Fixed Point Theory, Volume 5, No. 2, 2004, 165-180
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

CONTINUATION PRINCIPLES FOR FRACTALS

JAN ANDRES

Dept. of Math. Analysis, Fac. of Science, Palacký University,
Tomkova 40, 779 00 Olomouc-Hejčín, Czech Republic
E-mail: andres@risc.upol.cz

Abstract. Based on continuation principles for compact maps and contractions due to A. Granas [Gra72, Gra94], we shall present two respective continuation principles for (multivalued) fractals considered as fixed-points of the induced Hutchinson-Barnsley (union) operators in hyperspaces. The one for topological fractals (based on [Gra72]) is recalled, for the sake of completeness, from [AFGL05], while the one for metric fractals (based on [Gra94]) is newly developed here. Both principles are then randomized. We also briefly discuss possible generalizations related to systems of weak contractions and nonexpansive maps.
Key Words and Phrases: Multivalued fractals, continuation principles, Hutchinson-Barnsley operators, systems of contractions, systems of compact maps, fixed-points in hyperspaces, randomization.
2000 Mathematics Subject Classification: 28A80, 47H09, 47H10, 54C60.

1. Introduction

In [AFGL05], we have proved the following existence result for (multivalued) fractals.

Theorem 1. *Let (X, d) be a complete metric space and let $\{\varphi_i : X \multimap X, i = 1, \dots, n\}$ be a system of condensing (w.r.t. Kuratowski or Hausdorff measure of noncompactness) maps such that $\varphi_i(X)$ is bounded, for every $i = 1, \dots, n$. Then there exists a minimal, nonempty, compact, invariant set $A^* \subset X$ under*

Supported by the Council of Czech Government (MSM 6198959214).

This paper was presented at International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from August 24 to August 27, 2004.

the Hutchinson-Barnsley map

$$F(x) := \bigcup_{i=1}^{n} \varphi_i(x),\ x \in X,$$

i.e. a minimal (multivalued) fractal $A^* \in \mathcal{K}(X) := \{A \subset X \mid A$ *is nonempty and compact*$\}$ *with* $F^*(A^*) = A^*$*, where*

$$F^*(A) := \overline{\bigcup_{x\in A} F(x)},\ A \in \mathcal{K}(X),$$

is the Hutchinson-Barnsley operator.

As, furthermore, observed in [AFGL05], as a consequence of Theorem 1, for every system of compact maps or of (weak in the sense of [Rho01]) contractions with compact values, in a complete metric space, there exists a multivalued fractal in the above sense. We already arrived, in a different way, at the same conclusion in [AF04]. For the case of metric multivalued fractals, cf. also [And01, AG01, EP, Leś03, Pet01, Pet02, PR01, PR04].

If, for at least one $i \in \{1, \dots, n\}$, $\varphi_i : X \multimap X$ is not condensing on the whole X, but only on a subset $S \subset X$, where possibly $\varphi_j(S) \not\subset S$, for some $j \in \{1, \dots, n\}$, then Theorem 1 is no longer available. Therefore, similarly as in the degree theory, Theorem 1 must be replaced by a suitable continuation principle for (multivalued) fractals (whence the title). This aim will be separately treated for systems of compact multivalued maps (in Section 2) and of multivalued contractions (in Section 3). Then both continuation principles are randomized in Section 4. Possible extensions to systems of weak multivalued contractions and of nonexpansive multivalued maps are discussed in Section 5.

2. Continuation principle for topological fractals

Since the fundamental role will be played here by (metric) ANR-spaces (for their definition and properties, see e.g. [AG03, GD03]), we shall start with an important particular case in [Cur80].

Lemma 1 ([Cur80])**.** *If* (X, d) *is a locally continuum connected (a connected and locally continuum connected) metric space, then* $\mathcal{K}(X) \in$ ANR *(*$\mathcal{K}(X) \in$ AR*), where* $(\mathcal{K}(X), d_H)$ *is the hyperspace of compact subsets of* X *endowed with the Hausdorff metric* d_H*.*

Remark 1. Let us note that every ANR-space is locally continuum connected, but not vice versa (cf. [AG03, GD03]).

Hence, let (X, d) be a locally continuum connected metric space and let

$$F_\lambda(x) := \bigcup_{i=1}^{n} \varphi_i(\lambda, x), \ \lambda \in [0,1], x \in X, \tag{1}$$

be a one-parameter family of *Hutchinson-Barnsley maps*, where $\varphi_i : [0,1] \times X \multimap X$, $i = 1, \dots, n$, are compact Hausdorff-continuous (i.e., equivalently, upper and lower semicontinuous) maps. Then we have the induced *Hutchinson-Barnsley operators* $F^*_\lambda : \mathcal{K}(X) \to \mathcal{K}(X)$ ($\in$ ANR),

$$F^*_\lambda(A) := \bigcup_{x \in A} F_\lambda(x), \ \lambda \in [0,1], A \in \mathcal{K}(X), \tag{2}$$

and F^*_λ becomes a compact (continuous) homotopy (cf. [AF04]).

Thus, in view of Lemma 1, we can associate with F^*_λ the generalized Lefschetz number $\Lambda(F^*_\lambda) \in \mathbb{Z}$ as well as the fixed-point index $\mathrm{ind}(F^*_\lambda, U) \in \mathbb{Z}$, for every open set $U \subset \mathcal{K}(X)$ such that $\mathrm{Fix}(F^*_\lambda) \cap \partial U = \emptyset$ (for more details, see [Gra72, GD03]). If $\Lambda(F^*_\lambda) \neq 0$, for some $\lambda \in [0,1]$, then the Granas fixed-point theorem applies (see [Gra72, GD03]), and so we get a fixed-point A^* of F^*_λ (i.e. $F^*_\lambda(A^*) = A^*$), for such a $\lambda \in [0,1]$. Using the fixed-point index and the Nielsen equivalence relation in $\mathrm{Fix}(F^*_\lambda)$ (see e.g. [AG03, GD03]), we can distinguish between (homotopically) essential and inessential classes of fixed-points of F^*_λ; i.e. the class $C \subset \mathrm{Fix}(F^*_\lambda)$ (which can be verified to be isolated and compact) is (*homotopically*) *essential* if $\mathrm{ind}(F^*_\lambda, U) \neq 0$, for an open $U \subset \mathcal{K}(X)$ with $U \cap \mathrm{Fix}(F^*_\lambda) = C$. Let us note that if $\Lambda(F^*_\lambda) \neq 0$, then at least one of the Nielsen classes is (homotopically) essential.

