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DATA DEPENDENCE OF FIXED POINTS FOR MEIR-KEELER TYPE OPERATORS

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Abstract. The purpose of this note is to present data dependence results for the fixed points of Meir-Keeler type operators.

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1. Preliminaries

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. For the convenience of the reader we recall some of them.

Let (X, d) be a metric space and $f : X \to X$ be an operator. Then $f^0 := 1_X,\ f^1 := f, \ldots, f^{n+1} = f \circ f^n,\ n \in \mathbb{N}$ denote the iterate operators of f. A sequence of successive approximations of f starting from $x \in X$ is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X with $x_0 = x,\ x_{n+1} = f(x_n)$, for $n \in \mathbb{N}$. By $F_f := \{x \in X |\ x = f(x)\}$ we will denote the fixed point set of the operator f.

Also we will use the following symbols:

$P(X) = \{Y \subset X |\ Y \text{ is nonempty}\}$, $P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\}$.

If $T : X \to P(X)$ is a multivalued operator, then $F_T := \{x \in X |\ x \in T(x)\}$ denotes the fixed point set of the T.

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The Pompeiu-Hausdorff generalized functional will be denoted by H. Also, the generalized functional δ, is used in the main section of the paper, i. e. $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\delta(A, B) = \sup\{d(a, b)|\ a \in A,\ b \in B\}$.

It is well-known that $(P_{b,cl}(X), H)$ is a complete metric space provided (X, d) is a complete metric space.

For more details and basic results concerning the above notions see for example [3], [7] among others.

By definition, $f : X \to X$ is a Meir-Keeler operator if it satisfies the condition:

$$\text{for all } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } \varepsilon \leq d(x, y) < \varepsilon+\delta \Rightarrow d(f(x), f(y)) < \varepsilon.$$

Meir-Keeler operators were introduced in order to generalize the well-known Banach contraction principle. In fact Meir and Keeler [6] proved the following result.

Theorem 1.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a Meir-Keeler operator. Then $Fix f = \{x^*\}$ and for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations of f starting from x converges to the unique fixed point.*

For more details about generalizations of contractive-type conditions see Kirk [4] or Rus [9].

A function $\varphi : [0, \infty[\to [0, \infty[$ is said to be an L-function if $\varphi(0) = 0$, $\varphi(s) > 0$ for all $s > 0$, and for every $s > 0$ there exists $u > s$ such that:

$$\varphi(t) \leq s \text{ for } t \in [s, u].$$

Recall also the concept of modulus of uniform continuity. If (X, d) is a metric space and $f : X \to X$ is an operator then, by definition, the modulus of uniform continuity of f is defined by:

$$\delta(\varepsilon) := \sup\{\lambda : d(x, y) < \lambda \ \Rightarrow d(f(x), f(y)) < \varepsilon\}, \text{ for } \varepsilon > 0 \text{ and } \delta(0) = 0.$$

In [5] T.C. Lim proved the following characterization theorem:

Theorem 1.2. (Lim [5]) *Let X be a metric space. Let $f : X \to X$ and let $\delta(\varepsilon)$ be its modulus of uniform continuity. Then the following assertions are equivalent:*

(i) f is a Meir-Keeler operator

(ii) $\delta(\varepsilon) > \varepsilon$, for each $\varepsilon > 0$

(iii) there exists a nondecreasing and right continuous L-function φ such that $d(f(x), f(y)) < \varphi(d(x, y))$ for each $x, y \in X$, with $x \neq y$.

Remark 1.3. *From the proof of the above theorem, it follows that the mapping φ can be represented as follows: $\varphi(t) := \sup\{s : \frac{s+\delta(s)}{2} \leq t\}$, where $\delta(s)$ is the modulus of uniform continuity of f.*

For the multivalued case, the following notion was introduced by Reich [8].

By definition, $F : X \to P_{cl}(X)$ is called a multivalued Meir-Keeler operator if

$$\text{for all } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } \varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow H(F(x), F(y)) < \varepsilon.$$

Lim also proved that the above characterization of Meir-Keeler operators remains true in the multivalued case (Theorem 2 in [5]).

In the main section of the paper, we will need the notion of L-space in the sense of Fréchet.

Let X be a nonempty set and $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X,\ n \in \mathbb{N}\}$.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x,$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition an element of $c(X)$ is a convergent sequence and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we write $x_n \to x$ as $n \to \infty$.

In what follows we will denote an L-space by $(X, \to)$.

Actually, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces (in Perov's sense: $d(x, y) \in \mathbb{R}^m_+$, in Luxemburg-Jung's sense: $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$, $d(x, y) \in K$, K a cone in an ordered Banach space, $d(x, y) \in E$, E an ordered linear space with a notion of linear convergence, etc.), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details see Fréchet [2], Blumenthal [1].

2. Main results

Definition 2.1. Let $(X, \to)$ be an L-space. Then, $f : X \to X$ is called a weakly Picard operator if the sequence $(f^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f. If the fixed point is unique, then f is said to be a Picard operator.

