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ATTRACTORS AND FIXED POINTS OF WEAKLY CONTRACTING RELATIONS

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Abstract. The well known property of denseness of periodic points in Julia sets and in the attractors of hyperbolic IFS's is stated for weakly contracting multifunctions. This result is obtained as a consequence of the (periodic) Shadowing Property for weakly contracting compact valued multifunctions. The last property is also stated in this paper.

Key Words and Phrases: Attractor, set-valued maps, weak contractions, fixed point, shadowing property.

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1. Introduction

R.Williams [18] in 1971 has studied finite collections of contractions and the set of fixed points for various their compositions. He was concerned with topological properties of these sets (compactness, topological dimension, etc.). More precisely, R.Williams has proved that given contractions $f_1, \dots, f_m$ the set

$$K = \operatorname{cls} \Big(\bigcup_{n \geq 1} \bigcup_{1 \leq i_1, \dots, i_n \leq m} \operatorname{Fix} \left(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n} \right) \Big) \tag{1.1}$$

is the unique compact set satisfying the equation

$$K = f_1(K) \cup \cdots \cup f_m(K). \tag{1.2}$$

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Although this interesting work of Williams contains the main ideas of the modern fractal geometry (e.g. the equation (1.2) as a prototype of self-similarity, strings methods to prove the equality (1.1) as prototype of symbolic Iterated Function Systems, topological dimension of the attractor), it seems that this paper remained unobserved by many specialists in fractals.

In 1981 J.Hutchinson [7], when studying self-similar sets, also considered the equation (1.2) and solved it using Banach Fixed Point Principle in the Hausdorff-Pompeiu metric space. He proved that (1.1) represents the unique solution of the equation (1.2).

On the other hand, G.Julia [8], when studying iterations of rational functions, has proved that a set, later coined the Julia set, is completely invariant and equals the closure of repelling periodic points. The complete invariance corresponds to the equality (1.2) with $f_1, \ldots, f_m$ as branches of the inverse of the rational function, while denseness of repelling periodic points means the equality (1.1).

These two aspects of dynamics: a ordinary one, generated by (forward) iterations of an expanding function, and another one as iterations of finite collections of contracting functions (which can be regarded as a backward evolution of a rational function) have been evolved for a long time separately.

After the seminal Hutchinson's work [7] a series of researches dedicated to Iterated Function Systems (IFS) occurred (see, e.g. [3] and the bibliography therein).

In 1985 M.Hata [6] has been relaxed the contractivity condition up to a weak form of contraction and obtained similar results to those of Williams and of Hutchinson, i.e. he proved the existence of the attractor as the solution of the equation (1.2) and stated the equality (1.1).

In all these papers the equality (1.1) is proved by using symbolic IFS.

In [5] the authors have stated the existence of the attractor and have proved the equality (1.1) for general contracting relations on metric spaces. Denseness of periodic points on the attractor has been obtained as a consequence of the Shadowing Property of contracting relations. The latter in turn has been proved by using a Fixed Point Theorem for contracting multifunctions with bounded and closed values.

Here we are concerned with the equality (1.1) as the solution of an equation similar to (1.2) for compact valued weak contractions. The existence of a compact attractor for such a multifunction has been stated by A.Petruşel and I.A.Rus (see [12, 14, 15]).

We formulate and prove the Shadowing Property for a larger class of relations, namely for so called weakly contracting functions with compact values. As a consequence we obtain that the periodic points form a dense subset on the attractor. Thus, we obtain a equality similar to (1.1).

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2. Preliminaries

Let (X, d) be a complete metric space. A *relation on* X is a subset $f \subset X \times X$. Any relation can be regarded also as a (set-valued) function from X to the power set $\mathcal{P}(X)$, associating to each $x \in X$ a subset $f(x)$ of X. These two aspects of relations (set theoretical and functional) allows one to apply subset operations, such as union, intersection and closure, on the one hand, and the functional operations, such as composition, inverse and iterations, on the other hand.

A relation on a metric space is said to be *closed*, if it is a closed subset of the Cartesian product of the space with itself. In a compact space this is equivalent to the upper semicontinuity of the relation (see [1] or [10]).

For two relations $f, g : X \to \mathcal{P}(X)$ we define the *composition* $g \circ f : X \to \mathcal{P}(X)$ by: $(x, y) \in g \circ f$, if there exists $z \in X$ such that $(x, z) \in f$ and $(z, y) \in g$. The *inverse* of f is, by definition, $f^{-1} := \{(y, x) \mid (x, y) \in f\}$. The composition is associative, so for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ we define f^n to be the n-fold composition of f, and similarly $f^{-n} := \left(f^{-1}\right)^n$. From associativity it follows that $f^{m+n} = f^m \circ f^n$ for $m, n \geq 0$ or $m, n \leq 0$. We call $\mathcal{O}f := \bigcup_{n \geq 1} f^n$ the *orbit relation*.

Given a relation $f : X \to \mathcal{P}(X)$ one can construct the Hutchinson-Barnsley mapping $f_* : \mathcal{P}(X) \to \mathcal{P}(X)$, defined for any $A \in \mathcal{P}(X)$ by $f_*(A) := \overline{f[A]}$, where $f[A] = \bigcup_{a \in A} f(a)$ and $\overline{S}$ stands for the closure of the set S.

A subset $A \subset X$ is said to be *positive invariant* with respect to a relation f on X (written "A is f *+invariant*"), if $f[A] \subset A$. Further, A is called f *invariant*, if $f[A] = A$. The last means that A is f +invariant and, in addition, $f^{-1}(x) \cap A \neq \emptyset$ for all $x \in A$.

