*\}$, $F_{g_n} = \{x_n\}$ and $x_n \to x^*$ as $n \to \infty$.

Proof. From Perov's theorem we have that $F_{g^{n_0}} = \{x^*\}$ and $F_{g_n^{n_0}} = \{x_n\}$, $n \in \mathbb{N}$. From Lemma 1.3.3 in [56] it follows that

$$F_g = F_{g^{n_0}} = \{x^*\},\ F_{g_n} = F_{g_n^{n_0}} = \{x_n\},\ n \in \mathbb{N}.$$

From (1) we have that $g_n^{n_0} \xrightarrow{p} g^{n_0}$ as $n \to \infty$.

Now the proof follows from Theorem 3.4.

Theorem 3.6. (Rus [56]). *Let (X, d) (where $d(x, y) \in \mathbb{R}^m_+$) be a complete generalized metric space and $g, g_n : X \to X$. We suppose that:*

(1) $g_n \xrightarrow{u} g$ as $n \to \infty$;

(2) g is an S-contraction;

(3) $x_n \in F_{g_n},\ n \in \mathbb{N}$.

Then, $x_n \to x^*$ as $n \to \infty$.

Proof. We have that

$$\begin{aligned} d(x_n, x^*) &= d(g_n(x_n), g(x^*)) \\ &\leq d(g_n(x_n), g(x_n)) + d(g(x_n), g(x^*)) \\ &\leq d(g_n(x_n), g(x_n)) + Sd(x_n, x^*). \end{aligned}$$

Hence,

$$d(x_n, x^*) \leq (I - S)^{-1} d(g_n(x_n), g(x_n)) \to 0 \text{ as } n \to \infty.$$

Remark 3.2. For $m = 1$ Theorem 3.2 follows from Theorem 3.6.

For other results with respect to Problem 1 see L. S. Dube and S. P. Singh [23], R. B. Fraser and S. B. Nadler [27], S. B. Nadler [42], I. A. Rus [56]-[59], [62], W. Russell and S. P. Singh [68], S. Reich [53],...

4. Problem 2

Another aspect of our question is given by

Problem 2. Let $(X, \to)$ and $(\mathbb{M}(X), \Rrightarrow)$ be L-spaces. We suppose that

(i) $g_n \Rrightarrow g$ as $n \to \infty$;

(ii) $g, g_n, n \in \mathbb{N}$, are WPOs.

• In which conditions we have

(iii) $g_n \xrightarrow{f_p} g$ as $n \to \infty$?

• In which conditions we have

$$g_n^\infty(X) \xrightarrow{?} g^\infty(X) \text{ as } n \to \infty?$$

Example 4.1. In the case of Example 2.3 we have (see I. A. Rus [64])

$$g_n^\infty(x)(t) = B_n^\infty(x)(t) = x(0) + (x(1) - x(0))t, \ B^\infty(x)(t) = x(t).$$

So, $B_n^\infty \overset{p}{\nrightarrow} B^\infty$ as $n \to \infty$.

Example 4.2. In the case of Theorem 3.1 we have that, $g_n^\infty(x) = x_n$, and $g^\infty(x) = x^*$, $\forall\, x \in X$.

So, $g_n^\infty \overset{p}{\to} g^\infty$ as $n \to \infty$, i.e.,

$$g_n \overset{f_p}{\to} g \text{ as } n \to \infty.$$

We have

Theorem 4.1. *Let (X, d) be a metric space and $g, g_n : X \to X$. We suppose that:*

(i) $g_n \overset{u}{\to} g$ as $n \to \infty$;

(ii) $g, g_n, n \in \mathbb{N}$, are WPOs;

(iii) there exists $c > 0$ such that

$$d(x, g^\infty(x)) \le cd(x, g(x)) \text{ and } d(x, g_n^\infty(x)) \le cd(x, g_n(x)), \ \forall\, x \in X, \ \forall\, n \in \mathbb{N}.$$

Then, $H(F_{g_n}, F_g) \to 0$ as $n \to \infty$, where H is Pompeiu-Hausdorff functional (see L. Górniewicz [30], A. Petruşel [50]).

