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A NOTE ON THE EXISTENCE OF POSITIVE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

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Abstract. One studies the existence of positive continuous solutions of Fredholm integral equation

$$y(t) = h(t) + \int_0^T k(t,s) f(y(s))\, \mathrm{d}s, \quad t \in [0,T], \quad T > 0 \text{ fixed},$$

using limit type conditions for f in 0 and $+\infty$. The results obtained are applied to the study of the bilocal problem

$$\begin{cases} -y'' = f(y) \text{ on } [0,1] \\ y(0) = \alpha,\, y(1) = \beta \end{cases}.$$

Key Words and Phrases: Positive solutions, fixed point, cone, two point boundary value problem, nonlinear integral equation.

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1. Introduction

The purpose of this note is to find *nice* conditions which ensure *the existence of continuous positive solutions to the Fredholm nonlinear integral equation:*

$$y(t) = h(t) + \int_0^T k(t,s) f(y(s))\, \mathrm{d}s, \quad t \in [0,T], \quad T > 0 \text{ fixed}. \tag{1}$$

A very common approach is to make use of some fixed point principle, such as Schauder's fixed point theorem or Krasnoselskii's compression-expansion fixed point theorem in cones.

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Theorem 1 (Schauder, [11]). *Let X be a Banach space and $C \subset X$ a non-empty, closed, bounded, convex set. If $K : C \to C$ is a completely continuous operator, then K has a fixed point in C.*

Theorem 2 (Krasnoselskii, [5]). *Let $(X, |\cdot|)$ be a Banach space and $C \subset X$ a cone. Consider Ω_1, Ω_2 open sets in X such that $0 \in \overline{\Omega_1} \subset \Omega_2$, and*

$$K : C \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to C$$

a completely continuous operator such that either

(i) $|Ky| \le |y|,\ \forall\, y \in C \cap \partial\Omega_1$ *and* $|Ky| \ge |y|,\ \forall\, y \in C \cap \partial\Omega_2$

or

(ii) $|Ky| \ge |y|,\ \forall\, y \in C \cap \partial\Omega_1$ *and* $|Ky| \le |y|,\ \forall\, y \in C \cap \partial\Omega_2$

takes place. Then K has a fixed point in $C \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$.

Define $X = C[0, T]$ endowed with the sup-norm $|\cdot|_\infty$ and C_0 the positive cone in X, i.e. $C_0 = \{y \in X : y \ge 0 \text{ on } [0, T]\}$. Also take

$$Ky(t) = h(t) + \int_0^T k(t, s) f(y(s))\, \mathrm{d}s, \quad t \in [0, T].$$

First of all, it is needed that $K : C_0 \to C_0$, $y \to Ky$ is well defined and completely continuous (the first step in applying any of the fixed point theorems previously stated).

This takes place if the following conditions are satisfied:

(f1): $f : [0, +\infty) \to [0, +\infty)$ is continuous

(h1): $h : [0, T] \to [0, +\infty)$ is continuous

(k1): $k \in C\left([0, T]; L^1[0, T]\right)$ if considered as $t \xrightarrow{k} k(t) = k(t, \cdot)$, and $k(t)$ is positive a.e. on $[0, T]$.

Instead of **(k1)** a less general condition can be considered:

(k1'): $k : [0, T] \times [0, T] \to [0, +\infty)$ is continuous.

Notation 3. *For a function $f : D \subset \mathbb{R} \to \mathbb{R}$ we will denote* $\sup f(E) := \sup\limits_{t \in A} f(t)$ *and* $\inf f(E) := \inf\limits_{t \in E} f(t)$*, for any $E \subset D$. When E is a compact interval $[a, b]$, we will write* $\sup f[a, b]$ *instead of* $\sup f([a, b])$ *and* $\inf f[a, b]$ *instead of* $\inf f([a, b])$.