Definition 1. We call a (topological) fractal A^* (i.e. a fixed-point of F^*_λ) of the multivalued system $\{\varphi_i(\lambda, x); i = 1, \dots, n\}$, for a given $\lambda \in [0,1]$, *homotopically essential* if A^* belongs to some essential Nielsen class for F^*_λ.

Because of the invariance under homotopy of the generalized Lefschetz number $\Lambda(\mathcal{F}^*_\lambda)$ (cf. [Gra72, GD03]), we can formulate the first continuation principle for topological fractals.

Proposition 1. *Let (X, d) be a locally continuum connected metric space and $\{\varphi_i : [0,1] \times X \multimap X; i = 1, \dots, n\}$ be a system of Hausdorff-continuous* *compact maps. Then a (homotopically) essential fractal exists for the system* $\{\varphi_i(0,\cdot) : X \multimap X; i = 1,\dots,n\}$ *iff the same is true for* $\{\varphi_i(1,\cdot) : X \multimap X; i = 1,\dots,n\}$.

Remark 2. Because of $\mathcal{K}(X) \in \mathrm{ANR}$, the well-defined $\Lambda(F^*_\lambda)$, and the fixed-point set $\mathrm{Fix}(F^*_\lambda)$ being compact, we can even associate with F^*_λ, for every $\lambda \in [0,1]$, another invariant under homotopy, namely the Nielsen number $N(F^*_\lambda)$, allowing us to make a lower estimate of the number of fractals for the system $\{\varphi_i : [0,1] \times X \multimap X; i = 1,\dots,n\}$. On the other hand, if (X,d) is a (not necessarily locally continuum connected) complete metric space and φ_i, $i = 1,\dots,n$, are compact maps, then the sole existence can be already deduced (even under less restrictions) from Theorem 1. That is why we stated the first continuation principle only in the form of proposition.

To compute the fixed-point index on open subsets of $\mathcal{K}(X) \in \mathrm{ANR}$ (in order to avoid the handicap mentioned in Remark 2) is a delicate problem. Some possibilities are indicated in [RdPS01] (cf. also [And04]). It will be therefore useful to recall some notions which are typical in the Conley index theory (cf. [KM95, RdPS01, And04]).

Hence, defining the (semi)*invariant parts* of $N \subset U$, where U is locally compact, w.r.t. $F : X \supset U \multimap X$ as (cf. [KM95])

$$\begin{aligned}
\mathrm{Inv}^+(N, F_\lambda) &:= \{x \in N \mid \sigma(i+1) \in F_\lambda(\sigma(i)),\ \text{for all } i \in \mathbb{N} \cup \{0\},\\
&\qquad \text{where } \sigma : \mathbb{N} \cup \{0\} \to N \text{ is a single-valued map}\\
&\qquad \text{with } \sigma(0) = x\},\\
\mathrm{Inv}^-(N, F_\lambda) &:= \{x \in N \mid \sigma(i+1) \in F_\lambda(\sigma(i)),\ \text{for all } i \in \mathbb{Z} \setminus \mathbb{N},\\
&\qquad \text{where } \sigma : \mathbb{Z} \setminus \mathbb{N} \to N \text{ is a single-valued map}\\
&\qquad \text{with } \sigma(0) = x\},\\
\mathrm{Inv}(N, F_\lambda) &:= \mathrm{Inv}^+(N, F_\lambda) \cap \mathrm{Inv}^-(N, F_\lambda),
\end{aligned}$$

we say that a compact invariant set $K \subset U$ (i.e. $F_\lambda(K) = K$) is *isolated* w.r.t. F_λ if there exists a compact neighbourhood N of K such that

$$O_{\mathrm{diam}(N,F_\lambda)}(\mathrm{Inv}(N, F_\lambda)) \subset \mathrm{int}\, K,$$

where $\mathrm{diam}(N, F_\lambda) := \sup_{x \in N}\{\mathrm{diam}\, F_\lambda(x)\}$ and $O_\Delta(A) := \{x \in U \mid d(x, A) < \Delta\}$, or equivalently,

$$\mathrm{dist}(\mathrm{Inv}(N, F_\lambda), \partial N) > \mathrm{diam}(N, F_\lambda),$$

where ∂N stands for the boundary of N. The neighbourhood N is then called an *isolating neighbourhood* of K.

As already pointed out, we have proved in [AF04] that $F_\lambda : X \supset U \multimap X$ induces, for every $\lambda \in [0,1]$, the compact (continuous) single-valued map F_λ^* in the hyperspace $(\mathcal{K}(X), d_H)$, i.e. $F_\lambda^* |_{\mathcal{K}(U)} : \mathcal{K}(U) \to \mathcal{K}(U)$. Hence, let $K \subset U$ be a compact isolated invariant set and N be its isolating neighbourhood. Considering an open set W such that $K \subset W \subset N$, we have defined a locally compact (continuous) single-valued map $F_\lambda^* |_{\mathcal{K}(W)} : \mathcal{K}(W) \to \mathcal{K}(X)$. Since $\mathrm{Fix}(F_\lambda^* |_{\mathcal{K}(W)}) \subset \mathcal{K}(K)$, for each $\lambda \in [0,1]$, the set of fixed-points of $F_\lambda^* |_{\mathcal{K}(W)}$ is a compact subset of $\mathcal{K}(K)$. Moreover, if X is locally connected, then $\mathcal{K}(W)$ is obviously an open subset of the ANR-space $\mathcal{K}(X)$, and so the fixed-point index

$$\mathrm{ind}(F_\lambda^* |_{\mathcal{K}(W)}, \mathcal{K}(W)) \in \mathbb{Z}$$

of $F_\lambda^* |_{\mathcal{K}(W)} : \mathcal{K}(W) \to \mathcal{K}(X)$ in $\mathcal{K}(W)$ is well-defined (see e.g. [Gra72, GD03]).