Proof. Condition (i) implies that there exist $\eta_n > 0$, $n \in \mathbb{N}$, such that

$$d(g(x), g_n(t)) \le \eta_n \to 0 \text{ as } n \to \infty, \ \forall\, x \in X.$$

In the conditions (ii)+(iii), from a theorem in Rus-Mureşan [65] we have that

$$H(F_g, F_{g_n}) \le c\eta_n, \ \forall\, n \in \mathbb{N}.$$

So, $H(F_g, F_{g_n}) \to 0$ as $n \to \infty$, i.e.,

$$H(g^\infty(X), g_n^\infty(X)) \to 0 \text{ as } n \to \infty.$$

Theorem 4.2. *Let (X, d) be a complete metric space and $g, g_n : X \to X$ be closed operators. We suppose that:*

(i) $g_n \overset{u}{\to} g$ as $n \to \infty$;

(ii) there exists $\alpha \in]0, 1[$ such that

$$d(g^2(x), g(x)) \le \alpha d(x, g(x)), \ \forall\, x \in X,$$

$$d(g_n^2(x), g_n(x)) \le \alpha d(x, g_n(x)), \ \forall\, x \in X, \ \forall\, n \in \mathbb{N}.$$

Then, $H(F_{g_n}, F_g) \to 0$ *as* $n \to \infty$.

Proof. Condition (ii) implies that the operators g and g_n, $n \in \mathbb{N}$, are WPOs. From condition (ii) we also have condition (iii) in Theorem 4.1. So, Theorem 4.2 follows from Theorem 4.1.

Example 4.3. Let X be a Banach space and $f, f_n \in C([a,b] \times X, X)$, $n \in \mathbb{N}$. Moreover we suppose that $f(t,\cdot), f_n(t,\cdot) : X \to X$ are L-Lipschitz, for all $t \in [a,b]$. We consider the following differential equations

$(I)\ x' = f(t,x),\ t \in [a,b],\ x \in C^1([a,b],X)$,

$(I_n)\ x' = f_n(t,x),\ t \in [a,b],\ x \in C^1([a,b],X)$

and equivalent fixed point equations

$(E)\ x(t) = x(a) + \int_a^t f(s,x(s))ds,\ t \in [a,b],\ x \in C([a,b],X)$

$(E_n)\ x(t) = x(a) + \int_a^t f_n(s,x(s))ds,\ t \in [a,b],\ x \in C([a,b],X).$

Now we consider the operators

$$A, A_n : C([a,b],X) \to C([a,b],X)$$

defined by

$$A(x)(t) := \text{the second part of } (E) \text{ and}$$

$$A_n(x)(t) := \text{the second part of } (E_n).$$

Let $C([a,b],X)$ be endowed with a suitable Bielecki norm,

$$\|x\|_B := \sup_{t\in[a,b]} (\|x(t)\| e^{-\tau(t-a)}),\ \tau > 0.$$

Let $\lambda \in X$ and $X_\lambda := \{x \in C([a,b],X) |\ x(a) = \lambda\}$. Then $C([a,b],X) = \bigcup_{\lambda\in X} X_\lambda$ is a partition of $C([a,b],X)$.

Moreover,

$$A(X_\lambda) \subset X_\lambda,\ A_n(X_\lambda) \subset X_\lambda,\ \lambda \in X,\ n \in \mathbb{N},$$

and

$$A \text{ and } A_n \text{ are } \frac{L}{\tau} - \text{Lipschitz}.$$

So, for $\tau > L$, $A|_{X_\lambda}$, $A_n|_{X_\lambda}$ are $\frac{L}{\tau}$-contractions. Hence, if $f_n \xrightarrow{u} f$ we are in the conditions of the Theorem 4.2. From this theorem we have that

$$H(F_A, F_{A_n}) \to 0 \text{ as } n \to \infty.$$

If we denote by S and S_n (where $S, S_n \subset C([a,b],X)$) the solution sets of the equations $(I), (I_n)$, then in the above conditions on f, f_n,

$$H(S, S_n) \to 0 \text{ as } n \to \infty.$$

5. Problem 3

The following problem appears in *iterative approximation of fixed points* (V. Berinde [7], [8], C. E. Chidume and H. Zegeye [13], B. E. Rhoades [54], S. P. Singh and B. Watson [69],...), *fiber WPOs* (M. W. Hirsch and C. C. Pugh [32], I. A. Rus [60], [61], Sz. András [1], C. Bacoţiu [5], M. Şerban [70], [71], G. Dezso, V.Mureşan, A. Tămăşan (see I. A. Rus, A. Petruşel and G. Petruşel [66],...) and in *dynamical systems* (D. Chiorean, B. Rus, I. A. Rus and D. Trif [14], D. Constantinescu and M. Predoi [17], R. Kempf [36], K. Nakajo and W. Takahashi [44], R. D. Nussbaum [45], A. Petruşel [48],...).

Problem 3. Let $(X, \to)$ and $(\mathbb{M}(X), \Rrightarrow)$ be two L-spaces. Let $M(X) \subset \mathbb{M}(X)$ and $g, g_n \in M(X)$. We suppose that

(i) $g_n \Rrightarrow g$ as $n \to \infty$;

(ii) g is WPO;

(iii) $f, g \in M(X) \;\Rightarrow\; f \circ g \in M(X)$.