To apply Schauder's theorem, one only needs a closed ball (therefore a radius u) such that the intersection with the cone C_0 is invariant through K. Taking a radius $u > 0$, then for every $y \in C_0$, $|y|_\infty \le u$, we obtain that

$$Ky(t) = h(t) + \int_0^T k(t,s) f(y(s))\,\mathrm{d}s \le |h|_\infty + \left(\int_0^T k(t,s)\,\mathrm{d}s\right) \cdot \sup f\,[0,u]$$

for every $t \in [0,T]$, hence

$$|Ky|_\infty \le |h|_\infty + K_1 \cdot \sup f[0,u] \tag{2}$$

where $K_1 = \sup\limits_{t\in[0,T]} \int_0^T k(t,s)\,\mathrm{d}s$.

Therefore, the invariance in Schauder's theorem is achieved if we ask that:

(u): $\exists\, u > 0 : |h|_\infty + K_1 \cdot \sup f[0,u] \le u$

The result obtained is the following existence theorem:

Theorem 4. *If **(f1)**, **(h1)**, **(k1)**, **(u)** are satisfied, then the problem (1) has at least one solution y in C_0.*

Remark 5. *If y is a solution of (1) in C_0, then*

$$y(t) = Ky(t) = h(t) + \int_0^T k(t,s) f(y(s))\,\mathrm{d}\,s \ge h(t) + \left(\int_0^T k(t,s)\,\mathrm{d}\,s\right) \cdot \inf f[0,u],$$

therefore

$$|h|_\infty + K_1 \cdot \inf f[0,u] \le |y|_\infty\,.$$

Hence

$$|h|_\infty + K_1 \cdot \inf f[0,u] \le |y|_\infty \le |h|_\infty + K_1 \cdot \sup f\,[0,u]\,. \tag{3}$$

Remark 6. *If* $\begin{cases} h = 0 \text{ on } [0,T] \\ f(0) = 0 \end{cases}$*, then 0 is a solution for (1) in C_0. In this situation, there is no use in applying Schauder's theorem.*

For the existence of non-trivial solutions, Krasnoselskii's theorem is a useful tool, because the fixed point can not be 0 in this case.

Unfortunately, in the extreme situation as above, the cone C_0 is too "large" to achieve the expansion condition. Simply, there exists no radius $v > 0$ such that $|Ky|_\infty \ge |y|_\infty$, $\forall\, y \in C_0$, $|y|_\infty = v$. Therefore, the cone has to be made "small" enough. In [4] and [9], two examples of such "small" cones can be seen, together with existence results regarding our problem. In this paper, we will use the results from [4] since here the cone is more general than the one in [9]. We will also use the methods from [9] in order to obtain limit type results for the problem (1).

In [4], the chosen cone is $C := \{y \in C_0 : \min y[a,b] \geq M\, |y|_\infty\}$, where

(M): $0 < M < 1$ and $0 \leq a < b \leq T$

are 'a priori' chosen. The invariance condition of the cone C through K leads to the following conditions:

(h2): $h(t) \geq M\, |h|_\infty,\ \forall\, t \in [a,b]$;

(k2): $\kappa(s) := \sup\limits_{t\in[0,T]} k(t,s) < +\infty$, for *a.e.* $s \in [0,T]$ and $\kappa \in L^1[0,T]$.

(k3): $k(t,s) \geq M\kappa(s),\ \forall\, t \in [a,b]$, *a.e.* $s \in [0,T]$.

Notice that **(k2)**, like **(k1)**, is implied by **(k1')**.

The compression condition in Krasnoselskii's theorem, written for our problem, is the same with the invariance condition in Schauder's theorem, i.e. the condition **(u)**.

The expansion condition is satisfied by

(v): $\exists\, t^* \in [0,T],\ \exists\, v > 0 : v \leq h(t^*) + \left(\int_a^b k(t^*,s)\,\mathrm{d}s\right) \cdot \inf f[Mv, v]$

or by a simpler one:

(v'): $\exists\, v > 0 : v \leq K_2 \cdot \inf f[Mv,v]$, where $K_2 = \sup\limits_{t\in[0,T]} \int_a^b k(t,s)\,\mathrm{d}s$.