For relations among many possibilities of generalization of an orbit is the notion of the chain. The finite or infinite sequence $\{x_n\}$ in X is called a *chain* for a relation $f : X \to \mathcal{P}(X)$, if $x_{n+1} \in f(x_n)$ for all n, or, in other words, if $(x_n, x_{n+1}) \in f$.

Given a relation $f : X \to \mathcal{P}(X)$ a point $x \in X$ is called a *fixed point* for f, if $x \in f(x)$. Thus, $x \in \mathcal{O}f(x)$ if and only if there exists $n \geq 1$ such that $x \in f^n(x)$. Such point is called a *periodic point* for f.

Denote by $\mathcal{P}_{b,cl}(X)$ and $\mathcal{P}_{cp}(X)$ the set of all nonempty bounded and closed and respectively compact subsets of X.

Recall that for any two bounded and closed subsets B_1 and B_2 of a metric space (X, d) the quantity $\varrho(B_1, B_2)$, given by

$$\varrho(B_1, B_2) := \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} d(b_1, b_2),$$

defines the Hausdorff-Pompeiu metric as follows:

$$H(B_1, B_2) := \max\{\varrho(B_1, B_2),\ \varrho(B_2, B_1)\}.$$

This metric on the space $\mathcal{P}_{b,cl}(X)$ or $\mathcal{P}_{cp}(X)$ is complete, if d is complete.

The following lemmas will be useful in the sequel.

Lemma 2.1. *Let A and B be nonempty bounded subsets of X. Then for any $a \in A$ and any $\varepsilon > 0$ there exists $b \in B$ such that*

$$d(a, b) \leq \varrho(a, B) + \varepsilon \leq H(A, B) + \varepsilon. \tag{2.1}$$

Proof. Since $\varrho(a, B) = \inf\{d(a, b) \mid b \in B\}$, for any $a \in A$ and any $\varepsilon > 0$ there exists $b \in B$ such that (2.1) holds. □

Lemma 2.2. [14] *Let A and B be compacts in X. Then for any $a \in A$ there exists $b \in B$ such that*

$$d(a, b) = \varrho(a, B) \leq H(A, B).$$

Lemma 2.3. [5] *For any nonempty bounded subsets $B_1, B_2 \subset X$ and a point $x \in X$ the following inequality holds:*

$$\varrho(x, B_1) \leq \varrho(x, B_2) + H(B_1, B_2).$$

3. Weak contractions

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* [6, 14] if:

- φ is monotonically increasing, i.e. $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;
- $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$.

Lemma 3.1. [14] *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$ for all $t > 0$.*

Theorem 3.2. *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a right continuous comparison function, then for any $b > a > 0$ there exists $M = \max\{\varphi(t)/t \,|\, a \leq t \leq b\}$ and $0 \leq M < 1$.*

Proof. Let $b > a > 0$. Consider the function g, $g(t) = \varphi(t)/t$ for any $t \in [a, b]$. Since $\varphi(t)/t < 1$ for all $t > 0$, there exists $M = \sup\{g(t) \,|\, a \leq t \leq b\} < \infty$. Thus, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $[a, b]$ such that $\{g(t_n)\} \to M$ as $n \to \infty$. Due to the compactness of the segment $[a, b]$ one can choose a convergent subsequence. Therefore, we can consider a convergent sequence $\{t_n\}_{n\in\mathbb{N}} \to \tau \in [a, b]$ such that $\{g(t_n)\} \to M$ as $n \to \infty$.

We have only two possible cases.

a) If φ is continuous in τ, then g is also continuous in τ and $g(\tau) = M$.

b) If φ is discontinuous in τ, let denote $\alpha = \lim\limits_{t\to\tau-0} \varphi(t)$ and $\beta = \varphi(\tau)$. Since φ is monotonically increasing and right continuous, it follows that $\beta > \alpha \geq 0$.

Take $\varepsilon = (\beta - \alpha)\tau/\beta > 0$. For any $t \in (\tau - \varepsilon, \tau)$ one has $\varphi(t) \leq \alpha$ and

$$g(t) = \frac{\varphi(t)}{t} < \frac{\alpha}{\tau - \varepsilon} = \frac{\beta}{\tau} = g(\tau) < M.$$

Since $\{\tau_n\} \to \tau$ and $\{g(t_n)\} \to M$ as $n \to \infty$, it follows that $t_n \geq \tau$ for big enough n. Due to the right continuity of the function g one has $g(\tau) = \lim\limits_{n\to\infty} g(t_n) = M$.

Thus, in any case the function g reaches its greatest value on the segment $[a, b]$ and there exists $M = \max\{\varphi(t)/t \,|\, a \leq t \leq b\}$. Obviously, $0 \leq M < 1$. □

Following [6, 14], we will call a relation $f : X \to \mathcal{P}_{b,cl}(X)$ as a *weak contraction*, if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(f(x), f(y)) \leq \varphi(d(x, y)), \qquad \forall\, x, y \in X. \tag{3.1}$$

In this case we will say also that f is a *contraction with respect to* φ, or that f is a φ-*contraction* (see [14]).

Example 3.1. The relation $f : \mathbb{R}_+ \to \mathcal{P}_{cp}(\mathbb{R}_+)$, $f(x) = \left[\frac{x}{2(1+x)}, \frac{x}{1+x}\right]$, is a weak contraction with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \frac{t}{1+t}$, as a comparison function.

Given a weak contraction $f : X \to \mathcal{P}_{b,cl}(X)$ we will say that a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an *optimal comparison function for* f if the following conditions hold:

- f is a φ-contraction;
- for any other comparison function ψ such that f is a ψ-contraction one has that $\psi(t) \geq \varphi(t)$ for all $t \geq 0$.

Remark 3.1. Obviously, the optimal comparison function for f, if it exists, is unique.

