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A CONSTRUCTION OF TOPOLOGICAL INDEX FOR CONDENSING MAPS OF FINSLER MANIFOLDS

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Abstract. The topological index for maps of infinite-dimensional Finsler manifolds, condensing with respect to internal Kuratowski or Hausdorff measure of noncompactness, is constructed under the hypothesis that the manifold can be embedded into a certain Banach linear space as a neighbourhood retract so that the Finsler norm in tangent spaces and the restriction of the norm from enveloping space on the tangent spaces are equivalent. It is shown that the index is an internal topological characteristic, i.e., it does not depend on the choice of enveloping space, embedding, etc. The total index (Lefschetz number) is also introduced.

Key Words and Phrases: Topological index, condensing maps, Finsler manifolds.

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The aim of this paper is development of the topological theory of condensing maps on Banach manifolds. In such a theory the main difficulty is that formulations of many facts from linear spaces sound reasonably on manifolds

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but cannot be proved directly since the topological theory of condensing maps is essentially based on the notion of convex closure that is absolutely ill-posed on nonlinear manifolds.

In a series of previous works (see, e.g., [3, 2, 5] and references there) such a theory was constructed for the case of Finsler manifolds that can be embedded isometrically into a certain Banach linear space as a neighbourhood retract. It works perfectly on the manifold of continuous curves in a finite-dimensional manifold but it turns out that the condition of isometric embedding fails for the manifold of C^1-curves where, say, the shift operator of neutral type functional differential equation acts.

Here we modify the previous approach and construct the topological index for condensing maps of Finsler manifolds that can be embedded into a linear Banach space as a neighbourhood retract so that the Finsler norm in tangent spaces and the restriction of the norm from the enveloping Banach space onto those tangent spaces are equivalent. It is shown that the construction does not depend on the choice of enveloping space, embedding and other details. Thus the index is an internal topological characteristic in spite of the fact that in its construction the enveloping space is involved.

Let $\mathcal{M}$ be a Finsler manifold and $\mathcal{M}$ be embedded (possibly not isometrically) into a Banach space $\mathcal{E}$ with the norm $\|\cdot\|$ as a neighbourhood retract. Denote by $\|\cdot\|_I$ the internal (Finsler) norm in tangent spaces $T_m\mathcal{M}$, and by $\|\cdot\|_E$ the restriction of the norm in $\mathcal{E}$ onto T_mM. Below we suppose that the norms $\|\cdot\|_I$ and $\|\cdot\|_E$ are equivalent, i.e., that there exist real functions $0 < c(m) \le C(m)$ continuously depending on $m \in \mathcal{M}$ such that for any $Y \in T_m\mathcal{M}$ the relation

$$c(m)\|Y\|_I \le \|Y\|_E \le C(m)\|Y\|_I \tag{1}$$

takes place.

Starting from the norms in tangent spaces to $\mathcal{M}$, one can find the corresponding lengths of piece-wise smooth curves in $\mathcal{M}$ as integrals of norms of velocity (derivative) vectors and then define the distance functions on $\mathcal{M}$ as infimums of the lengths of curves connecting the points (standard constructions of Riemannian and Finsler geometry). Denote by ρ_I the distance generated with $\|\cdot\|_I$, and by ρ_E — the distance generated with $\|\cdot\|_E$. Besides, the distance can be measured directly in $\mathcal{E}$, as the norm of difference of vectors in

$\mathcal{E}$. Note that the latter two distance functions are related by obvious estimate

$$\rho_E(m^0, m^1) \geq \|m^0 - m^1\|, \tag{2}$$

for all couples m^0 and $m^1 \in \mathcal{M}$ since the lengths with respect to $\|\cdot\|_E$ and $\|\cdot\|$ coincide but ρ_E is the infimum of lengths of curves on M while $\|\cdot\|$ – in $\mathcal{E}$.

Recall (see details, e.g., in [1]) the notions of Kuratowski and Hausdorff measures of noncompactness in a metric space E. Let $\Omega \subset E$ be a bounded subset.

Definition 1. $\alpha(\Omega) = inf\{d > 0|\Omega$ permits its partition in E into a finite number of subsets with diameters less than $d\}$ *is called the Kuratowski measure of non-compactness of* Ω.

Definition 2. $\chi(\Omega) = inf\{\varepsilon > 0|\Omega$ has in E a finite ε-net$\}$ *is called the Hausdorff measure of non-compactness of* Ω.