The following theorem is a slight extension of the Theorem 2.1 from [9]:

Theorem 7. *If **(f1-2)**, **(h1-2)**, **(k1-3)**, **(M)**, **(u)** ,**(v)** take place and u and v found are distinct, then the problem has at least one solution y such that either*

(A) $0 < u < |y|_\infty < v$ *and* $y(t) \geq Mu$, $\forall t \in [a,b]$ *(if $u < v$)*

or

(B) $0 < v < |y|_\infty < u$ *and* $y(t) \geq Mv$, $\forall t \in [a,b]$ *(if $v < u$).*

The aim of this note is to give sufficient conditions to ensure **(u)** and **(v)**. Our results complement those from [1] and [9].

2. Limit type existence results

We will begin with a very simple lemma.

Lemma 8. *If f satisfies **(f1)** and is not bounded, then there exists $u > 0$ as large as needed such that* $\sup f[0,u] = f(u)$.

Proof. Assume that there exists $u_0 > 0$ such that

$$\sup f\,[0,u] > f(u),\ \forall u \geq u_0.$$

Fix $u \geq u_0$. Using the continuity of f, we can find $\overline{u} \in [u_0, u]$ such that $f\,(\overline{u}) = \sup f\,[u_0, u]$. Moreover,

$$\sup f\,[0,u] \geq \sup f\,[0,\overline{u}] > f(\overline{u}) = \sup f\,[u_0,u]\,.$$

Hence,

$$\forall\, u \geq u_0 : \sup f\,[0,u] > \sup f\,[u_0,u]\,. \tag{4}$$

But $\sup f\,[0,u] = \max\left\{\sup f\,[u_0,u]\,,\ \sup f\,[0,u_0]\right\}$ and using (4), we obtain that

$$\forall\, u \geq u_0 : \sup f\,[0,u_0] > \sup f\,[u_0,u] \geq f\,(u)\,.$$

Concluding,

$$f\,(u) < M := \sup f\,[0,u_0] < +\infty,\ \forall\, u \geq u_0$$
$$f\,(u) \leq M := \sup f\,[0,u_0] < +\infty,\ \forall\, u \leq u_0$$

which represents a contradiction with the unboundedness of f.

Therefore,

$$\forall u_0 > 0, \exists u \geq u_0 : \sup f\,[0,u] = f(u).$$

The lemma is proved. □

Assuming their existence, we make the following notations:

Notation 9. $L_\infty = \lim\limits_{u\to\infty} \frac{f(u)}{u}$, $L_0 = \lim\limits_{v\downarrow 0} \frac{f(v)}{v}$.

The following partial results take place:

Proposition 10. *If $L_\infty < \frac{1}{K_1}$ and **(f1)** is satisfied, then there exists $u > 0$ as large as needed such that **(u)** is satisfied.*

Proof. If f is bounded by some constant $M > 0$ (which means that $L_\infty = 0$), then **(u)** is satisfied for every $u > |h|_\infty + K_1 M$.

If f is unbounded, then $\lim\limits_{u\to\infty} \frac{f(u)}{u} < \frac{1}{K_1}$ implies that $\lim\limits_{u\to\infty} \frac{|h|_\infty + K_1 f(u)}{u} < 1$, which means that there exists some $u_0 > 0$ for which $\frac{|h|_\infty + K_1 f(u)}{u} \leq 1, \forall\, u \geq u_0$.

Using the above lemma, there exists u large enough (i.e. $u \geq u_0$) such that $f(u) = \sup f[0,u]$. Therefore, there exists $u > 0$ large enough such that $\frac{|h|_\infty + K_1 \sup f[0,u]}{u} \leq 1$, which concludes our proof. □

Proposition 11. *If $L_0 > \frac{1}{MK_2}$, **(f1)** and **(M)** take place, then there exists $v_0 > 0$ such that **(v')** is satisfied for every $v \in (0, v_0]$.*

Proof. $\lim\limits_{\substack{v \to 0 \\ v > 0}} \frac{f(v)}{v} > \frac{1}{MK_2}$ implies the existence of some $v_0 > 0$ such that $f(v) \geq \frac{v}{MK_2}$ for every $v \in (0, v_0]$.