Theorem 3.3. *Any weak contraction* $f : X \to \mathcal{P}_{b,cl}(X)$ *admits an optimal comparison function.*

Proof. Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction. Denote by Ψ_f the nonempty set of all comparison functions with respect to which f is a weak contraction, and put

$$\varphi(t) = \inf\{\psi(t) \,|\, \psi \in \Psi_f\}, \qquad \forall\, t \geq 0.$$

The function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is well defined, since $\psi(t) \geq 0$ for any $\psi \in \Psi_f$, $t \geq 0$. We will show that φ is an optimal comparison function for f.

It is easily seen that φ is monotonically increasing. For, given $0 < t_1 < t_2$ one has $\psi(t_1) \leq \psi(t_2)$ for any $\psi \in \Psi_f$ and, taking infimum on Ψ_f, the corresponding inequality for φ is obtained as well. Moreover, since $0 \leq \varphi(t) \leq \psi(t)$ for any $\psi \in \Psi_f$ and for all $t \geq 0$, one has that

$$\varphi^2(t) = \varphi(\varphi(t)) \leq \varphi(\psi(t)) \leq \psi(\psi(t)) = \psi^2(t).$$

One can show by induction that $0 \leq \varphi^n(t) \leq \psi^n(t)$ for all $t \geq 0$ and any $n \geq 0$. It follows that $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$. This means that φ is a comparison function.

Finally, we prove that $\varphi \in \Psi_f$, i.e. f is a φ-contraction. For any $x, y \in X$ and any $\psi \in \Psi_f$ we have

$$H(f(x), f(y)) \leq \psi(d(x, y)).$$

Taking infimum of the right hand side with respect to $\psi \in \Psi_f$, one obtains (3.1). Thus, φ is an optimal comparison function for f. $\square$

Denote by $B(x, r) = \{z \in X \mid d(x, z) < r\}$ the open ball of radius r centered at x.

We will say that the space (X, d) is *γ-convex*, if for any $x, y \in X$ and any $r_1, r_2 > 0$ such that $r_1 + r_2 > d(x, y)$ we have $B(x, r_1) \cap B(y, r_2) \neq \emptyset$.

Example 3.2. Examples of γ-convex metric spaces: any linear normed space, the space of rational points $\mathbb{Q}^m$ endowed with the standard metric from $\mathbb{R}^m$. At the same time, the space X of shape of the letter "H" endowed with the standard metric from $\mathbb{R}^2$ is not γ-convex.

Remark 3.2. C.Ursescu [17] studied a class of special metric spaces, having a property equivalent to the γ-convexity (see Appendix).

Theorem 3.4. *Let (X, d) be a γ-convex metric space. Any weak contraction $f : X \to \mathcal{P}_{b,cl}(X)$ admits a continuous optimal comparison function.*

Proof. Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be the optimal comparison function for f, the latter exists due to Theorem 3.3. We have to show that φ is continuous.

Assume, for the contrary, that there exists $c > 0$ such that

$$\lim_{t \to c-0} \varphi(t) = \alpha < \beta = \lim_{t \to c+0} \varphi(t).$$

The fact that φ is the optimal comparison function for f implies that for any $\varepsilon > 0$, in particular for $\varepsilon = (\beta - \alpha)/2 > 0$, there exist $x_0, y_0 \in X$ such that $c \leq d(x_0, y_0) < c + \varepsilon$ and

$$H(f(x_0), f(y_0)) > \varphi(d(x_0, y_0)) - \varepsilon \geq \beta - \varepsilon = \frac{\alpha + \beta}{2}. \tag{3.2}$$

Since X is γ-convex and $c + \varepsilon > d(x_0, y_0)$, it follows that there exists $z_0 \in X$ such that $z_0 \in B(x_0, c) \cap B(y_0, \varepsilon) \neq \emptyset$. From $d(x_0, z_0) < c$ and $d(z_0, y_0) < \varepsilon$

one obtains, using Lemma 3.1,

$$H(f(x_0), f(y_0)) \leq H(f(x_0), f(z_0)) + H(f(z_0), f(y_0)) \leq$$
$$\varphi(d(x_0, z_0)) + \varphi(d(z_0, y_0)) \leq \lim_{t \to c-0} \varphi(t) + \varphi(\varepsilon) < \alpha + \varepsilon = \frac{\alpha + \beta}{2}.$$

The latter contradicts (3.2). Therefore, φ is continuous. □

Remark 3.3. The condition of γ-convexity, imposed on the space (X, d), is essential for the continuity of the optimal comparison function as the following examples show.

Example 3.3. Let $X = [0, 1] \cup [3, 4]$ be a metric space, endowed with the standard metric from $\mathbb{R}$. The space X is not γ-convex. Consider the function $f : X \to X$,

$$f(x) = \begin{cases} 0, & x \in [0, 1], \\ 1, & x \in [3, 4]. \end{cases}$$

The function f is a weak contraction with the optimal comparison function

$$\varphi(t) = \sup_{|x-y| \leq t} |f(x) - f(y)| = \begin{cases} 0, & 0 \leq t < 2, \\ 1, & t \geq 2, \end{cases}$$

which is not continuous. At the same time, f admits a continuous comparison function.

Example 3.4. Let $X = \{x_n\}_{n \in \mathbb{N}}$, with $x_0 = 0$, $x_{n+1} = x_n + 1 + \frac{1}{n+1}$ $(n \geq 0)$, be a metric space endowed with the standard metric from $\mathbb{R}$. The space X is not γ-convex. Consider the function $f : X \to X$, defined by $f(x_n) = x_{n+1}$ for all $n \geq 0$. One can show that f is a weak contraction, which does not admit a continuous (nor yet a right continuous) comparison function.