Below the words "a measure of noncompactness ψ" will mean either α or χ.

Having defined the distances in $\mathcal{M}$, denote by ψ_I the measure of noncompactness with respect to ρ_I, by ψ_E – the measure of noncompactness with respect to ρ_E and by $\psi_{\|\cdot\|}$ — measure of noncompactness with respect to $\|\cdot\|$.

Definition 3. *A continuous operator $F : \mathcal{M} \to \mathcal{M}$ is called condensing with respect to ψ with constant $q < 1$ if for any bounded set $\Omega \subset \mathcal{M}$ the inequality*

$$\psi(F\Omega) < q\psi(\Omega) \tag{3}$$

holds.

Definition 4. *A continuous operator $F : \mathcal{M} \to \mathcal{M}$ is called locally condensing with respect to ψ, if any point $x \in \mathcal{M}$ has a neighbourhood U_x such that for any bounded set $\Omega \subset U_x$ the inequality*

$$\psi(F\Omega) < q\psi(\Omega), \quad q < 1. \tag{4}$$

is satisfied.

Let the operator F be condensing with respect to ψ_I with a constant $q < 1$ and let $\Omega \subset \mathcal{M}$ be a bounded domain with boundary $\dot{\Omega}$. Consider the set $F^\infty\Omega = \bigcap\limits_{k=1}^{\infty} F^k\Omega$, where F^k is the k-th iteration of F. Sometimes we shall introduce the additional assumption that F sends the entire $\mathcal{M}$ into a domain having finite diameter with respect to the distance ρ_I. In this case we can consider the set $F^\infty\mathcal{M} = \bigcap_{k=1}^{\infty} F^k\mathcal{M}$.

Lemma 5. *The set $F^\infty\Omega$ is compact. If F sends the entire $\mathcal{M}$ into a domain having finite diameter with respect to the distance ρ_I, the set $F^\infty\mathcal{M}$ is compact.*

The proof can be found, e.g., in [3], [5]. Notice that the set $F^\infty\mathcal{M}$ contains all fixed points of F from $\mathcal{M}$ and $F^\infty\Omega$ contains all fixed points of F from Ω.

Since the functions $C(m)$ and $c(m)$ from (1) are continuous and the sets $F^\infty\Omega$ and $F^\infty\mathcal{M}$ from Lemma 6 are compact, there exist constants $C > c > 0$ and a neighbourhood $\mathcal{A}$ of $F^\infty\Omega$ or of $F^\infty\mathcal{M}$ such that for any $m \in \mathcal{A}$, $Y \in T_m\mathcal{M}$

$$c\|Y\|_I \leq \|Y\|_E \leq C\|Y\|_I. \tag{5}$$

Let $V \subset \mathcal{A}$ be bounded. Then from (5) and (3) we obtain the following sequence of inequalities:

$$\psi_I(F(V)) \leq q\psi_I(V) \leq Cq\psi_E(V). \tag{6}$$

But on the other hand, from (1) and (3) it follows that

$$\psi_I(F(V)) \geq c\psi_E(F(V)). \tag{7}$$

Then from (4), (6) and (7) we get

$$\psi_E(F(V)) \leq \frac{C}{c}q\psi_E(V). \tag{8}$$

Thus form (8) it follows that with respect to ψ_E the operator F is condensing with another constant $\frac{C}{c}q$ that may be greater than 1.

Denote by $R : \bar{U} \to \mathcal{M}$ a smooth retraction of a certain tubular neighbourhood $\bar{U} \subset \mathcal{E}$ of $\mathcal{M}$ and by $TR : T\bar{U} \to T\mathcal{M}$ its tangent map. Recall that the tangent map sends the vector $X \in T_xU$ into $TRX = d_xRX \in T_{Rx}\mathcal{M}$, where the linear operator $d_xR : T_x\bar{U} \to T_{Rx}\mathcal{M}$ is the Frechet derivative of R at the point $x \in \bar{U}$.

Theorem 6. *For any $m \in \mathcal{M} \subset \bar{U}$ and $Q > 1$ there exists a neighbourhood V_m^Q of m in $\bar{U}$ such that for any $x \in V_m^Q$ the inequality*

$$Q > \|d_xR\| > \frac{1}{Q}, \tag{9}$$

holds, where $\|d_xR\|$ is the norm of operator d_xR.

The proof follows from continuity of d_xR in x and from the fact that for $x \in \mathcal{M}$ the derivative d_xR is obviously the unit operator, see details in [5].