Fix $v \in (0, v_0]$ and take any $v' \in [Mv, v]$. We will have that $Mv \leq v' \leq v \leq v_0$, which implies $\frac{v}{K_2} = \frac{Mv}{MK_2} \leq \frac{v'}{MK_2}$; since $\frac{v'}{MK_2} \leq f(v')$, we obtain that $v \leq K_2 f(v')$. Since v' is arbitrary chosen in $[Mv, v]$, we can conclude that $v \leq K_2 \cdot \inf f[Mv, v]$. □

Using the same arguments as in the proofs of Propositions 10 and 11, we can easily prove also the following two results.

Proposition 12. *If $L_0 < \frac{1}{K_1}$, $h = 0$ on $[0, T]$ and **(f1)** is satisfied, then there exists $u_0 > 0$ such that **(u)** is satisfied for every $u \in (0, u_0]$.*

Proposition 13. *If $L_\infty > \frac{1}{MK_2}$, **(f1)** and **(M)** take place, then there exists $v_0 > 0$ such that **(v')** is satisfied for every $v \in [v_0, +\infty)$*

Using the results form Propositions 10 and 11, we can choose $u > v$ such that the conditions **(u)** and **(v')** are satisfied. Also, using the results form Propositions 12 and 13, we can choose $u < v$ such that the conditions **(u)** and **(v')** are satisfied. Applying Theorem 7, we obtain the following existence results for our problem.

Theorem 14. *If $\begin{cases} L_\infty < \frac{1}{K_1},\ L_0 > \frac{1}{MK_2} \\ \textbf{(f1), (h1-2), (k1-3), (M)} \end{cases}$ take place, then there exists a non-trivial solution y such that $0 < v < |y|_\infty < u$ and $y(t) \geq Mv$, $\forall\, t \in [a, b]$, where u comes from the condition **(u)** large enough and v comes from **(v')** small enough.*

Theorem 15. *If $\begin{cases} L_0 < \frac{1}{K_1},\ L_\infty > \frac{1}{MK_2} \\ \textbf{(f1), (k1-3), (M),}\ h \equiv 0 \end{cases}$ take place, then there exists a non-trivial solution y such that $0 < u < |y|_\infty < v$ and $y(t) \geq Mu$, $\forall\, t \in [a, b]$, where u comes from the condition **(u)** small enough and v comes from **(v')** large enough.*

3. Applications

We study the existence of positive non-trivial solutions for the two point boundary value problem

$$\begin{cases} -y'' = f(y) \text{ on } [0,1] \\ y(0) = \alpha,\, y(1) = \beta \end{cases}, \quad y \in C^2[0,1]. \tag{5}$$

using the two final results from the previous section.

This problem can be written as a Fredholm integral equation:

$$y(t) = h(t) + \int_0^1 G(t,s) f(y(s))\, \mathrm{d}s, \quad y \in C[0,1] \tag{6}$$

where

$$h(t) = (1-t)\alpha + t\beta$$

and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1 \\ t(1-s), & 0 \le t \le s \le 1 \end{cases}$$

is the Green function associated to this problem.

We proceed by checking the conditions of Theorem 14 and Theorem 15.

Since $G : [0,1] \times [0,1] \longrightarrow [0,+\infty)$ is continuous, conditions **(f1)**, **(h1)** and **(h1)** hold if:

$$\begin{cases} f : [0,+\infty) \longrightarrow [0,+\infty) \text{ is continuous} \\ \alpha \ge 0,\, \beta \ge 0 \end{cases} \tag{7}$$

Moreover, $|h|_\infty = \max\{\alpha,\beta\}$ and

$$K_1 = \sup_{t\in[0,1]} \int_0^1 G(t,s)\, \mathrm{d}s = \sup_{t\in[0,1]} \frac{t(1-t)}{2} = \frac{1}{8}. \tag{8}$$

The conditions **(h2)**, **(k2-3)**, **(M)** and $L_0 > \frac{1}{MK_2}$ (respectively, $L_\infty > \frac{1}{MK_2}$) remain to be fulfilled.