Theorem 3.5. *Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction with respect to a right continuous comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then the Hutchinson-Barnsley mapping $f_* : \mathcal{P}_{b,cl}(X) \to \mathcal{P}_{b,cl}(X)$, $f_*(A) = \overline{f[A]}$, is also a φ-contraction, i.e. for any $A, B \in \mathcal{P}_{b,cl}(X)$ the following inequality holds*

$$H(f_*(A), f_*(B)) \leq \varphi(H(A, B)). \tag{3.3}$$

Proof. Firstly, the mapping f_* is well defined. If $A \in \mathcal{P}_{b,cl}(X)$, then $f_*(A) = \overline{f[A]}$ is a bounded and closed subset of X.

Let $A, B \in \mathcal{P}_{b,cl}(X)$. Fix an arbitrary $\varepsilon > 0$. By Lemma 2.1 for any $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$. Analogously, for any $\xi \in f(a) \subset f[A]$ there exists $\eta \in f(b) \subset f[B] \subset f_*(B)$ such that

$$d(\xi, \eta) \leq H(f(a), f(b)) + \varepsilon \leq \varphi(d(a, b)) + \varepsilon \leq \varphi(H(A, B) + \varepsilon) + \varepsilon.$$

Therefore, for any $\xi \in f_*(A) = \overline{f[A]}$ there exists $\xi' \in f[A]$ and $\eta \in f_*(B)$ such that $d(\xi, \xi') \leq \varepsilon$ and

$$d(\xi, \eta) \leq d(\xi, \xi') + d(\xi', \eta) \leq \varphi(H(A, B) + \varepsilon) + 2\varepsilon. \tag{3.4}$$

Taking infimum with respect to $\eta \in f_*(B)$ in (3.4), and subsequently supremum with respect to $\xi \in f_*(A)$, we obtain

$$\varrho(f_*(A), f_*(B)) \leq \varphi(H(A, B) + \varepsilon) + 2\varepsilon.$$

Changing places of A and B, we have

$$\varrho(f_*(B), f_*(A)) \leq \varphi(H(A, B) + \varepsilon) + 2\varepsilon.$$

This means that

$$H(f_*(A), f_*(B)) \leq \varphi(H(A, B) + \varepsilon) + 2\varepsilon.$$

Due to the arbitrariness of $\varepsilon > 0$ and to the right continuity of φ we obtain (3.3). □

Corollary 3.6. *Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction with respect to a right continuous comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then for any $A, B \in \mathcal{P}_{b,cl}(X)$ one has*

$$H(f_*^n(A), f_*^n(B)) \leq \varphi^n(H(A, B)) \to 0, \qquad \text{as} \quad n \to \infty. \tag{3.5}$$

For a compact valued weakly contracting relations the condition of right continuity of the comparison function can be dropped.

Theorem 3.7. *Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact valued weakly contracting relation with respect to a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then the Hutchinson-Barnsley mapping $f_* : \mathcal{P}_{cp}(X) \to \mathcal{P}_{cp}(X)$, $f_*(A) = \overline{f[A]}$, is also a φ-contraction.*

Proof. Since the image of a compact set under a compact valued continuous mapping is a compact set, the mapping f_* is well defined. Use the proof of Theorem 3.5, taking $\varepsilon = 0$. □

Remark 3.4. As a consequence we obtain that for compact valued weakly contracting relations the inequality (3.5) with A, B as compact sets holds true without the condition of right continuity of the comparison function.

4. Attractors in weak contractions

The dynamics of a contracting mapping is trivial: the unique fixed point attracts all points and even all bounded subsets of the phase space. As for set-valued functions the dynamics is much more complicated. Firstly, a fixed point, if it exists, need not be unique, nor attractive. Moreover, the set of fixed points is not even invariant. Secondly, the attractor of a hyperbolic IFS, a particular case of a contracting relation, is an invariant set, although containing the fixed points, it is a scene of a much more complicated dynamics, e.g. it contains a dense subset of periodic points as well as a dense chain (see [3]).

There are various definitions of attractor in dynamical systems. In ordinary dynamics (e.g. iterations of mappings) one usually means by an attractor an invariant set, which is dynamically indivisible and whose basin – the set of attracted points – is a large set. The dynamical indivisibility sometimes is understood as the existence of a dense orbit. As for the basin, it must contain a neighborhood of the attractor, or at least the nonvoid interior, sometimes positive Lebesgue measure is required.

In the case of relations in compact spaces in [1] (see also [10]) the following definition has been proposed: A is an attractor, if it is invariant and there exists a closed neighborhood V of A such that $\bigcap_{n\geq 0} f^n[V]$ is contained in A. For another definition of attractor see [9].

In [11] the invariance $f[A] = A$ is relaxed up to the condition $f[A] \supset A$ with the assumption that A attracts any bounded subset of a neighborhood of A.

Let X be a metric space and let $f : X \to \mathcal{P}(X)$ be a closed relation. A nonempty closed subset $A \subset X$ is called an *attractor* for f, if:

- $f[A] \supset A$;
- there is a closed neighborhood $V = \text{cls}\,\{x \in X \mid \varrho(x, A) < \delta\}$ of A such that $\bigcap_{n\geq 0} f^n[V] \subset A$.

Remark 4.1. Both of inclusions are, in fact, equalities [5].