Specify a point $m \in \mathcal{M}$ and a number $Q > 1$. Since V_m^Q is an open set, it contains a ball $B_m \subset V_m^Q$ of $\mathcal{E}$, centered at m with a certain radius ρ.

Theorem 7. *The retraction R is Lipschitz continuous on B_m:*

$$\rho_E(R(u_0), R(u_1)) \le Q\|u_0 - u_1\|, \tag{10}$$

where u_0 $u_1 \in B_m$.

The proof can be found in [5].

Introduce $\bar{F} : \bar{U} \to \mathcal{M} \subset \bar{U}$, by the formula $\bar{F} = F \circ R$. From (2) it follows that for any $u_0, u_1 \in B_m$

$$\|\bar{F}(u_0) - \bar{F}(u_1)\| = \|FR(u_0) - FR(u_1)\| \le \rho_E(FR(u_0), FR(u_1)).$$

From this and from (1), (3), (7), (9) and (10) we get that for a bounded set $V \subset B_m$

$$\psi_{\|\cdot\|}(\bar{F}(V)) \le \psi_E(FR(V)) \le \frac{C}{c} q\psi_E(R(V)) \le Q\frac{C}{c} q\psi_{\|\cdot\|}(V).$$

Consider a bounded domain $\Omega \subset \mathcal{M}$ such that F has no fixed points on its boundary $\dot{\Omega}$, or the entire $\mathcal{M}$ under the assumption that $F\mathcal{M}$ is bounded (see above). For any $x^* \in F^\infty\Omega$ ($x^* \in F^\infty\mathcal{M}$, respectively) take the ball $B_{x^*} \subset \bar{U}$ as above. Cover the set $F^\infty\Omega \bigcap \Omega$ ($F^\infty\mathcal{M}$, respectively) with the balls B_{x^*}. This is an open covering of the compact set $F^\infty\Omega \bigcap \Omega$ ($F^\infty\mathcal{M}$, respectively) and so there exists its finite subcovering B_{x_i}. Let $V_B = \bigcup\limits_i B_{x_i}$. For any $V \subset V_B$ there exists a finite number of sets $V_i = V \bigcap B_{x_i}$ such that $\bar{F}$ is condensing with the constant $\frac{C}{c}qQ$ with respect to $\psi_{\|\cdot\|}$ on every V_i. Then

$$\psi_{\|\cdot\|}(F(V)) = \max_i \psi_{\|\cdot\|}(F(V_i)) \le \max_i \frac{C}{c} qQ\psi_{\|\cdot\|}(V_i) = \frac{C}{c} qQ\psi_{\|\cdot\|}(V).$$

By the construction, on the boundary of V_B there are no fixed points of $\bar{F}$. In spite of the fact that $\bar{F}$ is condensing on V_B with the constant $\frac{C}{c}qQ$ that is greater than 1 we can show that the topological index of condensing operator type is well-posed for $\bar{F}$ on the boundary of V_B.

Lemma 8. *There exists an integer k such that $\bar{F}^k$ is condensing with respect to $\psi_{\|\cdot\|}$ with a constant $\bar{q} < 1$.*

Proof. Since F is condensing with the constant $q < 1$, F^k is obviously condensing with the constant q^k. Notice that from the properties of a retraction it follows that $\bar{F}^k = (F \circ R)^k = F^k \circ R$. Then with the same scheme of arguments as above we can show that $\bar{F}^k$ is condensing with respect to $\psi_{\|\cdot\|}$ with the constant $\bar{q} = \frac{C}{c} q^k Q$. Since $q < 1$, for k large enough $q^k < \frac{1}{\frac{C}{c}Q}$. For such k we get $\bar{q} < 1$. □

Recall the following

Definition 9. *A set $S \subset \mathcal{E}$ is called fundamental for an operator $F : \bar{U} \to \mathcal{E}$, if:*
(i) $S \neq \emptyset$ is convex and compact;
(ii) $F(\bar{U} \cap S) \subset S$;
(iii) if $x_0 \in \bar{U} \setminus S$, then $x_0 \notin \bar{co}[\{F(x_0)\} \cup S]$.

In the standard theory of topological index for condensing maps in Banach spaces (see, e.g., [1]) the index is defined as that for the contraction of the operator to a certain fundamental set, containing the set of fixed points of this operator. The key fact here is that for an operator, condensing with a constant less than 1, such a fundamental set exists. We shall show that for the above-mentioned operator $\bar{F}$ a fundamental set, containing fixed points of $\bar{F}$, does exist in spite of the fact that $\bar{F}$ is condensing with a constant greater than 1.