The condition **(h2)** becomes

$$\begin{aligned} (1-a)\alpha + a\beta \ge M\beta, \text{ if } \alpha \le \beta \\ (1-b)\alpha + b\beta \ge M\alpha, \text{ if } \alpha > \beta \end{aligned} \tag{9}$$

The conditions **(k2-3)** give

$$G(t,s) \ge M s(1-s),\ \forall\, t,\ s \in [0,1] \tag{10}$$

since $\sup_{t\in[0,1]} G(t,s) = s(1-s)$ (attained for $t = s$). It can be shown easily that together **(k2-3)** and **(M)** are equivalent to

$$0 < M \le a < b \le 1 - M \tag{11}$$

which also gives

$$0 < M < \frac{1}{2} \tag{12}$$

Also, it is not difficult to prove that (11) and (12) assure that (9) (hence **(h2)**) is satisfied.

In order to have in $L_0 > \frac{1}{MK_2}$ (respectively, in $L_\infty > \frac{1}{MK_2}$) a less restrictive condition, we will choose a, b and M such that MK_2 becomes maximum. Therefore, $[a,b]$ is the largest possible ($a = M$, $b = 1 - M$). We obtain from simple computation that:

$$M \sup_{t\in[0,1]} \int_M^{1-M} G(t,s)\,\mathrm{d}s = \frac{M}{8} - \frac{M^3}{2} \text{ for } t^* = \frac{1}{2}$$

$$\sup_{M\in\left(0,\frac{1}{2}\right)} \frac{M}{8} - \frac{M^3}{2} = \frac{\sqrt{3}}{72} \text{ for } M = \frac{\sqrt{3}}{6} \text{ and } \frac{1}{MK_2} = 24\sqrt{3}$$

Concluding, by applying Theorem 14, the following result takes place:

Theorem 16. *Assume that:*

(i) $f : [0,+\infty) \to [0,+\infty)$ *is continuous*
(ii) $\alpha,\ \beta \ge 0$
(iii) $\lim_{u\to 0} \frac{f(u)}{u} > 24\sqrt{3}$ *and* $\lim_{u\to\infty} \frac{f(u)}{u} < 8$

Then there exists $u > 0$ *as large as wanted such that*

$$\max\{\alpha,\beta\} + \tfrac{1}{8}\sup f\,[0,u] \le u,$$

$v \in (0,u)$ *small enough such that*

$$v \le \tfrac{1}{12} \inf f\left[\tfrac{\sqrt{3}}{6}v, v\right]$$

and a solution y *of the problem (5) such that:*

$$0 < v < |y|_\infty < u \text{ and } y(t) \ge \tfrac{\sqrt{3}}{6}v,\ \forall\, t \in \left[\tfrac{\sqrt{3}}{6}, 1 - \tfrac{\sqrt{3}}{6}\right].$$

Also, by applying Theorem 15, the following result takes place:

Theorem 17. *Assume that:*

(i) $f : [0, +\infty) \to [0, +\infty)$ *is continuous*

(ii) $\alpha = \beta = 0$

(iii) $\lim\limits_{u \to 0} \frac{f(u)}{u} < 8$ *and* $\lim\limits_{u \to \infty} \frac{f(u)}{u} > 24\sqrt{3}$

Then there exists $u > 0$ *small enough such that*

$$\tfrac{1}{8} \sup f\,[0, u] \le u,$$

$v > u$ *large enough such that*

$$v \le \tfrac{1}{12} \inf f \left[\tfrac{\sqrt{3}}{6} v, v\right]$$

and a solution y *of the problem (5) such that*

$$0 < u < |y|_\infty < v \text{ and } y(t) \ge \tfrac{\sqrt{3}}{6} u, \ \forall\, t \in \left[\tfrac{\sqrt{3}}{6}, 1 - \tfrac{\sqrt{3}}{6}\right].$$

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