Theorem 4.1. *For any weak contraction $f : X \to \mathcal{P}_{b,cl}(X)$ with respect to a right continuous comparison function there exists an unique bounded and closed set A such that $\overline{f[A]} = A$.*

Proof. Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction with respect to a right continuous comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. By Theorem 3.5 the Hutchinson-Barnsley mapping f_* is also a φ-contraction. It is known [14], that in a complete metric space any weakly contracting mapping has an unique fixed point. Therefore, the equation $f[A] = A$ has a solution in $\mathcal{P}_{b,cl}(X)$ and this solution is unique. □

Theorem 4.2. *A bounded and closed subset $A \subset X$ is an attractor for a compact valued weakly contracting relation $f : X \to \mathcal{P}_{cp}(X)$ with respect to a right continuous comparison function, if and only if A is a compact invariant set for f.*

Proof. Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact valued weakly contracting relation with respect to a right continuous comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

If A is a bounded attractor for f, then A is a bounded and closed invariant set for f (see Remark 4.1). Since A is closed, one has $\overline{f[A]} = A$. So A is the unique fixed point for the corresponding Hutchinson-Barnsley mapping f_* on $\mathcal{P}_{b,cl}(X)$. On the other hand, f_* restricted to $\mathcal{P}_{cp}(X)$ takes values also in $\mathcal{P}_{cp}(X)$. As a weak contraction it has an unique fixed point in $\mathcal{P}_{cp}(X)$ [14]. Therefore, the bounded and closed subset $A \subset X$ is actually compact.

Conversely, assume that A is a compact (and so, a bounded and closed) invariant set for f. Take a closed neighborhood V of A of small enough radius $\delta > 0$. It is bounded as well.

By Corollary 3.6, we have for any $n \geq 0$

$$H(f_*^n(V), A) = H(f_*^n(V), f_*^n(A)) \leq \varphi^n(H(V, A)). \tag{4.1}$$

We will prove that $f[V] \subset V$. If $x \in V$, then $\varrho(x, A) < \delta$. This means that there exists $a \in A$ such that $d(x, a) < \delta$. Using Theorem 3.7, we obtain

$$H(f(x), f(a)) \leq \varphi(d(x, a)) \leq \varphi(\delta) < \delta.$$

Since $f(a) \subset f[A] = A$, it follows that

$$\varrho(f(x), A) \leq \varrho(f(x), f(a)) \leq H(f(x), f(a)) < \delta.$$

As a consequence, $f(x) \subset V$. Therefore, $f[V] \subset V$.

Because of the inclusion $f^{n+1}[V] \subset f^n[V]$, the inequality (4.1) implies the following one:

$$H(\bigcap_{k=0}^{n} f_*^k(V), A) \leq \varphi^n(H(V, A)) \qquad (n \geq 0).$$

Passing to the limit as $n \to \infty$, one obtains $H(\bigcap_{k\geq 0} f_*^k(V), A) = 0$, which is equivalent to the equality $\bigcap_{k\geq 0} f_*^k(V) = A$. Hence, A is an attractor. □

Remark 4.2. The condition on the relation to take compact values is necessary. As a counterexample one can take a constant relation on an infinitely dimensional Banach space with the unit closed ball as value. Unfortunately, this example invalidates Theorem 2.2 from [5], in which the compact valued condition should be added.

Corollary 4.3. *Any compact valued weakly contracting relation with respect to a right continuous comparison function has a nonempty compact attractor and this attractor is unique.*

Theorem 4.4. *Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact valued weakly contracting relation with respect to a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then for any chain $\{x_n\}_{n\in\mathbb{N}}$ in X and any $y_0 \in X$ there exists a chain $\{y_n\}_{n\in\mathbb{N}}$ in X, starting at y_0, such that*

$$d(x_n, y_n) \leq \varphi^n(d(x_0, y_0)) \qquad (n \geq 0).$$

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ in X and let $y_0 \in X$ be arbitrary. By Lemma 2.2, given $x_1 \in f(x_0)$ there exists $y_1 \in f(y_0)$ such that

$$d(x_1, y_1) \leq H(f(x_0), f(y_0)) \leq \varphi(d(x_0, y_0)).$$

Similarly, for any $x_2 \in f(x_1)$ there exists $y_2 \in f(y_1)$ such that

$$d(x_2, y_2) \leq H(f(x_1), f(y_1)) \leq \varphi(d(x_1, y_1)) \leq \varphi^2(d(x_0, y_0)).$$

Moreover, by induction, for any $n \geq 0$ one can choose $y_n \in f(y_{n-1})$ such that $d(x_n, y_n) \leq \varphi^n(d(x_0, y_0))$. The chain $\{y_n\}_{n\in\mathbb{N}}$ is a required one. □

As a consequence we obtain the following result.

Theorem 4.5 (Asymptotic phase theorem for weakly contracting relations)**.** *Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact valued weakly contracting relation with respect to a right continuous comparison function and let A stand for its attractor. Then for any chain $\{x_n\}_{n\in\mathbb{N}}$ in X and any $a_0 \in A$ there exists a chain $\{a_n\}_{n\in\mathbb{N}}$ in A, starting at a_0, such that $d(x_n, a_n) \to 0$ as $n \to \infty$.*

Proof. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a right continuous comparison function with respect to which f is a weak contraction. Due to the invariance of A and to Theorem 4.4 one has that given a chain $\{x_n\}_{n\in\mathbb{N}}$ and $a_0 \in A$ there exists a chain $\{a_n\}_{n\in\mathbb{N}}$ in A, starting at a_0, such that for all $n \geq 0$ one has $d(x_n, a_n) \leq \varphi^n(d(x_0, y_0))$. The latter converges to 0 as $n \to \infty$. □

Theorem 4.6. *Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction with respect to a right continuous comparison function, and let A stand for the fixed point of the corresponding Hutchinson-Barnsley mapping $f_* : \mathcal{P}_{b,cl}(X) \to \mathcal{P}_{b,cl}(X)$. Then for any $a \in A$ one has $\overline{\mathcal{O}f(a)} = A$.*

Proof. Since $\mathcal{O}f = \bigcup_{n\geq 1} f^n$, the inclusion $\overline{\mathcal{O}f(a)} \subset A$ follows from the positive invariance of A and from its closeness.