Following [RdPS01], we can also define the *Conley-type* (integer-valued) *index* $I_X(K, F_\lambda)$ of the pair (K, F_λ) just by identifying

$$I_X(K, F_\lambda) := \mathrm{ind}(F_\lambda^* |_{\mathcal{K}(W)}, \mathcal{K}(W)).$$

Because of the definition, this index has all usual properties as a standard fixed-point index. The only a bit exceptional property is the additivity property which reads as follows:

$$I_X(K, F_\lambda) = I_X(K_1, F_\lambda) + I_X(K_2, F_\lambda) + I_X(K_1, F_\lambda) \cdot I_X(K_2, F_\lambda),$$

where K is a compact isolated invariant set which is a disjoint union of two compact isolated invariant sets K_1 and K_2, i.e. $K = K_1 \cup K_2$, $K_1 \cap K_2 = \emptyset$. Moreover, it follows from the excision property of the related fixed-point index that $I_X(K, F_\lambda)$ depends neither on the choice of the isolating neighbourhood N of K, nor on the open set W.

Proposition 1 can be therefore improved in terms of the Conley-type indices as follows.

Theorem 2. *Let (X, d) be a locally continuum connected metric space, $U \subset X$ be its locally compact subset and $\{\varphi_i : [0,1] \times X \supset [0,1] \times U \multimap X; i = 1, \ldots, n\}$ be a system of Hausdorff-continuous compact maps. Assume that $N \subset U$ is an isolating neighbourhood which is common for all the Hutchinson-Barnsley maps F_λ defined in (1). Then a (homotopically) essential fractal $K_0 \subset N$ (i.e.*

$I_X(K_0, F_0) \neq 0$) exists for the system $\{\varphi_i(0,\cdot) : X \supset U \multimap X, i = 1,\dots,n\}$ iff a (homotopically) essential fractal $K_1 \subset N$ (i.e. $I_X(K_1, F_1) \neq 0$) exists for $\{\varphi_i(1,\cdot) : X \supset U \multimap X\}$. Moreover, $I_X(K_0, F_0) = I_X(K_1, F_1)$.

3. Continuation principle for metric fractals

At first, we recall a continuation principle for contractions, including the appropriate notions of homotopy and (topological) essenciality, in [Gra94] (cf. [GD03]).

Let (X, d) be a complete metric space and $U \subset X$ its subset. Denoting by $\mathcal{C}(U)$ the set of all contractions $f : \overline{U} \to X$ and, by $\{h_t : \overline{U} \to X\}$ in $\mathcal{C}(U)$, a one-parameter family of (L, M)-*Lipschitz maps*, where $L \in [0, 1)$ and $M > 0$, whenever

(i) $d(h_t(x), h_t(y)) \leq Ld(x, y)$, for all $t \in [0, 1]$ and $x, y \in \overline{U}$, and
(ii) $d(h_t(x), h_s(x)) \leq M|t - s|$, for some $M > 0$, all $x \in \overline{U}$ and $t, s \in [0, 1]$.

Remark 3. Obviously, as pointed out in [Gra94], if $\{h_t\}$ is a family of (L, M)-Lipschitz maps, then the map $h : [0, 1] \times \overline{U} \to X$ given by $h(t, x) = h_t(x)$ is continuous.

Consider still the set

$$\mathcal{C}_0(U) := \{f \in \mathcal{C}(U) \mid \operatorname{Fix} f \cap \partial U = \emptyset\},$$

where $\operatorname{Fix} f := \left\{x \in \overline{U} \mid x = f(x)\right\}$.

Definition 2. By a *homotopy* in $\mathcal{C}_0(U)$, it is meant a family of (L, M)-Lipschitz maps (see (i), (ii)) $\{h_t : \overline{U} \to X\}$, where $L \in [0, 1)$, such that $h_t \in \mathcal{C}_0(U)$. Two maps $f, g \in \mathcal{C}_0(U)$ are *homotopic* (written, $f \sim g$) if there is a homotopy $\{h_t\}$ in $\mathcal{C}_0(U)$ such that $h_0 = f$ and $h_1 = g$. Obviously, "$\sim$" is an equivalence relation in $\mathcal{C}_0(U)$ and, under this relation, $\mathcal{C}_0(U)$ decomposes into disjoint homotopy classes of contractions.

Definition 3. A map $f \in \mathcal{C}_0(U)$ is said to be *topologically essential* if f has a fixed-point. Otherwise, it is called *topologically inessential*.

The following (topological essentiality) result was proved in [Gra94].

Proposition 2. *Let $\{h_t\}$ be a homotopy in $\mathcal{C}_0(U)$. If h_0 is (topologically) essential, then so is h_t, for every $t \in [0, 1]$.*

Now, Proposition 2 will be applied for obtaining metric fractals regarded again as compact invariant subsets of $\overline{S} \subset X$ under the *Hutchinson-Barnsley maps* (cf. (1))

$$F_\lambda(x) := \bigcup_{i=1}^{n} \varphi_i(\lambda, x), \ \lambda \in [0,1], x \in \overline{S}, \tag{3}$$

or, equivalently, as fixed-points in the hyperspace $(\mathcal{K}(\overline{S}), d_H)$, where $\mathcal{K}(\overline{S}) := \{A \subset \overline{S} \mid A \neq \emptyset \text{ is compact}\}$ and d_H stands for the Hausdorff metric, of the *Hutchinson-Barnsley operators* (cf. (2))

$$F_\lambda^*(A) := \bigcup_{x \in A} F_\lambda(x), \ \lambda \in [0,1], \ A \in \mathcal{K}(\overline{S}). \tag{4}$$