Lemma 10. *There exists a fundamental set for operator $\bar{F}$, constructed above, that contains all fixed points of $\bar{F}$ in Ω (in $\mathcal{M}$, respectively).*

Proof. We shall deal here with the bounded domain Ω, the case of $\mathcal{M}$ with $F(\mathcal{M})$ bounded is absolutely analogous. Choose k from Lemma 8. Denote by $\aleph$ the collection of all closed sets containing $F^\infty(\Omega)$ and satisfying all conditions from the definition of fundamental set for $\bar{F}$ and $\bar{F}^k$ together except maybe compactness.

The collection $\aleph$ is not empty since at least the set $T_0 = \bar{co}[F^\infty(\Omega) \cup F(\Omega)] = \bar{co}(F(\Omega))$ belongs to $\aleph$. Indeed, since $T_0 = \bar{co}(F(\Omega))$, $F(T_0 \cap \Omega) \subset F(\Omega) \subset T_0$ and analogously $F^k(T_0 \cap \Omega) \subset F(\Omega) \subset T_0$. Let $x_0 \in \Omega \setminus T_0$, then, since $F(x_0) \in F(\Omega) \subset T_0$ and $F^k(x_0) \in F(\Omega) \subset T_0$, $x_0 \notin \bar{co}[F(x_0) \cup T_0]$ means that $x_0 \notin T_0$, and $x_0 \notin \bar{co}[F^k(x_0) \cup T_0]$ also means that $x_0 \notin T_0$. But these two conditions are satisfied by the hypothesis $x_0 \in \Omega \setminus T_0$.

Let a set $T \in \aleph$. This means that $F^{\infty}(\Omega) \subset T$, if $x_0 \in \Omega \cap T$, then $F(x_0)$ and $F^k(x_0)$ belongs to the set T and that if $x_0 \in \Omega \setminus T$, then $x_0 \notin \bar{co}[F(x_0) \cup T]$ and $x_0 \notin \bar{co}[F^k(x_0) \cup T]$.

Consider the set $T_1 = \bar{co}[F^{\infty}(\Omega) \cup F(\Omega \cap T)]$. By the construction $T \supset T_1 = \bar{co}[F(\Omega \cap T)] \supset \bar{co}[F^k(\Omega \cap T)]$. Hence $F(\Omega \cap T_1) \subset F(\Omega \cap T) \subset \bar{co}[F(\Omega \cap T)] = T_1$, and consequently $F^k(\Omega \cap T_1) \subset T_1$.

Let $x_0 \in \Omega \setminus T_1$. Consider two cases:

1) $x_0 \notin T$, then $x_0 \notin \bar{co}[F(x_0) \cup T]$, and so $x_0 \notin \bar{co}[F(x_0) \cup T_1]$ and from $x_0 \notin \bar{co}[F^k(x_0) \cup T]$, it follow that $x_0 \notin \bar{co}[F^k(x_0) \cup T_1]$;

2) $x_0 \notin T_1$ and $x_0 \in T$, hence $x_0 \in \Omega \cap T$. Thus $F(x_0) \in T$ and $F^k(x_0) \in T$. From this it follows that $F(x_0) \in F(\Omega \cap T) \subset \bar{co}[F(\Omega \cap T)] \subset T_1$ and $F^k(x_0) \in F^k(\Omega \cap T) \subset \bar{co}[F^k(\Omega \cap T)] \subset T_1$. Then since $x_0 \notin T_1$ and $F(x_0) \in T_1$, we get $x_0 \notin \bar{co}[F(x_0) \cup T_1]$ since $\bar{co}[F(x_0) \cup T_1] \subset T_1$. Analogously $x_0 \notin \bar{co}[F^k(x_0) \cup T_1] \subset T_1$.

Thus conditions (ii) and (iii) of Definition 9 are fulfilled both for $\bar{F}$ and $\bar{F}^k$, i.e., $T_1 \in \aleph$.