To prove the inverse inclusion we use Corollary 3.6 and observe that

$$H(f^n(a), A) = H(f_*^n(\{a\}), f_*^n(A)) \leq \varphi^n(H(a, A)). \tag{4.2}$$

Since the right hand side of (4.2) converges to 0 as $n \to \infty$, for any $b \in A$ and any $\varepsilon > 0$ there exists a natural n such that $\varrho(b, f^n(a)) < \varepsilon/2$.

Therefore, for any $b \in A$ and any $\varepsilon > 0$ there exist a natural n and a point $a' \in f^n(a)$ such that $d(b, a') < \varepsilon$. This implies the equality $\overline{\mathcal{O}f(a)} = A$. □

5. Shadowing in weak contractions

In [4] a generalization of the concept of Shadowing has been proposed for IFS's. In [5] this concept has been developed for any contracting relation.

Given $\delta > 0$ a finite or infinite sequence $\{x_n\}$ in X is called a *δ-chain* (or a *δ-pseudo-orbit*) for a relation $f : X \to \mathcal{P}(X)$, if $\varrho(x_{n+1}, f(x_n)) \leq \delta$ for all n.

One says that the relation $f : X \to \mathcal{P}(X)$ has the *Shadowing Property* on X, if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-chain $\{x_n\}_{n\in\mathbb{N}}$ in X there exists a chain $\{y_n\}_{n\in\mathbb{N}}$ in X such that $d(x_n, y_n) \leq \varepsilon$ for all n (one says that the δ-chain $\{x_n\}$ is *ε-shadowed* by the chain $\{y_n\}$).

Remark 5.1. In [16] another concept of δ-pseudo-orbit has been proposed. More precisely, treating relations as subsets of the Cartesian product of the space with itself, a δ-pseudo-orbit is called a sequence $\{z_n\}$ of points in X, which are close to the relation as a subset, i.e. $\mathrm{dist}((z_n, z_{n+1}), f) < \delta$ for all n. In these terms a Shadowing Property is stated for so called hyperbolic smooth relations. For Shadowing in some abelian group actions see [13].

Remark 5.2. In [2] the Shadowing Property has been considered for the corresponding Hutchinson-Barnsley mapping on the hyperspace and an interesting notion of "digitalization", as a kind of "collage" by using finite sets, has been introduced and studied with applications to computer construction of fractals.

Lemma 5.1. *Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact valued weakly contracting relation with respect to a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then for any $\delta > 0$, any δ-chain $\{x_n\}_{n\in\mathbb{N}}$ of f and any $y_0 \in X$ there exists a chain $\{y_n\}_{n\in\mathbb{N}}$, starting at y_0, which verifies the following inequality for any $n \geq 0$:*

$$d(x_{n+1}, y_{n+1}) \leq \delta + \varphi(d(x_n, y_n)). \tag{5.1}$$

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a δ-chain. In what follows we construct inductively the desired chain $\{y_n\}_{n\in\mathbb{N}}$.

Assume that given y_0 we have constructed a finite segment $y_0, y_1, \ldots, y_n$ of the chain. We choose the next term as follows: by Lemma 2.2 there exists $z \in f(x_n)$ such that

$$d(x_{n+1}, z) = \varrho(x_{n+1}, f(x_n)) \leq \delta.$$

Moreover, for this $z \in f(x_n)$ there exists $y_{n+1} \in f(y_n)$ such that

$$d(z, y_{n+1}) \leq H(f(x_n), f(y_n)).$$

Using (3.1), we obtain

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq d(x_{n+1}, z) + d(z, y_{n+1}) \leq \\ &\delta + H(f(x_n), f(y_n)) \leq \delta + \varphi(d(x_n, y_n)). \end{aligned}$$

This completes the proof. ∎

Theorem 5.2. *Any compact valued weakly contracting relation $f : X \to \mathcal{P}_{cp}(X)$ with respect to a right continuous comparison function has the Shadowing Property on X.*

Proof. Let $f : X \to \mathcal{P}_{cp}(X)$ be a weak contraction with respect to a right continuous comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

Fix $\varepsilon > 0$. Due to the right continuity of the function φ and to Theorem 3.2 there exists

$$M = \max\{\varphi(t)/t \,|\, \varepsilon/2 \leq t \leq \varepsilon\}, \tag{5.2}$$

and $0 \leq M < 1$. Moreover, (5.2) implies that

$$\varphi(t) \leq Mt, \qquad \forall\, t \in [\varepsilon/2,\, \varepsilon]. \tag{5.3}$$

Take $\delta = \varepsilon(1 - M)/2 \leq \varepsilon/2$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a δ-chain of f. This means that $\varrho(x_{n+1}, f(x_n)) \leq \delta$ for all $n \geq 0$. Take an arbitrary $y_0 \in X$ such that $d(x_0, y_0) \leq \varepsilon/2$. By Lemma 5.1 there exists a chain $\{y_n\}_{n\in\mathbb{N}}$, starting at y_0, which verifies (5.1). We will prove inductively that the chain $\{y_n\}_{n\in\mathbb{N}}$ ε-shadows $\{x_n\}_{n\in\mathbb{N}}$, i.e. for all $n \geq 0$

$$d(x_n, y_n) \leq \varepsilon. \tag{5.4}$$

For $n = 0$ the equality (5.4) holds due to the choice of y_0.

Assume that (5.4) holds for any $0 \leq n \leq k$. We have to prove that (5.4) holds for $n = k + 1$.