Hence, considering the family $\{\varphi_i : [0,1] \times \overline{S} \multimap X; i = 1, \dots, n\}$ of multivalued (L_i, M_i)-Lipschitz maps with compact values (cf. (i), (ii)), where $L_i \in [0,1)$, $M_i > 0$ and $S \subset X$ is a subset of a complete metric space (X,d), i.e. $(i = 1, \dots, n)$

$$d_H(\varphi_i(\lambda, x), \varphi_i(\lambda, y)) \leq L_i d(x,y), \text{ for all } \lambda \in [0,1] \text{ and } x, y \in \overline{S}, \tag{5}$$

and

$$d_H(\varphi_i(\lambda_1, x), \varphi_i(\lambda_2, x)) \leq M_i |\lambda_1 - \lambda_2|, \tag{6}$$

for some $M_i > 0$, all $x \in \overline{S}$ and $\lambda_1, \lambda_2 \in [0,1]$, F_λ becomes a family of multivalued (L,M)-Lipschitz maps with compact values, where $L = \max_{i=1,\dots,n}\{L_i\}$, $M = \max_{i=1,\dots,n}\{M_i\}$, and subsequently $F_\lambda^* : \mathcal{K}(\overline{S}) \to \mathcal{K}(X)$, for all $\lambda \in [0,1]$. Moreover, $\mathcal{K}(\overline{S}) \subset \mathcal{K}(X)$ are complete metric spaces, and $F^* : [0,1] \times \mathcal{K}(\overline{S}) \to \mathcal{K}(X)$, given by $F^*(\lambda, A) = F_\lambda^*(A)$, can be proved (cf. Remark 3) to be an (L,M)-Lipschitz map, too. More precisely, $F^*(\lambda, \cdot) : \mathcal{K}(\overline{S}) \to \mathcal{K}(X)$ can be proved as in [AF04] to be an L-contraction, for every $\lambda \in [0,1]$, and $F^*(\cdot, A) : [0,1] \to \mathcal{K}(X)$ to be an M-Lipschitz, for every $A \in \mathcal{K}(\overline{S})$, namely

$$\begin{aligned} d_H(F^*(\lambda_1, A), F^*(\lambda_2, A)) &= d_H\Big(\bigcup_{x \in A} F(\lambda_1, x), \bigcup_{x \in A} F(\lambda_2, x)\Big) \leq \\ &\leq \sup_{x \in A} d_H(F(\lambda_1, x), F(\lambda_2, x)) := \Delta \leq M|\lambda_1 - \lambda_2|, \end{aligned}$$

because

$$\forall x \in A : O_\Delta(F(\lambda_1, x)) \supset F(\lambda_2, x) \text{ and } O_\Delta(F(\lambda_2, x)) \supset F(\lambda_1, x),$$

and subsequently

$$\bigcup_{x\in A} O_\Delta(F(\lambda_1,x)) \supset \bigcup_{x\in A} F(\lambda_2,x) \text{ and } \bigcup_{x\in A} O_\Delta(F(\lambda_2,x)) \supset \bigcup_{x\in A} F(\lambda_1,x),$$

by which

$$O_\Delta(\bigcup_{x\in A} F(\lambda_1,x)) \supset F(\lambda_2,A) \text{ and } O_\Delta(\bigcup_{x\in A} F(\lambda_2,x)) \supset F(\lambda_1,A),$$

i.e.

$$O_\Delta(F(\lambda_1,A)) \supset F(\lambda_2,A) \text{ and } O_\Delta(F(\lambda_2,A)) \supset F(\lambda_1,A),$$

where $O_\Delta(B) := \{x \in X \mid d(x,B) < \Delta\}$. In other words,

$$d_H(F^*(\lambda_1,A), F^*(\lambda_2,A)) \le \Delta,$$

as claimed.

Taking, therefore, $\overline{\mathcal{U}} \subset \mathcal{K}(\overline{S})$, the family of (L,M)-Lipschitz maps $\{F^*_\lambda \mid_{\overline{\mathcal{U}}} \overline{\mathcal{U}} \to \mathcal{K}(X)\}$ in $\mathcal{C}(\mathcal{U})$ should still satisfy $\mathrm{Fix}(F^*_\lambda) \cap \partial\mathcal{U} = \emptyset$, for all $\lambda \in [0,1]$, i.e. we require that $F^*_\lambda \in \mathcal{C}_0(\mathcal{U})$ (see Definition 2). Applying Proposition 2, we can give immediately

Proposition 3. *Let $\{\varphi_i : [0,1] \times \overline{S} \multimap X; i = 1,\dots,n\}$ be a family of multivalued (L_i, M_i)-Lipschitz maps with compact values satisfying (5), (6), where $L_i \in [0,1)$, $M_i \in (0,\infty)$, for $i = 1,\dots,n$, and $S \subset X$ is a subset of a complete metric space (X,d). Assume there is $\overline{\mathcal{U}} \subset \mathcal{K}(\overline{S})$ such that $F^*_\lambda \in \mathcal{C}_0(\mathcal{U})$, $\lambda \in [0,1]$. Then if F_0 has a compact invariant set (a metric fractal), say $K_0 \in \mathcal{U}$, i.e. $F_0(K_0) = K_0$, then F_1 also has a compact invariant set (a metric fractal), say $K_1 \in \mathcal{U}$, i.e. $F_1(K_1) = K_1$.*

The requirement $F^*_\lambda \in \mathcal{C}_0(\mathcal{U})$, $\lambda \in [0,1]$, can be satisfied if a locally compact $\overline{S}$ contains an isolating neighbourhood $N \subset \overline{S}$ w.r.t. F_λ, as already employed in the foregoing section. Hence, let $F_\lambda : \overline{S} \multimap X$ be a family of (L,M)-Lipschitz maps with compact values, $K_0 \subset \overline{S}$ be a compact isolated invariant set, for $\lambda = 0$, and N be an isolating neighbourhood, for all $\lambda \in [0,1]$. We have defined a family of (L,M)-Lipschitz maps $F^*_\lambda \mid_{\mathcal{K}(N)}: \mathcal{K}(N) \to \mathcal{K}(X)$, where $\overline{\mathcal{U}} := \mathcal{K}(N)$ and the boundary $\partial\mathcal{U} = \partial\mathcal{K}(N)$ is fixed-point free w.r.t. F^*_λ, for all $\lambda \in [0,1]$, i.e. $F^*_\lambda \in \mathcal{C}_0(\mathcal{U})$, as required.

We are in position to apply Proposition 2, (or more precisely, Proposition 3) for formulating a continuation principle for metric fractals.