In which conditions we have that

$$g_n \circ g_{n-1} \circ \cdots \circ g_0 \Rrightarrow g^{\infty} \text{ as } n \to \infty?$$

In what follow we present some partial results for this problem.

Theorem 5.1. (Y.-Z. Chen [12]). *Let (X, d) be a complete metric space and $g_n : X \to X$, $n \in \mathbb{N}$, a sequence which converges pointwise to g. Suppose that for $0 < a < b < +\infty$, there exists $L(a,b) \in]0,1[$ such that*

$$d(g_n(x), g_n(y)) \le L(a,b)d(x,y)$$

for all $x, y \in X$, $a \le d(x,y) \le b$ and $n \in \mathbb{N}$. If for each $x \in X$, there exists $y \in X$ and $R(x) > 0$ such that $d((g_n \circ g_{n-1} \circ \cdots \circ g_0)(x), y) \le R(x)$, for $n \in \mathbb{N}$, then $(g_n \circ g_{n-1} \circ \cdots \circ g_0)(x) \to g^{\infty}(x)$ as $n \to \infty$, $\forall\, x \in X$.

Theorem 5.2. *Let (X, d) (where $d(x,y) \in \mathbb{R}^m_+$) be a generalized complete metric space and $g, g_n : X \to X$ be S-contractions. If $g_n \xrightarrow{p} g$ as $n \to \infty$, then*

$$g_n \circ g_{n-1} \circ \cdots \circ g_0 \xrightarrow{p} g^{\infty}.$$

Proof. If we denote by x^* the unique fixed point of g we have that $g^\infty(x) = x^*,\ \forall\ x \in X$ and

$$d((g_n \circ \cdots \circ g_0)(x), x^*) \leq d((g_n \circ \cdots \circ g_0)(x), g_n(x^*)) + d(g_n(x^*), x^*) \leq$$

$$\leq Sd((g_{n-1} \circ \cdots \circ g_0)(x), x^*) + d(g_n(x^*), x^*) \leq$$

$$\leq S^2 d((g_{n-2} \circ \cdots \circ g_0)(x), x^*) + Sd(g_{n-1}(x^*), x^*) + d(g_n(x^*), x^*) \leq \cdots \leq$$

$$\leq S^n d(g_0(x), x^*) + S^{n-1} d(g_1(x^*), x^*) + \cdots + Sd(g_{n-1}(x^*), x^*) + d(g_n(x^*), x^*).$$

Now the proof follows from the following

Lemma 5.1. (I. A. Rus [61]). *Let $A_n \in M_{mm}(\mathbb{R}_+)$ and $B_n \in \mathbb{R}^m_+$, $n \in \mathbb{N}$. We suppose that*

(i) $B_n \to 0$ as $n \to \infty$;

(ii) $\sum_{n\in\mathbb{N}} A_n$ converges.

Then

$$\sum_{i=0}^{n} A_{n-i} B_i \to 0 \text{ as } n \to \infty.$$

Remark 5.1. For the case of φ-contractions see M. Şerban [70].

Remark 5.2. For the case $m = 1$ see I. A. Rus [60].

Remark 5.3. The following result is in connection with Theorem 5.2.

Lemma 5.2. *Let (X, d) (where $d(x, y) \in \mathbb{R}^m_+$) be a generalized complete metric space and $g_n : X \to X$ be an S_n-contraction, $n \in \mathbb{N}$, such that, $S_n \to 0$ as $n \to \infty$ ($F_{g_n} = \{x^*_n\}$). Let $x^* \in X$. The following statements are equivalent:*

(i) there exists $\widetilde{x} \in X$ such that $g_n(\widetilde{x}) \to x^*$, as $n \to \infty$.

(ii) $g_n(x) \to x^*$ as $n \to \infty$, $\forall\ x \in X$;

(iii) $x^*_n \to x^*$ as $n \to \infty$.

Proof. (i) $\Rightarrow$ (ii).

$$d(g_n(x), x^*) \leq d(g_n(x), g_n(\widetilde{x})) + d(g_n(\widetilde{x}), x^*) \leq$$

$$\leq S_n d(x, \widetilde{x}) + d(g_n(\widetilde{x}), x^*) \to 0 \text{ as } n \to \infty.$$

(ii) $\Rightarrow$ (iii).

$$d(x^*_n, x^*) = d(g_n(x^*_n), x^*) \leq d(g_n(x^*_n), g_n(x)) + d(g_n(x), x^*) \leq$$

$$\leq S_n d(x^*_n, x) + d(g_n(x), x^*).$$

Hence, we have

$$d(x_n^*, x^*) \leq (I - S_n)^{-1} d(g_n(x), x^*) \to 0 \text{ as } n \to \infty.$$

(iii) $\Rightarrow$ (i).

We take $\widetilde{x} := x^*$.