There are only two possible cases.

a) If $d(x_k, y_k) < \varepsilon/2$, then

$$d(x_{k+1}, y_{k+1}) \leq \delta + \varphi(d(x_k, y_k)) \leq \delta + d(x_k, y_k) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Here we have used (5.1) and Lemma 3.1.

b) If $d(x_k, y_k) \geq \varepsilon/2$, then let m, $0 \leq m < k$, denote the maximal natural such that $d(x_m, y_m) < \varepsilon/2$. Using successively (5.1) and (5.3), one obtains:

$$\begin{gathered} d(x_{k+1}, y_{k+1}) \leq \delta + \varphi(d(x_k, y_k)) \leq \delta + Md(x_k, y_k) \leq \\ \delta + M(\delta + Md(x_{k-1}, y_{k-1})) = \delta + M\delta + M^2 d(x_{k-1}, y_{k-1}) \leq \cdots \leq \\ \delta + M\delta + M^2\delta + \cdots + M^{k-m}\delta + M^{k-m}\varphi(d(x_m, y_m)) \leq \\ \frac{\delta}{1 - M} + d(x_m, y_m) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{gathered}$$

Therefore, the equality (5.4) holds for all $n \geq 0$. This means that the chain $\{y_n\}_{n\in\mathbb{N}}$ ε-shadows the δ-chain $\{x_n\}_{n\in\mathbb{N}}$. □

Corollary 5.3. *Let (X, d) be a γ-convex metric space. Then any compact valued weakly contracting relation on X has the Shadowing Property.*

Corollary 5.4. *A finite IFS, consisting of weakly contracting mappings, such that each of them admits a right continuous comparison function, has the Shadowing Property.*

Proof. Let $\mathcal{F} = \{X; f_1, \ldots, f_m\}$ be an IFS, consisting of weakly contracting mappings $f_i, i = \overline{1, m}$. Let φ_i be a right continuous comparison function for $f_i, i = \overline{1, m}$. Obviously, the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \sup_{1 \le i \le m} \varphi_i(t)$, is also a right continuous comparison function for each function $f_i, i = \overline{1, m}$, and one can apply Theorem 5.2. □

Remark 5.3. The Corollary 5.4 holds for an Infinite IFS (IIFS) under an additional assumption that all mappings of the IIFS admits a common right continuous comparison function. Given a family of continuous comparison functions $\{\varphi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \alpha \in \mathcal{A}\}$ the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \sup\{\varphi_\alpha(t) \,|\, \alpha \in \mathcal{A}\}$ $(t \ge 0)$, need not necessarily be a continuous comparison function.

Example 5.1. For any $k \ge 1$ the function $\varphi_k : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\varphi_k(t) = \begin{cases} t/2, & 0 \le t \le 1, \\ (k-1)t/2 + (2-k)/2, & 1 < t \le (k+2)/(k+1), \\ k/(k+1), & t > (k+2)/(k+1), \end{cases}$$

is a continuous comparison function. Moreover, for any $t \ge 0$ we have $\varphi_k^n(t) \to 0$ as $n \to \infty$ uniformly on $k \ge 1$. At the same time, the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\varphi(t) = \sup_{k \ge 1} \varphi_k(t) = \begin{cases} t/2, & 0 \le t \le 1, \\ 1, & t > 1, \end{cases}$$

is not yet a right continuous comparison function.

Remark 5.4. In [5] the Shadowing Property for contracting multifunctions with bounded and closed values has been stated, using a Fixed Point Theorem for contracting multifunctions with bounded and closed values. As for weakly contracting multifunctions, at our knowledge, a Fixed Point Theorem has been proved only in a compact valued case. So we give a direct proof.

6. Periodic Shadowing in weak contractions

The equality (1.1) for IFS, i.e. denseness of periodic points in the attractors, has been obtained in [18] as a consequence of the codification of the attractor. The last means that the IFS restricted to the attractor is a factor of a symbolic IFS. For the latter denseness of periodic points on the attractor is well known. Such a method doesn't work in the general case of contracting multifunctions.

In what follows we state the equality (1.1), using the property of periodic Shadowing for weak contractions. The latter is proved by using a Fixed Point Theorem for compact valued weakly contracting multifunctions (see e.g. [14]).

Theorem 6.1. *Let (X, d) be a complete metric space and let $f : X \to \mathcal{P}_{cp}(X)$ be a compact valued weakly contracting relation. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that for any periodic δ-chain $\{x_n\}_{n\in\mathbb{N}}$ there exists a periodic chain $\{y_n\}_{n\in\mathbb{N}}$ such that $d(x_n, y_n) \leq \varepsilon$ for all $n \in \mathbb{N}$.*

Proof. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function with respect to which f is a weak contraction. Fix $\varepsilon > 0$ and take $\delta = \varepsilon - \varphi(\varepsilon)$.

Let $\alpha = \{x_n\}_{n\in\mathbb{N}}, x_{n+m} = x_n$, be a m-periodic δ-chain. Define by $\hat{X} := X^m$ the space of all m-periodic sequences $\beta = \{z_n\}_{n\in\mathbb{N}}$ with the metric $\hat{d}$,

$$\hat{d}(\beta_1, \beta_2) := \max_{1\leq j\leq m} d(z'_j, z''_j) \quad \text{for all} \quad \beta_1 = \{z'_n\}_{n\in\mathbb{N}}, \beta_2 = \{z''_n\}_{n\in\mathbb{N}} \in \hat{X}.$$