Determine the set S as $S = \bigcap_{T \in \aleph} T$ that belongs to $\aleph$. Hence, as it is proved above, the set $S_1 = \bar{co}[F^{\infty}(\Omega) \cup F^k(\Omega \cap S)]$ also belongs to $\aleph$. Show that S is fundamental for F. Conditions (i) and (ii) of the definition are fulfilled both for the F and for F^k by the construction. The set $S \in \aleph$ is minimal in $\aleph$. Hence $S_1 = \bar{co}[F^{\infty}(\Omega) \cup F^k(\Omega \cap S)]$ coincides with S. Then since by Lemma 8 $\bar{F}^k$ is condensing with a constant less than 1, from the equality $S = \bar{co}[F^{\infty}(\Omega) \cup F^k(\Omega \cap S)]$ it follows that S is compact. □

Thus the index of vector field $I - \bar{F}$ on the boundary of V_B is well-posed.

Definition 11. *For the case of entire $\mathcal{M}$ with $F(\mathcal{M})$ bounded we call the index of $I - \bar{F}$ on the boundary of V_B* the Lefschetz number Λ_F *of F on $\mathcal{M}$.*

For the case of bounded $\Omega \subset \mathcal{M}$ we call the same index the index $ind_F(\dot{\Omega})$ *of F on $\dot{\Omega}$.*

Notice that the index of an isolated fixed point is also well-posed.

Our definition of Lefschetz number is compatible with the usual terminology since in the finite-dimensional case the Lefschetz number (in the sense of usual homological definition) is equal to the total index of fixed points. Suppose that F has only isolated fixed points and denote by j_i the index of F in a neighbourhood of the fixed point x_i. One can easily see that Λ_F is equal to the sum of indices, i.e., $\Lambda_F = \sum_i j_i$.

Notice that we have reduced a condensing map of the manifold to a completely continuous map from a certain domain into a Banach space. The same construction is also applicable to homotopies of condensing maps in the manifold that are reduced to completely continuous homotopies in the Banach space. All this allow us to prove the following statements in complete analogy with [5].

Lemma 12. *Λ_f does not depend on the choice of $\bar{U}$, R, $\mathcal{E}$ and embedding.*

Lemma 13. *Let $\mathcal{M}_i$ be a submanifold in $\mathcal{M}$, such that $F : \mathcal{M} \to \mathcal{M}_i$. Then $\Lambda_F = \Lambda_{F_{|\mathcal{M}_i}}$, where $F_{|\mathcal{M}_i}$ is the restriction of F on $\mathcal{M}_i$.*

The next statements follow from the construction, the above arguments and routine facts of the topological fixed point theory for condensing maps in Banach linear spaces.

Theorem 14. *The Lefschetz number is constant under homotopies in the class of condensing maps.*

Theorem 15. *If $\Lambda_F \neq 0$, F has a fixed point in $\mathcal{M}$.*

Analogues of Lemmas 12 and 13 and of Theorems 14 and 15 are evidently true also for $ind_F(\dot{\Omega})$ (cf. [5]) (of course in analogue of Theorem 14 we suppose that the homotopy has no fixed points of F on $\dot{\Omega}$).

It should be pointed out that the construction of index and Lefschetz number, described above, can be obviously generalized for locally condensing maps F of Finsler manifolds of the same sort under the assumption that for a certain integer l, $0 < l \leq \infty$, the iteration F^l sends a bounded domain Ω or the entire $\mathcal{M}$, respectively, into a compact set.

Consider a basic example of the situation, described above: the manifold of C^1-curves on a compact Riemannian manifold. This manifold is a natural phase space for functional-differential equations of neutral type (see this theory in linear spaces, e.g., in [6]).

Let M be a compact Riemannian manifold. By Nash's theorem it can be isometrically embedded into a Euclidean space R^N, where N is large enough, as a neighbourhood retract. Denote by $i : M \to R^N$ this embedding and by Ti its tangent map. Notice that all tangent spaces to R^N are canonically isomorfic to R^N itself and that is why we consider Ti as a map sending TM into R^N.

Denote by $C^1([-h,0],M)$ the Banach manifold of C^1-curves in M, given on the interval $[-h,0]$. For a curve $x(\cdot)$ from $C^1([-h,0],M)$ the tangent space $T_{x(\cdot)}C^1([-h,0],M)$ is the set of C^1 vector fields along $x(\cdot)$.