Remark 5.4. For other results for Problem 3 see Y.-Z. Chen [12], R. Kannan and Z. Vorel [35], R. Kempf [36].

6. Problem 4

Another aspect of our basic problem is given by

Problem 4. Let $(X, \to)$ and $(\mathbb{M}(X), \rightrightarrows)$ be two L-spaces. Let $g, g_n \in \mathbb{M}(X)$. We suppose that

(i) $g_n \rightrightarrows g$ as $n \to \infty$;

(ii) $F_{g_n} \neq \emptyset$, $\forall\, n \in \mathbb{N}$.

In which conditions we have that $F_g \neq \emptyset$?

Problem 4a. Let X be a Banach space, $Y \subset X$ a compact subset of X and $g, g_n \in (\mathbb{M}(Y), \rightrightarrows)$.

We suppose that

(i) $g_n \rightrightarrows g$;

(ii) $F_{g_n} \neq \emptyset$;

(iii) $g \in C(Y, Y)$.

In which conditions we have that $F_g \neq \emptyset$?

Problem 4b. Let (X, d) be a complete K-metric space (see P. P. Zabrejko [75]) and $g, g_n \in (\mathbb{M}(X), \rightrightarrows)$.

We suppose that

(i) $g_n \rightrightarrows g$ as $n \to \infty$;

(ii) there exist $x_n \in X$, $n \in \mathbb{N}$, such that

$$d(g_n(x_n), x_n) \to 0 \text{ as } n \to \infty;$$

(iii) $g \in C(X, X)$.

In which conditions we have that $F_g \neq \emptyset$?

Problem 4c. Let $(X, \to)$, $(\mathbb{M}(X), \rightrightarrows)$ be two L-spaces, and $g, g_n \in \mathbb{M}(X)$. We suppose that

(i) $g_n \rightrightarrows g$ as $n \to \infty$;

(ii) $x_n \in F_{g_n},\ n \in \mathbb{N}$.

In which conditions we have that

$$x_n \to x^* \text{ as } n \to \infty \ \Rightarrow\ x^* \in F_g?$$

In which conditions we have that

$$g(x_n) \to x^* \text{ as } n \to \infty \ \Rightarrow\ x^* \in F_g?$$

Problem 4d. Use the results of the above problems for the study of the following problem:

Let (X, τ) be a topological space, $Y \subset X$ a compact subset. In which conditions we have that

$$g \in C(Y, Y) \ \Rightarrow\ F_g \neq \emptyset?$$

First of all, we present some simple and useful remarks:

Lemma 6.1. (G. Vidossich [73]). *Let (X, U) be a uniform space, $Y \subset X$ a subset of X, $g \in C(Y, X)$ and $g_n \in \mathbb{M}(Y, X)$. We suppose that*

(i) $g_n \overset{u}{\to} g$ as $n \to \infty$;

(ii) $x_n \in F_{g_n},\ n \in \mathbb{N}$.

Then, every cluster point of $(x_n)_{n\in\mathbb{N}}$ is a fixed point of g.

Lemma 6.2. (W. G. Dotson [21]). *Let X be a Banach space, $Y \subset X$ a starshaped subset of X and $g : Y \to Y$ a nonexpansive operator. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$, $g_n : Y \to Y$, such that:*

(i) $g_n \overset{u}{\to} g$

(ii) g_n is $\left(1 - \dfrac{1}{n}\right)$-contraction, $n \in \mathbb{N}$.

Lemma 6.3. *Let (X, d) be a K-metric space, $Y \subset X$ a compact subset of X and $g \in C(Y, Y)$. Then the following statements are equivalent:*

(i) $F_g \neq \emptyset$;

(ii) there exist $g_n \in C(Y, Y)$, $n \in \mathbb{N}$, such that $F_{g_n} \neq \emptyset$ and $g_n \overset{u}{\to} g$ as $n \to \infty$;

(iii) there exist $g_n : Y \to Y$, $n \in \mathbb{N}$ such that $F_{g_n} \neq \emptyset$ and $g_n \overset{c}{\to} g$ as $n \to \infty$;

(iv) there exist $g_n : Y \to Y$ and $x_n \in Y$ such that $g_n \overset{u}{\to} g$ as $n \to \infty$ and $d(g_n(x_n), x_n) \to 0$ as $n \to \infty$.

Proof. (i) $\Rightarrow$ (ii). We take $g_n := g$.

(ii) $\Rightarrow$ (i). Let $x_n \in F_{g_n}$. Then there exist a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$,

$$x_{n_i} \to x^* \text{ as } n \to \infty.$$

We have

$$d(x^*, g(x^*)) \leq d(x^*, x_{n_i}) + d(g_{n_i}(x_{n_i}), g(x^*)) \to 0 \text{ as } i \to \infty.$$

So, $x^* \in F_g$.