Theorem 3. *Let* $\{\varphi_i : [0,1] \times \overline{S} \multimap X; i = 1, \dots, n\}$ *be a family of multivalued* (L_i, M_i)*-Lipschitz maps with compact values satisfying (5), (6), where* $L_i \in [0,1)$*,* $M_i \in (0,\infty)$*, for* $i = 1, \dots, n$*, and* $S \subset X$ *is a locally compact subset of a complete metric space* (X,d)*. Assume there exists an isolating neighbourhood* $N \subset \overline{S}$ *which is common for all the Hutchinson-Barnsley maps* $F_\lambda : \overline{S} \multimap X$*,* $\lambda \in [0,1]$*, defined by (3). If* F_0 *has a compact invariant set (a metric fractal), say* $K_0 \subset N$*, i.e.* $F_0(K_0) = K_0$*, then* F_1 *also has a compact invariant set (a metric fractal), say* $K_1 \subset N$*, i.e.* $F_1(K_1) = K_1$*.*

4. Continuation principle for random fractals

Now, both continuation principles from foregoing sections will be randomized. For this, we need to recall the related definitions and an important transformation to the deterministic case (cf. [AFGL05, AG03]).

Definition 4. We say that a multivalued mapping $\varphi : \Omega \times X \multimap Y$, where X, Y are metric spaces and Ω is a complete probabilistic space, is a *random mapping* if $\varphi(\cdot, x)$ is measurable, for every $x \in X$, and $\varphi(\omega, \cdot)$ is Hausdorff-continuous (e.g. Lipschitz), for every $\omega \in \Omega$ ($\implies \varphi$ is product-measurable).

Definition 5. A measurable map $\hat{x} : \Omega \to X \cap Y$, where Ω is a complete probabilistic space, is called a *random fixed-point* of a random mapping $\varphi : \Omega \times X \multimap Y$ if $\hat{x}(\omega) \in \varphi(\omega, \hat{x}(\omega))$, for a.a. $\omega \in \Omega$. A measurable map $\hat{A} : \Omega \multimap X \cap Y$ is similarly called a *random invariant set* of a random operator $\varphi : \Omega \times X \multimap Y$ if $\hat{A}(\omega) = \varphi(\omega, \hat{A}(\omega))$, for a.a. $\omega \in \Omega$.

The following Proposition 4, which is due to F. S. DeBlasi, L. Górniewicz and G. Pianigiani (cf. Proposition 4.20 in Chapter III.4 [AG03]), will allow us to transform random problems into the deterministic setting.

Proposition 4. *Let* $\varphi : \Omega \times \overline{X_0} \multimap X$*, where* $\overline{X_0}$ *is a closed subset of* X *and* Ω *is a complete probabilistic space, be a random mapping with compact values such that, for every* $\omega \in \Omega$*, the set of fixed-points of* $\varphi(\omega, \cdot)$ *is nonempty. Then* φ *has a random fixed-point.*

Hence, let $\overline{X_0}$ be a closed subset of X and Ω be a complete probabilistic space and consider the one-parameter family of systems $\{\varphi_i : [0,1] \times \Omega \times \overline{X_0} \multimap X\}$ of random maps with compact values such that, for each $i \in \{1, \dots, n\}$:

(i) $\varphi_i(\cdot, \omega, \cdot) : [0,1] \times \overline{X_0} \multimap X$ is a Hausdorff-continuous compact map, for every $\omega \in \Omega$,

(ii) $\varphi_i(\lambda, \cdot, A) : \Omega \to \mathcal{K}(X)$ is a measurable map, for every $\lambda \in [0,1]$ and $A \in \mathcal{K}(\overline{X_0})$,

or (i'), (i"), (ii) hold, where

(i') $\varphi_i(\lambda, \omega, \cdot) : \overline{X_0} \multimap X$ is a contraction with compact values, for every $\lambda \in [0,1]$ and $\omega \in \Omega$,

(i") $d_H(\varphi_i(\lambda_1, \omega, x), \varphi_i(\lambda_2, \omega, x)) \leq M_i|\lambda_1 - \lambda_2|$, for all $\lambda_1, \lambda_2 \in [0,1]$, $\omega \in \Omega$ and $x \in \overline{X_0}$, where $M_i > 0$ is a constant.

Remark 4. It is a question whether or not condition (ii) can be replaced by (see below)

$\varphi_i(\lambda, \cdot, x) : \Omega \to \mathcal{K}(X)$ is a measurable map, for every $\lambda \in [0,1]$ and $x \in \overline{X_0}$.

Defining the *random Hutchinson-Barnsley map* $F_{\lambda,\omega}$ as

$$F(\lambda, \omega, x) := \bigcup_{i=1}^{n} \varphi_i(\lambda, \omega, x), \quad (\lambda, \omega, x) \in [0,1] \times \Omega \times \overline{X_0}, \tag{7}$$

and the induced *random Hutchinson-Barnsley operator* $F^*_{\lambda,\omega}$ as

$$F^*(\lambda, \omega, A) := \bigcup_{x \in A} F(\lambda, \omega, x), \quad A \in \mathcal{K}(\overline{X_0}), \tag{8}$$

one can check that $F(\cdot, \omega, \cdot) : [0,1] \times \overline{X_0} \multimap X$ is also a Hausdorff-continuous compact map (cf. (i)), for every $\omega \in \Omega$, resp. $F(\lambda, \omega, \cdot) : \overline{X_0} \multimap X$ is a contraction with compact values (cf. (i')), for every $\lambda \in [0,1]$, $\omega \in \Omega$, and (cf. (i")) $d_H(F(\lambda_1, \omega, x), F(\lambda_2, \omega, x)) \leq M|\lambda_1 - \lambda_2|$, for all $\lambda_1, \lambda_2 \in [0,1]$, $\omega \in \Omega$ and $x \in \overline{X_0}$, where $M = \max_{i=1,\ldots,n}\{M_i\}$. Furthermore, $F^*_\omega : [0,1] \times \Omega \times \mathcal{K}(\overline{X_0}) \to \mathcal{K}(X)$ can be verified (cf. [AFGL05]) to be (product-) measurable.

One can also prove, exactly in the same way as in [AF04] (cf. Section 2), that $F^*_\omega : [0,1] \times \mathcal{K}(\overline{X_0}) \to \mathcal{K}(X)$ is a compact (continuous) mapping, for every $\omega \in \Omega$, i.e. that $\overline{F^*([0,1], \omega, \mathcal{K}(\overline{X_0}))}$ is, for every $\omega \in \Omega$, a compact set in $\mathcal{K}(X)$, resp. that (cf. Section 3) $F^*_{\lambda,\omega} : \mathcal{K}(\overline{X_0}) \to \mathcal{K}(X)$ is a contraction with compact values, for every $\lambda \in [0,1]$ and $\omega \in \Omega$, and $d_H(F^*(\lambda_1, \omega, A), F^*(\lambda_2, \omega, A)) \leq M|\lambda_1 - \lambda_2|$, for all $\lambda_1, \lambda_2 \in [0,1]$, $\omega \in \Omega$ and $A \in \mathcal{K}(\overline{X_0})$. Propositions 1 and 3 can be, therefore, randomized, on the basis of Proposition 4 as follows.