Let $\overline{B}(\alpha, \varepsilon) = \{\beta \in \hat{X} \mid \hat{d}(\alpha, \beta) \leq \varepsilon\}$ be the closed ball of radius ε centered at α. Define a compact valued function $\Phi : \overline{B}(\alpha, \varepsilon) \to \mathcal{P}(\hat{X})$ as follows:

$$[\Phi(\beta)]_j = f(z_{j-1}) \quad (j = 1, 2, \dots, m),$$

where $\beta = \{z_n\}_{n\in\mathbb{N}} \in \overline{B}(\alpha, \varepsilon)$ and $z_0 = z_m$. Each value of this function is a compact subset of $\hat{X}$. We claim that, in fact, each value is a compact subset of $\overline{B}(\alpha, \varepsilon)$. To prove this, fix $j \in \{1, 2, \dots, m\}$. Due to Lemma 2.3 for chosen $\delta = \varepsilon - \varphi(\varepsilon)$ and for any $\beta = \{z_n\}_{n\in\mathbb{N}} \in \overline{B}(\alpha, \varepsilon)$ we have:

$$\varrho(x_j, f(z_{j-1})) \leq \varrho(x_j, f(x_{j-1})) + H(f(x_{j-1}), f(z_{j-1})) \leq$$
$$\delta + \varphi(d(x_{j-1}, z_{j-1})) \leq \delta + \varphi(\hat{d}(\alpha, \beta)) \leq \delta + \varphi(\varepsilon) = \varepsilon.$$

Thus, the function $\Phi : \overline{B}(\alpha, \varepsilon) \to \mathcal{P}_{cp}(\overline{B}(\alpha, \varepsilon))$ is well defined. Moreover, Φ is a φ-contraction. For, let $\beta_1 = \{z'_n\}_{n\in\mathbb{N}}, \beta_2 = \{z''_n\}_{n\in\mathbb{N}} \in \overline{B}(\alpha, \varepsilon)$. One has the following inequalities:

$$H(f(z'_j), f(z''_j)) \leq \varphi(d(z'_j, z''_j)) \quad (j = 1, 2, \ldots, m).$$

Therefore,

$$H(\Phi(\beta_1), \Phi(\beta_2)) \leq \varphi(\hat{d}(\beta_1, \beta_2)) \quad (j = 1, 2, \ldots, m).$$

By virtue of the Fixed Point Theorem for compact valued weak contractions (see [14]), there exists $\theta \in \overline{B}(\alpha, \varepsilon)$ such that $\theta \in \Phi(\theta)$, or, in other words, $\theta = \{y_n\}_{n\in\mathbb{N}}$, $y_{j+m} = y_j$ $(j = 1, 2, \ldots, m)$, is a m-periodic chain for f. □

Theorem 6.2. *For any compact valued weakly contracting relation $f : X \to \mathcal{P}_{cp}(X)$ with respect to a right continuous comparison function the periodic points form a dense subset of the attractor.*

Proof. For any $\varepsilon > 0$ and any $x \in A$, where A denotes the attractor, we have to find a periodic point $y \in A$ such that $d(x, y) < \varepsilon$. Assume that x itself is not a periodic point (otherwise it is nothing to prove). This means that $x \notin f^n(x)$ for any $n = 1, 2, \ldots$. Due to the compactness of A and to Theorem 4.6, for any $\delta > 0$ there exists p such that $\varrho(x, f^p(x)) \leq \delta$.

For $\delta < (\varepsilon - \varphi(\varepsilon))/2$ take a finite chain $x_0 = x, x_1, \ldots, x_p \in f^p(x)$ such that $d(x, x_p) \leq \delta$ and extend it up to a periodic δ-chain $\{x_0, x_1, \ldots, x_p, x_{p+1} = x_0, \ldots\}$. By virtue of Theorem 6.1 for this periodic δ-chain $\{x_n\}_{n\in\mathbb{N}}$ there exists a periodic chain $\{y_n\}_{n\in\mathbb{N}}$ such that $d(x_n, y_n) \leq \varepsilon/2 < \varepsilon$ for any $n \in \mathbb{N}$. Thus, $y = y_0$ is the desired periodic point. □

7. Appendix

Recently, C.Ursescu has communicated us that he stated and studied some property for metric spaces, which is equivalent to γ-convexity. With his kind permission we bring some results from an unpublished yet his paper [17].

Definition. [17] The metric space Y is said to *resemble normed spaces* if for every $y \in Y$, for every $\delta > 0$, and for every $\delta' > 0$ there is satisfied the equality

$$B(B(y, \delta), \delta') = B(y, \delta + \delta').$$

Proposition 7.1. [17] *The metric space Y resembles normed spaces if and only if $B(y, \delta) \cap B(y', \delta') \neq \emptyset$ whenever $\delta + \delta' > d(y, y')$.*

This means that the property of metric space to resemble normed spaces is equivalent to γ-convexity.

Theorem 7.2. [17] *Let Y be a metric space. The following three conditions are equivalent to each other:*

1. *the metric space Y resembles normed spaces;*
2. *for every $y \in Y$, for every $y' \in Y$, for every $\mu > 0$, and for every $\lambda > d(y, y')$ there exists a finite sequence $\chi = \{\chi_0, \chi_1, \ldots, \chi_n\}$ of points of Y such that*
$$\text{length}(\chi) := d(\chi_0, \chi_1) + \cdots + d(\chi_{n-1}, \chi_n) \le \lambda,$$
$$\text{mesh}(\chi) := \max\{d(\chi_0, \chi_1), \ldots, d(\chi_{n-1}, \chi_n)\} \le \mu,$$
as well as $\chi_0 = y$ and $\chi_n = y'$;
3. *for every $y \in Y$, for every $y' \in Y$, and for every $\lambda > d(y, y')$ there exists a λ-lipschitzean function $\psi : D \to Y$, where $D = \big\{i/2^n \,|\, n \in \{0, 1, \ldots\}, i \in \{0, 1, \ldots, 2^n\}\big\}$ stands for the set of all diadic numbers, such that $\psi(0) = y$, $\psi(1) = y'$ and $d(\psi(t), \psi(s)) \le \lambda|t - s|$ for any $t, s \in D$.*

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