(i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv). We take $g_n := g$.

(iii) $\Rightarrow$ (i). Follows from the notion of convergence with continuity.

(iv) $\Rightarrow$ (i). Y being a compact subset of X it implies that there exists $x_{n_i} \to x^*$ as $n \to \infty$.

We have

$$d(x^*, g(x^*)) \leq d(x^*, x_{n_i}) + d(x_{n_i}, g_{n_i}(x_{n_i}))+$$

$$+d(g_{n_i}(x_{n_i}), g(x_{n_i})) + d(g(x_{n_i}), g(x^*)) \to 0 \text{ as } i \to \infty.$$

Remark 6.1. In the case $K = \mathbb{R}_+$, (iv) $\Rightarrow$ (i) is Lemma 1 in M. Furi and M. Martelli [28].

Remark 6.2. From Lemma 6.2 we have

Theorem 6.1. (W. G. Dotson [21]). *Let X be a Banach space and $Y \subset X$ a compact starshaped subset of X. Then any nonexpansive operator $g : Y \to Y$ has a fixed point.*

Remark 6.2. For some generalization of the above theorem see W. G. Dotson [22], A. Petruşel (1987), A. Ganguly and H. K. Jadnov (1991), L. F. Guseman and B. C. Peters (1975) (see I. A. Rus, A. Petruşel and G. Petruşel [66]).

7. Sequences of operators and common fixed points

Problem 5. Let $(X, \to)$, $(\mathbb{M}(X), \Rrightarrow)$ be two L-spaces and $f, g, f_n, g_n : X \to X$, $n \in \mathbb{N}$, be such that

(i) $f_n \Rrightarrow f, g_n \Rrightarrow g$ as $n \to \infty$;

(ii) $F_f = F_g = \{x^*\}$;

(iii) $x_n \in F_{f_n},\ y_n \in F_{g_n},\ n \in \mathbb{N}$.

In which conditions we have that

$$x_n \to x^*,\ y_n \to x^* \text{ as } n \to \infty?$$

Let (X, d) (where $d(x, y) \in \mathbb{R}^m_+$) be a complete generalized metric space. We take on X, $\to := \xrightarrow{d}$ and on $\mathbb{M}(X)$, $\rightrightarrows := \xrightarrow{u}$. In this case we have

Theorem 7.1. (I. A. Rus [56]). *Let $f, g, f_n, g_n : X \to X$ be as in Problem 5. If there exists $S \in \mathbb{M}_{mm}(\mathbb{R}_+)$ such that*

(1) $[(I - S)^{-1}S]^n \to 0$ *as* $n \to \infty$,

and

(2) $d(f(x), g(y)) \leq S[d(x, f(x)) + d(y, g(y))]$, $\forall\ x, y \in X$

then

$$x_n \to x^*,\ y_n \to x^* \text{ as } n \to \infty.$$

Proof. From (1)+(2) we have that $F_f = F_g = \{x^*\}$. On the other hand,

$$d(x_n, x^*) = d(f_n(x_n), g(x^*)) \leq$$

$$\leq d(f_n(x_n), f(x_n)) + d(f(x_n), g(x^*)) \leq$$

$$\leq d(f_n(x_n), f(x_n)) + S[d(x_n, f(x_n)) + d(x^*, g(x^*))] \leq$$

$$\leq (I + S)d(f_n(x_n), f(x_n)).$$

Hence

$$x_n \to x^* \text{ as } n \to \infty.$$

In a similar way we prove that

$$y_n \to x^* \text{ as } n \to \infty.$$

Remark 7.1. For other properties of the pair (f, g) which satisfies (1)+(2) see I. A. Rus [56], [58].

8. Multivalued operators

Let X be a set. We denote by $\mathbb{M}^0(X)$ the set of all multivalued mappings $T : X \multimap X$.

Problem 6. Let $(X, \to)$ and $(M^0(X), \rightrightarrows)$ (where $M^0(X) \subset \mathbb{M}^0(X)$) be L-spaces. Let $T, T_n \in M(X)$. We suppose that

(i) $T_n \rightrightarrows T$ as $n \to \infty$;

(ii) $x_n \in F_{T_n}$, $n \in \mathbb{N}$.

In which conditions we have that $(x_n)_{n\in\mathbb{N}}$ converges and the limit $x^* \in F_T$?

As a partial result for Problem 6 we present the following:

Theorem 8.1. (S. B. Nadler [43]) *Let (X, d) be a complete metric space and $T, T_n : X \to P_{cp}(X)$. We suppose that*

(i) $T, T_n, n \in \mathbb{N}$ are α-contractions;

(ii) $T_n \xrightarrow{p} T$ as $n \to \infty$.