Proposition 5. *Let (X,d) be a locally continuum connected metric space and Ω be a complete probabilistic space. Let $\{\varphi_i : [0,1]\times\Omega\times X \multimap X;\ i = 1,\dots,n\}$ be a system of random maps with compact values, satisfying conditions (i), (ii), for $\overline{X_0} := X$. Then a (homotopically) essential fractal exists, for each $\omega \in \Omega$, to the system $\{\varphi_i(0,\omega,\cdot) : X \multimap X; i = 1,\dots,n\}$, and subsequently a random fractal, say $A_0^* : \Omega \to \mathcal{K}(X)$, exists to the system $\{\varphi_i(0,\cdot,\cdot) : \Omega\times X \multimap X; i = 1,\dots,n\}$, i.e. $A_0^*(\omega) = F^*(0,\omega,A_0^*(\omega))$, for a.a. $\omega \in \Omega$, iff a (homotopically) essential fractal exists, for each $\omega \in \Omega$, to the system $\{\varphi_i(1,\omega,\cdot) : X \multimap X; i = 1,\dots,n\}$, and subsequently a random fractal, say $A_1^* : \Omega \to \mathcal{K}(X)$, exists to the system $\{\varphi_i(1,\cdot,\cdot) : \Omega\times X \multimap X; i = 1,\dots,n\}$, i.e. $A_1^*(\omega) = F^*(1,\omega,A_1^*(\omega))$, for a.a. $\omega \in \Omega$.*

Proposition 6. *Let (X,d) be a complete metric space and $\overline{X_0} \subset X$ its closed subset. Let $\{\varphi_i : [0,1]\times\Omega\times\overline{X_0} \multimap X; i = 1,\dots,n\}$ be a family of random maps with compact values, satisfying conditions (i'), (i") and (ii). Assume there is $\overline{\mathcal{U}} \subset \mathcal{K}(\overline{X_0})$ such that (cf. (8)) $F^*_{\lambda,\omega} \in \mathcal{C}_0(\mathcal{U}), \lambda \in [0,1], \omega \in \Omega$. Then if $F_{0,\omega}$ (cf. (7)) has a (metric) fractal, for each $\omega \in \Omega$, and subsequently if $F(0,\cdot,\cdot)$ has a random fractal, say $A_0^* : \Omega \to \mathcal{K}(X)$, i.e. $A_0^*(\omega) = F^*(0,\omega,A_0^*(\omega))$, for a.a. $\omega \in \Omega$, then so has $F_{1,\omega}$ (cf. (7)), and subsequently $F(1,\cdot,\cdot)$ has a random fractal, say $A_1^* : \Omega \to \mathcal{K}(X)$, i.e. $A_1^*(\omega) = F^*(1,\omega,A_1^*(\omega))$, for a.a. $\omega \in \Omega$.*

Using the property of an isolating neighbourhood (cf. Section 2), Propositions 5 and 6 can be still respectively improved as follows.

Theorem 4. *Let (X,d) be a locally continuum connected metric space, $\overline{X_0} \subset X$ its closed, locally compact subset, and Ω be a complete probabilistic space. Let $\{\varphi_i : [0,1]\times\Omega\times\overline{X_0} \multimap X;\ i = 1,\dots,n\}$ be a system of random maps with compact values, satisfying conditions (i), (ii). Assume that $N \subset \overline{X_0}$ is an isolating neighbourhood which is common for all the random Hutchinson-Barnsley maps $F_{\lambda,\omega}$ defined in (7). Then a (homotopically) essential fractal $A_{0,\omega} \subset N$ exists (i.e. $I_X(A_{0,\omega},F_{0,\omega}) \neq 0$), for each $\omega \in \Omega$, to the system $\{\varphi_i(0,\omega,\cdot) : \overline{X_0} \multimap X; i = 1,\dots,n\}$, and subsequently a random fractal, say $A_0^* : \Omega \to \mathcal{K}(X)$, exists to the system $\{\varphi_i(0,\cdot,\cdot) : \Omega\times\overline{X_0} \multimap X; i = 1,\dots,n\}$, iff a (homotopically) essential fractal $A_{1,\omega} \subset N$ exists (i.e. $I_X(A_{1,\omega},F_{1,\omega}) \neq 0$), for each $\omega \in \Omega$, to the system $\{\varphi_i(1,\omega,\cdot) : \overline{X_0} \multimap X; i = 1,\dots,n\}$, and subsequently a random fractal, say $A_1^* : \Omega \to \mathcal{K}(X)$, exists to the system $\{\varphi_i(1,\cdot,\cdot) : \Omega\times\overline{X_0} \multimap X; i = 1,\dots,n\}$.*

Theorem 5. *Let (X, d) be a complete metric space and $\overline{X_0} \subset X$ its closed, locally compact subset. Let $\{\varphi_i : [0,1] \times \Omega \times \overline{X_0} \multimap X; i = 1, \dots, n\}$ be a family of random maps with compact values, satisfying conditions (i'), (i") and (ii). Assume there exists an isolating neighbourhood $N \subset \overline{X_0}$ which is common for all the random Hutchinson-Barnsley maps $F_{\lambda,\omega}$ defined in (7). If $F_{0,\omega}$ (cf. (7)) has a (metric) fractal, for each $\omega \in \Omega$, and subsequently a random fractal, say $A_0^* : \Omega \to \mathcal{K}(X)$, i.e. $A_0^*(\omega) = F^*(0, \omega, A_0^*(\omega))$, for a.a. $\omega \in \Omega$, then so has $F_{1,\omega}$ (cf. (7)), and subsequently $F(1, \cdot, \cdot)$ has a random fractal, say $A_1^* : \Omega \to \mathcal{K}(X)$, i.e. $A_1^*(\omega) = F^*(1, \omega, A_1^*(\omega))$, for a.a. $\omega \in \Omega$.*

Remark 5. Propositions 5 resp. 6 and Theorems 4 resp. 5 can be (formally, but more effectively) expressed directly in terms of homotopically resp. topologically essential random fractals, when defining the random fixed-point index resp. when formulating the random continuation principle for contractions, on the basis of Proposition 4.