Example 2.2. *Let (X, d) be a complete metric space and $f : X \to X$ an* **a-contraction***, i. e. $a \in]0, 1[$ and $d(f(x), f(y)) \leq a \cdot d(x, y)$, for each $x, y \in X$. Then the operator f is Picard. (Banach-Caccioppoli)*

Example 2.3. *Let (X, d) be a complete generalized metric space ($d(x, y) \in \mathbb{R}^m_+$) and $A \in M_{mm}(\mathbb{R}_+)$, such that, $A^n \to 0$ as $n \to \infty$. If $f : X \to X$ is an* **A-contraction***, i. e., $d(f(x), f(y)) \leq Ad(x, y)$, for all $x, y \in X$, then it is Picard operator. (Perov)*

Example 2.4. *Let (X, d) be a complete metric space and $f : X \to X$ be a Meir-Keeler type operator, i. e. for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\eta \leq d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$. Then f is a Picard operator. (Meir-Keeler)*

In [10] the basic theory of Picard operators is presented. Some data dependence results for several classes of singlevalued and multivalued (weakly) Picard operators are proved in A. Petruşel [7], Rus [10], I. A. Rus and S. Mureşan [11], Rus, A. Petruşel and A. Sîntămărian [13] and Sîntămărian [14]. The data dependence of the fixed points for singlevalued and multivalued Meir-Keeler type operators was announced as an open problem in I. A. Rus, A. Petruşel and G. Petruşel [12]. A partial answer to this question is the main purpose of this note.

We start the main section of the paper by presenting a data dependence result for a singlevalued Meir-Keeler operator.

Theorem 2.5. *Let (X, d) be a complete metric space and $f_i : X \to X$ be Meir-Keeler operators, for $i \in \{1, 2\}$. Denote by φ the mapping associated with f_1 from Lim's theorem. Assume that:*

(a) $t - \varphi(t) \to +\infty$, as $t \to +\infty$.

(b) there exists $\eta > 0$ such that $d(f_1(x), f_2(x)) \leq \eta$, for all $x \in X$.

Then $d(x^*_1, x^*_2) \leq t_\eta$, where $t_\eta := \sup\{t > 0 | t - \varphi \leq \eta\}$ and x^*_i denotes the unique fixed point of f_i, for $i \in \{1, 2\}$.

Proof. Using Lim's characterization theorem, we have successively: $d(x_1^*, x_2^*) = d(f_1(x_1^*), f_2(x_2^*)) \leq d(f_1(x_1^*), f_1(x_2^*)) + d(f_1(x_2^*), f_2(x_2^*)) < \varphi(d(x_1^*, x_2^*)) + \eta$. Then $d(x_1^*, x_2^*) \leq t_\eta$. □

For the multivalued case we have:

Theorem 2.6. *Let (X, d) be a complete metric space and $T_i : X \to P_{cl}(X)$ be multivalued Meir-Keeler operators, for $i \in \{1, 2\}$. Denote by φ_i $(i \in \{1, 2\})$ the corresponding mapping from Lim's characterization result.*

Suppose that:

ii) $\sum_{k=1}^{n} \varphi_i^k(t) < +\infty$, *for all* $t \in \mathbb{R}_+$.

i) there exists $\eta_1, \eta_2 > 0$ *such that*

$$\delta(x, T_2(x)) \leq \eta_1, \text{ for all } x \in F_{T_1}$$

and

$$\delta(y, T_1(y)) \leq \eta_2, \text{ for all } y \in F_{T_2}.$$

Then $H(F_{T_1}, F_{T_2}) \leq \eta + \max\{s_1(\eta + \frac{1}{u}), s_2(\eta + \frac{1}{u})\}$, *where* $\eta := \max\{\eta_1, \eta_2\}$ *and* $s_i(t)$ *denotes the sum of the series* $\varphi_i^k(t)$.

Proof. The conclusion follows from the multivalued version of Lim's characterization theorem for a multivalued Meir-Keeler operator and a slightly modified version of Theorem 2 from Sîntămărian [14]. □

Appendix. For the convenience of the reader, we recall here the modified version of Theorem 2 in Sîntămărian [14].

Theorem 2. *Let (X, d) be a complete metric space and $T_i : X \to P_{cl}(X)$ be a φ_i-contraction, for $i \in \{1, 2\}$. Assume $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and* $\sum_{k=1}^{n} \varphi^k(t) < +\infty$, *for all* $t \in \mathbb{R}_+$ *and for* $i \in \{1, 2\}$.

Suppose that:

i) there exists $\eta_1, \eta_2 > 0$ *such that*

$$\delta(x, T_2(x)) \leq \eta_1, \text{ for all } x \in F_{T_1}$$

and

$$\delta(y, T_1(y)) \leq \eta_2, \text{ for all } y \in F_{T_2}.$$

Then

a) F_{T_1} *and* F_{T_2} *are nonempty and closed* *b)* $H(F_{T_1}, F_{T_2}) \leq \eta + \max\{s_1(\eta + \frac{1}{u}), s_2(\eta + \frac{1}{u})\}$*, where* $\eta := \max\{\eta_1, \eta_2\}$ *and* $s_i(t)$ *denotes the sum of the series* $\varphi_i^k(t)$*.*

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