Then, if $x_n \in F_{T_n}$, $n \in \mathbb{N}$, there is a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{n_i})_{i\in\mathbb{N}}$ converges to a fixed point of T.

Theorem 8.2. (T.-C. Lim [40]) *Let (X, d) be a complete metric space and $T, T_n : X \to P_{b,cl}(X)$, $n \in \mathbb{N}$ be α-contractions. If*

$$H(T(x), T_n(x)) \to 0 \text{ as } n \to \infty, \quad \text{uniformly for all } x \in X,$$

then

$$H(F_T, F_{T_n}) \to 0 \text{ as } n \to \infty.$$

In what follow we need the following notions.

Let $(X, \to)$ be an L-space and $T : X \to P(X)$ be a multivalued operator. By definition, T is a multivalued Picard (briefly MWP) operator iff for each $x \in X$ and each $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_0 = x$, $x_1 = y$, $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$, and $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

For a MWP operator T we define the operator $T^\infty : G(T) \to P(F_T)$, by

$T^\infty(x, y) := \{z \in F_T |$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

Let (X, d) be a metric space and $T : X \to P(X)$ an MWP operator. By definition T is a c-multivalued weakly Picard operator ($c > 0$) iff there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x, y)) \leq cd(x, y), \ \forall\ (x, y) \in G(T).$$

We have

Theorem 8.3. *Let (X, d) be a metric space and $T, T_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$. We suppose that*

(i) there exists $\eta_n > 0$, $\eta_n \to 0$ as $n \to \infty$ such that

$$H(T(x), T_n(x)) \leq \eta_n, \ \forall\ n \in \mathbb{N}, \ \forall\ x \in X;$$

(ii) $T, T_n, n \in \mathbb{N}$ *are c-MWP operators.*
Then

$$H(F_T, F_{T_n}) \to 0 \text{ as } n \to \infty.$$

Proof. From Theorem 2.1 in [67] we have that

$$H(F_T, F_{T_n}) \le c\eta_n,\ n \in \mathbb{N}.$$

So,

$$H(F_T, F_{T_n}) \to 0 \text{ as } n \to \infty.$$

Remark 8.1. For the Problem 6 in uniform spaces see V. G. Angelov and I. A. Rus [3].

Remark 8.2. For other results see R. Espinola and A. Petruşel [24], T.-C. Lim [40], S. B. Nadler [42], [43], I. A. Rus, A. Petruşel and A. Sîntămărian [67], T. Wang [74].

References

[1] Sz. András, *Picard operators and convex contractions*, Fixed Point Theory, **4**(2003), No. 2, 121-129.

[2] J. Andres and L. Górniewicz, *On the Banach contraction principle for multivalued mappings*, Approximation, optimization and mathematical economics (Point-à-Pitre 1999), 1-23, Physica, Heidelberg, 2001.

[3] V. G. Angelov and I. A. Rus, *Data dependence of the fixed point set of multivalued weakly Picard operators in uniform spaces*, Studia Univ. Babeş-Bolyai Math., **45**(2000), 3-10.

[4] M. Angrisani and M. Clavelli, *Synthetic approaches to problems of fixed points in metric space*, Ann. Mat. Pura ed Appl., **170**(1996), 1-12.

[5] C. Bacoţiu, *Data dependence of the fixed points set of weakly Picard operators in generalized metric spaces*, Studia Univ. Babeş-Bolyai Math., **49**(2004), Nr. 1, 15-17.

[6] V. Berinde, *Contracţii generalizate şi aplicaţii*, Cub Press, Baia Mare, 1997.

[7] V. Berinde, *Iterative approximations of fixed points*, Efemeride, Baia Mare, 2002.

[8] V. Berinde, *On the convergence of the Ishikawa iteration*, Acta Math. Univ. Comenianae, **73**(2004), Nr. 1, 119-126.

[9] F. F. Bonsal, *Lectures on some fixed point theorems of functional analysis*, Tata Inst. Fund. Res., Bombay, 1962.

[10] A. Buică, *Principii de coincidenţă şi aplicaţii*, Presa Univ. Clujeană, Cluj-Napoca, 2001.

[11] T. A. Burton, *Integral equation, implicit functions and fixed points*, Proc. Amer. Math. Soc., **124**(1996), 2383-2390.

[12] Y.-Z. Chen, *Inhomogeneous iterates of contraction mappings and nonlinear ergodic theorems*, Nonlinear Analysis, **39**(2000), 1-10.

[13] C. E. Chidume and H. Zegeye, *Approximate fixed sequences and convergence theorems for asymptotically pseudocontractive mappings*, J. Math. Anal. Appl., **278**(2003), 354-366.

[14] D. Chiorean, B. Rus, I. A. Rus and D. Trif, *Rezultate şi probleme în dinamica unui operator*, Univ. Babeş-Bolyai, Cluj-Napoca, 1997.