5. Some further possibilities

As pointed out in Introduction, Theorem 1 applies in particular to systems of weak contractions with compact values in the sense of the following (cf. [Rho01])

Definition 6. Assume that (X, d) is a complete metric space and let h be a function such that

(i) $h : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing,
(ii) $h(t) = 0 \iff t = 0$ (i.e. $h(t) > 0$, for $t \in (0, \infty)$),
(iii) $\lim_{t\to\infty} h(t) = \infty$.

A mapping $\varphi : X \multimap X$ with compact values is said to be a *weak contraction* (in the sense of [Rho01]) if, for any $x, y \in X$,

$$d_H(\varphi(x), \varphi(y)) \leq d(x, y) - h(d(x, y)). \tag{9}$$

We proved in [AF04] (cf. [AG03]) that if $\{\varphi_i : X \multimap X;\ i = 1, \dots, n\}$ is a system of weak contractions in the sense of Definition 6, then the Hutchinson-Barnsley operator $F^* : \mathcal{K}(X) \to \mathcal{K}(X)$, where

$$F^*(A) := \bigcup_{x \in A} F(x),\ A \in \mathcal{K}(X),$$

is also a weak contraction in the sense of Definition 6, where (9) is replaced by

$$d_H(F^*(A), F^*(B)) \leq d_H(A,B) - h(d_H(A,B)), \text{ for all } A, B \in \mathcal{K}(X). \quad (10)$$

A natural question arises, whether Proposition 3 can be generalized for systems $\{\varphi_i : [0,1] \times \overline{S} \multimap X; i = 1, \ldots, n\}$, where $\varphi_i(\lambda, \cdot) : \overline{S} \multimap X, i = 1, \ldots, n$, are weak contractions in the sense of Definition 6 and $\varphi_i(\cdot, x) : [0,1] \multimap X$, $i = 1, \ldots, n$, satisfy condition (6). For this, we would need an appropriate analogy of Proposition 2.

Such an analogy was established in [Fri01], but the notion of a weak contraction is understood in a (bit different) sense of [DG78] (cf. [Fri01]):

Definition 7. Assume that (X, d) is a complete metric space and $\overline{U} \subset X$ its closed subset. A mapping $f : \overline{U} \to X$ is said to be a *weak contraction* (in the sense of [DG78]; cf. [Fri01]) if there exists a *compactly positive* mapping $\psi : X \times X \to (0, \infty)$, i.e.

$$\inf\{\psi(x,y) \mid a \leq d(x,y) \leq b\} := \theta(a,b) > 0, \text{ for every } 0 < a \leq b, \quad (11)$$

such that

$$d(f(x), f(y)) \leq d(x,y) - \psi(x,y), \text{ for any } x, y \in \overline{U}. \quad (12)$$

The mentioned extension of Proposition 2 in [Fri01] takes the following form:

Proposition 7. *Let $f, g : \overline{U} \to X$ be two homotopic weak contractions in the sense of Definition 7, i.e. there exists $h : [0,1] \times \overline{U} \to X$ such that $h(0, \cdot) = f$, $h(1, \cdot) = g$; $h(t,x) \neq x$, for every $x \in \partial U$, and $t \in [0,1]$; $h(t, \cdot)$ satisfies (12), for every $t \in [0,1]$, and $h(\cdot, x)$ satisfies (cf. (6))*

$$d(h(t,x), h(s,x)) \leq |r(t) - r(s)|, \text{ for all } x \in \overline{U} \text{ and } t, s \in [0,1],$$

where $r : [0,1] \to \mathbb{R}$ is a continuous function.

If a positively compact function ψ associated to h (cf. (12)) still satisfies (cf. (11))

$$\inf\{\theta(a,b) \mid b \geq a\} > 0, \text{ for all } a > 0, \quad (13)$$

then f has a fixed-point iff so has g.

Defining $\tilde{h} : (0, \infty) \to (0, \infty)$ as $\tilde{h}(a) := \theta(a, a)$, i.e. for $(0 <)a = b = d(x, y)$, we obtain the composed function $\tilde{h}(a) = \tilde{h}(d(x, y))$. Thus, if $\psi(x, y) \leq d(x, y)$, for all $x, y \in \overline{U}(\subset X)$, $x \neq y$, then under (11), (12):

$$\begin{aligned} d(f(x), f(y)) &\leq d(x, y) - \psi(x, y) \leq d(x, y) - \theta(a, b) \leq \\ &\leq d(x, y) - \inf\{\theta(a, b) \mid b \geq a\} = d(x, y) - \theta(a, a) = \\ &= d(x, y) - \tilde{h}(a) = d(x, y) - \tilde{h}(d(x, y)), \end{aligned}$$

for all $x, y \in \overline{U}, x \neq y$.

It follows that, for a continuous ψ, conditions (11), (12) seem to be, for the first glance, more restrictive than (9), for single-valued maps, with a continuous h satisfying $h(t) > 0$, for $t \in (0, \infty)$ (cf. (ii) in Definition 6), i.e. than those for weak contractivity in the sense of [KS69]. However, this type of weak contractivity was shown in [Jac97] to be equivalent with the one in the sense of Definition 7. Thus, because of additional requirements in (i) and (iii), it is Definition 6 which is more restrictive than Definition 7. Although there are (unlike in Definition 6) practically no regularity restrictions imposed on ψ in Definition 7, Proposition 7 need not apply, in view of (13), to single-valued weak contractions in the sense of Definition 6.

So, in order to obtain a generalization of Proposition 3, for a suitable notion of weak contractions, we should either develop the analogy of Proposition 7 for single-valued weak contractions in the sense of Definition 6, or to show as in [AF04] that the respective analogy of (10) (cf. (12)) holds for the Hutchinson-Barnsley operator $F^* : \mathcal{K}(\overline{U}) \to \mathcal{K}(X)$ related to (multivalued) contractions with compact values in the sense of an appropriately modified Definition 7, where ψ resp. θ satisfies (13). For the first case, it would be enough to find an additional condition imposed on h which is equivalent to (13). Condition (6) can be used without any change as well.