Define the internal Finsler metric on $C^1([-h,0],M)$ by constructing the norm in $T_{x(\cdot)}C^1([-h,0],M)$ of the form:

$$\|Y(\cdot)\|_I^{C^1} = \sup_{t\in[-h,0]} \|Y(t)\| + \sup_{t\in[-h,0]} \|\frac{D}{dt}Y(t)\|, \tag{11}$$

where $\frac{D}{dt}Y(t)$ is the covariant derivative of Levi-Civitá connection (see, e.g., [4]) on M of the vector field $Y(t)$ along $x(\cdot)$ (emphasize that norm (11) is given in intrinsic terms);

The map i generates the embedding of the manifold $C^1([-h,0],M)$ into the Banach space $C^1([-h,0],R^N)$ as a neighbourhood retract. Introduce the following norm in $T_{x(\cdot)}C^1([-h,0],M)$:

$$\|Y(\cdot)\|_E^{C^1} = \sup_{t\in[-h,0]} \|Y(t)\| + \sup_{t\in[-h,0]} \|(TiY(t))'\|, \tag{12}$$

where $(TiY(t))'$ is the derivative of curve $TiY(t)$ in R^N.

Introduce the norm, analogous to (12) in $C^1([-h,0],R^N)$. Then (12) is its restriction onto tangent spaces to $C^1([-h,0],M)$.

By standard procedure (see above) construct the distance functions in $C^1([-h,0],M)$, corresponding to norms (11) and (12) and denote them by ρ_I and ρ_E, respectively.

Denote by P the orthogonal projection of R^N onto T_mM. It is well-known that $\frac{D}{dt}Y(t) = P(TiY(t))'$, thus

$$\|Y(\cdot)\|_I^{C^1} \le \|Y(\cdot)\|_E^{C^1}.$$

One can easily see that

$$(TiY(t))' = \frac{D}{dt}Y(t) + (I-P)(T^2i(\frac{d}{dt}y(t),Y(t))),$$

where T^2i is the bilinear operator of second derivative of embedding i.

For $x(\cdot) \in C^1([-h,0],M)$ its velocity vector field $x'(t)$ is continuous and so its norm $\|x'(t)\|$ is bounded of the compact interval $[-h,0]$ by a certain constant $k_{x(\cdot)} > 0$. The operator norm of $(I-P)T^2i$ on M is bounded as a continuous function on the compact manifold M, i.e., $\|(I-P)Ti^2\| \le \Xi$ for some $\Xi > 0$. Hence, using the above estimates and the obvious fact that $\sup_{t\in[-h,0]} \|Y(t)\| \le \|Y(t)\|_I^{C^1}$, we see that

$$\|Y(\cdot)\|_E^{C^1} = \sup_{t\in[-h,0]} \|Y(t)\| + \sup_{t\in[-h,0]} \|\frac{D}{dt}Y(t) + ((I-P)Ti^2(\frac{d}{dt}x(t), Y(t)))\| \le$$

$$\le \sup_{t\in[-h,0]} \|Y(t)\| + \sup_{t\in[-h,0]} \|\frac{D}{dt}Y(t)\| + \Xi \sup_{t\in[-h,0]} (\|\frac{d}{dt}x(t)\|, \|Y(t)\|) \le$$

$$\le \|Y(t)\|_I^{C^1}(1 + \Xi k_{x(\cdot)}).$$

So, we obtain the following estimate for the norms:

$$\|Y(\cdot)\|_I^{C^1} \le \|Y(\cdot)\|_E^{C^1} \le \|Y(\cdot)\|_I^{C^1}(1 + \Xi k_{x(\cdot)}). \tag{13}$$

Inequality (13) means that the norms $\|\cdot\|_I^{C^1}$ and $\|\cdot\|_E^{C^1}$ are equivalent. Evidently (13) can be transformed to the form (1). In particular, consider a set $\bar\Omega \in C^1([-h,0], M)$ having finite diameter with respect to ρ_I. Then for all curves $x(\cdot) \in \bar\Omega$ the velocity vector filed is bounded: $\|x'(t)\| \le k$ for some $k \ge 0$ independent of $x(\cdot)$. Hence from (13) we obtain on $\bar\Omega$

$$\|Y(\cdot)\|_I^{C^1} \le \|Y(\cdot)\|_E^{C^1} \le \|Y(\cdot)\|_I^{C^1}(1 + \Xi k). \tag{14}$$

Thus we can apply the construction of index to condensing maps of $C^1([-h,0], M)$. For example, the shift operator along the trajectories of neutral type equations is condensing with respect to Kuratowski measure of noncompactness of some modification of the above-mentioned metrics on $C^1([-h,0], M)$. See this statement for equations in linear spaces in [7].

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