[15] J. E. Cohen, *Ergodic theorems in demography*, Bull. Amer. Math. Soc., **1**(1979), 275-295.

[16] P. L. Combettes and H. Puh, *Iterations of parallel convex projections in Hilbert spaces*, Num. Funct. Anal. and Optimiz., **15**(1994), 225-243.

[17] D. Constantinescu and M. Predoi, *An extension of the Banach fixed point theorem and some applications in the theory of dynamical systems*, Studia Univ. Babeş-Bolyai, **44**(1999), Nr. 4, 11-22.

[18] S. Czerwik, *Fixed point theorems and special solutions of functional equations*, Sc. Publ. of Univ. of Silezia, Nr. 428, Katowice, 1980.

[19] E. De Giorgi, *Γ-convergenza e G-convergenza*, Boll. U.M.I., 14A(1977), 213-220.

[20] I. Del Prete and C. Esposito, *Elementi uniti di transformazioni e teoremi di dipendenza continua*, Rend. Circ. Matemat. di Palermo, 23(1974), 187-207.

[21] W. G. Dotson, *Fixed point theorems for nonexpansive mappings on starshaped subsets of Banach spaces*, J. London Math. Soc., 4(1972), part 3, 408-410.

[22] W. G. Dotson, *On fixed points of nonexpansive mappings in nonconvex sets*, Proc. Amer. Math. Soc., **38**(1973), Nr. 1, 155-156.

[23] L. S. Dube and S. P. Singh, *On sequence of mappings and fixed points*, Nanta Math., **5**(1972), 84-89.

[24] R. Espinola, A. Petruşel, *Existence and data dependence of fixed points for multivalued operators in gauge spaces*, J. Math. Anal. Appl., 2005 (to appear).

[25] Y.-P. Fang, J. K. Kim and N.-J. Huang, *Stable iterative procedures with errors for strong pseudocontractions and nonlinear equations of accretive operators without Lipschitz assumption*, Nonlinear Funct. Anal. Appl., **7**(2002), No. 4, 497-507.

[26] R. Fiorenza, *Sull' esistenza di elementi uniti per una classe di transformazioni funzionali*, Ric. di Mat., **15**(1966), 127-153.

[27] R. B. Fraser and S. B. Nadler, *Sequences of contractive maps and fixed points*, Pacific J. Math., **31**(1969), No. 3, 659-667.

[28] M. Furi and M. Martelli, *Successioni di transformazioni in uno spazio metrico e punti fissi*, Rend. Cl. Sc. Acad. Naz. Lincei, **47**(1969), 27-31.

[29] H. Gonska and I. Raşa, *The limiting semigroup of the Bernstein iterates*, Univ. Duisburg, Essen, 2004.

[30] L. Górniewicz, *Topological fixed point theory of multivalued mappings*, Kluwer, London, 1999.

[31] A. M. Harder and T. L. Hicks, *Stability results for fixed point procedures*, Math. Japan. **33**(1988), 693-706.

[32] M. W. Hirsch and C. C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. Symp. in Pure Math., AMS (1970), 133-143.

[33] G. Isac and S. Z. Nemeth, *Fixed points and positive eigenvalues for nonlinear operators*, Proc. Amer. Math. Soc. (to appear).

[34] J. E. Joseph and M. R. Kwack, *Fixed point sets of normal maps*, Scientaiae Math. Japonicae, **57**(2003), 271-277.

[35] R. Kannan and Z. Vorel, *Continuous dependence of approximate solutions of operator equations*, J. Integral Eq., **9**(1985), No. 2, 153-159.

[36] R. Kempf, *On Ω-limit sets of discrete time dynamical systems*, J. Difference Eq. Appl., **8**(2002), 1121-1131.

[37] R. P. Kelisky and T. J. Rivlin, *Iterates of Bernstein polynomials*, Pacific J. Math., **21**(1967), 511-520.

[38] W. A. Kirk and B. Sims (eds.), *Handbook of metric fixed point theory*, Kluwer, 2001.

[39] M. A. Krasnoselskii, G. M. Vainikko, P. P. Zabreiko, Ja. B. Rutickii and V. Ja. Stecenko, *Approximate solution of operator equations*, Nauka, Moscow, 1969.

[40] T.-C. Lim, *On fixed point stability for set valued contactive mappings with applications to generalized differential equations*, J. Math. Appl., **110**(1985), 436-441.

[41] L. S. Liu, *Ishikawa and Mann iterative processes with error for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194**(1995), 114-125.

[42] S. B. Nadler, *Sequences of contractions and fixed points*, Pacific J. Math., **27**(1968), No. 3, 579-585.

[43] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math., **30**(1969), No. 2, 475-488.