Let us add that the notion of a weak contraction in Definition 6 can be significantly weaken (see Remark 1 in [AFGL05] and the references therein). For further continuation theorems for (weak) contractions, see e.g. [Che01, FG94, Pre02].

As concerns the systems of nonexpansive maps, the situation is even more delicate, because a possible direct analogy of Proposition 2 fails, as observed in [Fri01]. On the other hand, if a system of nonexpansive self-maps on a complete ANR-space satisfies a certain version of the Palais-Smale condition, then, as observed in [AFGL05], an R_δ-set of fractals can exist. Another question so arises, whether under additional assumptions, including the mentioned Palais-Smale condition for a system of nonexpansive maps and X being a complete ANR-space, the desired analogy of Proposition 2 holds. If so, then the analogy of Proposition 3 holds as well.

One can also ask whether the Conley index theory can be developed for discrete multivalued dynamical systems, which would directly apply to the Hutchinson-Barnsley maps, for obtaining compact invariant sets (i.e. multivalued fractals). A promisible step in this direction was done for upper semicontinuous maps with compact values in locally compact metric spaces in [KM95]. Unfortunately, these maps are assumed to be determined by a givem morphism, i.e. in particular that the set of values is connected, which excludes the application of the Conley index in [KM95] to the Hutchinson-Barnsley maps whose sets of values are disconnected.

References

[AF04] J. Andres and J. Fišer, *Metric and topological multivalued fractals*, Int. J. Bifurc. Chaos **14** (2004), no. 4, 1277–1289.

[AFGL05] J. Andres, J. Fišer, G. Gabor, and K. Leśniak, *Multivalued fractals*, Chaos, Solitons and Fractals **24** (2005), no. 3, 665–700.

[AG01] J. Andres and L. Górniewicz, *On the Banach contraction principle for multivalued mappings*, In: Approximation, Optimization and Mathematical Economics. Proceedings of the 5th international conference on "Approximation and Optimization in the Caribbean", Guadeloupe, French West Indies, March 29 - April 2, 1999 (M. Lassonde, ed.), Heidelberg: Physica-Verlag, 2001, pp. 1–23.

[AG03] ———, *Topological Fixed Point Principles for Boundary Value Problems*, Serie: Topological Fixed Point Theory and Its Applications, Kluwer, Dordrecht, 2003.

[And01] J. Andres, *Some standard fixed-point theorems revisited*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), 455–471.

[And04] ———, *Applicable fixed point principles*, In: "Handbook of Topological Fixed Point Theory" (R. F. Brown, M. Furi, L. Górniewicz, and B. Jiang, eds.), Kluwer, Dordrecht, 2004, to appear.

[Che01] Y.-Z. Chen, *Continuation method for α-sublinear mappings*, Proc. Amer. Math. Soc. **129** (2001), no. 1, 203–210.

[Cur80] D. W. Curtis, *Hyperspaces of noncompact metric spaces*, Compositio Math. **40** (1980), 139–152.

[DG78] J. Dugundji and A. Granas, *Weakly contractive maps and elementary domain invariance theorem*, Bull. Greek Math. Soc. **19** (1978), no. 1, 141–151.

[EP] R. Espínola and A. Petruşel, *Existence and data dependence of fixed points for multivalued operators on gauge spaces*, Preprint (2004).

[FG94] M. Frigon and A. Granas, *Résultats du type de Leray-Schauder pour des contractions multivoques*, Topol. Math. Nonlin. Anal. **4** (1994), 197–208.

[Fri01] M. Frigon, *On continuation methods for contractive and nonexpansive mappings*, In: "Recent Advances on Metric Fixed Point Theory" (T. Domínguez Benavides, ed.), Cienca, vol. 48, Univ. of Sevilla, Sevilla, 2001, pp. 19–30.

[GD03] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, Berlin, 2003.

[Gra72] A. Granas, *The Leray-Schauder index and the fixed point theory for arbitrary ANR's*, Bull. Soc. Math. France **100** (1972), 209–228.

[Gra94] ———, *Continuation method for contractive maps*, Topol. Math. Nonlin. Anal. **3** (1994), 375–379.

[Jac97] J. R. Jachymski, *Equivalence of some contractivity properties over metrical spaces*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 2327–2335.

[KM95] T. Kaczyński and M. Mrozek, *Conley index for discrete multivalued dynamical systems*, Topol. Appl. **65** (1995), 83–96.

[KS69] M. A. Krasnosel'skii and V. J. Stetsenko, *About the theory of equations with concave operators*, Sib. Mat. Zh. **10** (1969), 565–572, in Russian.

[Leś03] K. Leśniak, *Towards computing Lifshits constant for hyperspaces*, Semin. Fixed Point Theory Cluj-Napoca **4** (2003), no. 2, 159–163.

[Pet01] A. Petruşel, *Single-valued and multi-valued Meir-Keeler type operators*, Rev. d'Anal. Num. et de Thèorie de l'Approx. **30** (2001), no. 1, 75–80.

[Pet02] ———, *Fixed point theory with applications to dynamical systems and fractals*, Semin. Fixed Point Theory Cluj-Napoca **3** (2002), 305–316.

[PR01] A. Petruşel and I. A. Rus, *Dynamics on* $(P_{cp}(X), H_d)$ *generated by a finite family of multi-valued operators on* (X, d), Math. Moravica **5** (2001), 103–110.

[PR04] ———, *Multivalued Picard and weakly Picard operators*, In: Fixed Point Theory and Applications. Proceedings of the International Conference on "Fixed Point Theory and Applications", Valencia (Spain), 13 - 19 July, 2003 (J. García Falset, E. Llorens Fuster, and B. Sims, eds.), Yokohama Publishers, 2004, pp. 207–226.

[Pre02] R. Precup, *Continuation results for mappings of contractive type*, Semin. Fixed Point Theory Cluj-Napoca **2** (2002), 23–40.

[RdPS01] F. R. Ruiz del Portal and J. M. Salazar, *Fixed point index in hyperspaces: a Conley-type index for discrete semidynamical systems*, J. London Math. Soc. **64** (2001), no. 2, 191–204.

[Rho01] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis **47** (2001), 2683–2693.