[44] K. Nakajo and W. Takahashi, *A nonlinear strong ergodic theorem for asymptotically nonexpansive mappings with compact domains*, Dynamics of Continuous, Discrete and Impulsive Systems, Serie A, **9**(2002), 257-270.

[45] R. D. Nussbaum, *Hilbert's projective metric and iterated nonlinear maps*, Memoirs of AMS, No. 391, 1988.

[46] M. O. Osilike, *Stability of the Mann and Ishikawa iteration procedures for ϕ-strong pseudocontractions and nonlinear equations of the ϕ-strongly accretive type*, J. Math. Anal. Appl., **227**(1998), 319-334.

[47] A. Petruşel, *Generalized multivalued contractions*, Nonlinear Analysis, **47**(2001), 649-659.

[48] A. Petruşel, *Dynamic systems, fixed points and fractals*, Pure Math. Appl., **13**(2002), 275-281.

[49] A. Petruşel, *Multifunctions and applications*, Cluj University Press, 2002.

[50] A. Petruşel, *Multivalued weakly Picard operators and applications*, Scientiae Math. Japonicae, **59**(2004), No. 1, 169-202.

[51] A. Petruşel and I. A. Rus, *Multivalued Picard and weakly Picard operators*, Proc. Int. Conf. Fixed Point Theory and Applications, 207-226, Yokohama Publishers, 2004.

[52] W. V. Petryshyn, *On the approximation-solvability of equations involving A-proper and pseudo-A-proper mappings*, Bull. Amer. Math. Soc., **81**(1975), 223-311.

[53] S. Reich, *Kannan's fixed point theorem*, Boll. U.M.I., **4**(1971), 1-11.

[54] B. E. Rhoades, *Fixed point iteration methods for certain nonlinear mappings*, J. Math. Anal. Appl., **183**(1994), 118-120.

[55] I. A. Rus, *Approximation of fixed points of generalized contractions mappings*, in Topics in Numerical Analysis, Acad. Press, Dublin, 1975, 157-161.

[56] I. A. Rus, *Metrical fixed point theorems*, Univ. of Cluj-Napoca, 1979.

[57] I. A. Rus, *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

[58] I. A. Rus, *Results and problems in the metrical common fixed point theory*, Mathematica, **21**(1979), 189-194.

[59] I. A. Rus, *Basic problems of the metric fixed point theory revisited*, I, Studia Univ. Babeş-Bolyai, **34**(1989), 61-69, II, **36**(1991), 81-89.

[60] I. A. Rus, *Fiber Picard operators theorem and applications*, Studia Univ. Babeş-Bolyai, **44**(1999), 89-97.

[61] I. A. Rus, *A fiber generalized contraction theorem and applications*, Mathematica, **41**(1999), Nr. 1, 85-90.

[62] I. A. Rus, *Generalized contractions and applications*, Cluj Univ. Press, Cluj-Napoca, 2001.

[63] I. A. Rus, *Picard operators and applications*, Scientiae Math. Japonicae, **58**(2003), 191-219.

[64] I. A. Rus, *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl., **292**(2004), 259-261.

[65] I. A. Rus and S. Mureşan, *Data dependence of the fixed point set of some weakly Picard operators*, Itinerant Seminar, Cluj-Napoca, 2000, 201-208.

[66] I. A. Rus, A. Petruşel and G. Petruşel, *Fixed point theory 1950-2000: Romanian Contributions*, House of the Book of Science, Cluj-Napoca, 2002.

[67] I. A. Rus, A. Petruşel and A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Analysis, **52**(2003), 1947-1959.

[68] W. Russell and S. P. Singh, *A note on a sequence of contraction mapping*, Bull. Canad. Math., **12**(1969), 513-516.

[69] S. P. Singh and B. Watson, *On convergence results in fixed point theory*, Rend. Sem. Mat. Univ. Politec. Torino, **51**(1993), 73-91.

[70] M.-A. Şerban, *Fiber φ-contractions*, Studia Univ. Babeş-Bolyai, **44**(1999), No. 3, 99-108.

[71] M.-A. Şerban, *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Univ. Clujeană, Cluj-Napoca, 2002.

[72] J. Stepräns, S. Watson and W. Just, *A topological Banach fixed point theorem for compact Hausdorff spaces*, Canad. Math. Bull.

[73] G. Vidossich, *Approximation of fixed points of compact mappings*, J. Math. Anal. Appl., **33**(1971), 111-115.

[74] W. Wang, *Fixed point theorems and fixed point stability for multivalued mappings on metric spaces*, J. Nanjing Univ. Math. Biq., **6**(1989), 16-23.

[75] P. P. Zabrejko, *K-metric and K-normed linear spaces: survey*, Collect. Math., **48**(1997), 825-859.