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STRICT FIXED POINT THEORY

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Abstract. In this paper, we survey some results on strict fixed point theory for multivalued operators. Some new results are given.

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1. Introduction

Let X be a nonempty set and $T : X \multimap X$ a multivalued operator. An element $x \in X$ is a fixed point of T if $x \in T(x)$ and a strict fixed point if $T(x) = \{x\}$. We denote by F_T the fixed point set of T and by $(SF)_T$ the strict fixed point set of T.

Let (X, d) be a metric space. We denote:

$$P(X) := \{Y \subset X |\ Y \neq \emptyset\},$$

$$P_p(X) := \{Y \in P(X) |\ Y \text{ has the property } p\},$$

where p could be: b = bounded, cl = closed, cp = compact, etc.

In what follow we consider the following functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+, \quad D(Y,Z) := \inf\{d(y,z)|\ y \in Y,\ z \in Z\},$$

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad \delta(Y,Z) := \sup\{d(y,z)|\ y \in Y,\ z \in Z\},$$

$$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad \rho(Y,Z) := \sup\{D(y,Z)|\ y \in Y\},$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad H(Y,Z) := \sup(\rho(Y,Z), \rho(Z,Y)).$$

For the basic properties of these functionals see [26] and [14].

The aim of this paper is to survey strict fixed point theorems for multivalued operators and to give some new results. For the reader convenience some of our results which appeared in less accessible publications will be presented together with their proofs.

2. Strict fixed point theorems

We begin our considerations with the following result for generalized δ-contraction:

Theorem 2.1 ([22], [23]). *Let (X,d) be a complete metric space, $T : X \to P_b(X)$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ a function. We suppose that:*

(i) $r, s \in \mathbb{R}^5_+,\ r \le s$ implies $\varphi(r) \le \varphi(s)$;

(ii) there exists $p > 1$ such that

$$\varphi(r, pr, pr, r, r) < r,\ \forall\ r > 0;$$

(iii) $r - \varphi(r, pr, pr, r, r) \to +\infty$ as $r \to +\infty$;

(iv) φ is continuous;

(v) for all $x, y \in X$,

$$\delta(T(x), T(y)) \le \varphi(d(x,y), \delta(x,T(x)), \delta(y,T(y)), D(x,T(y)), D(y,T(x))).$$

Then

$$F_T = (SF)_T = \{x^*\}.$$

Proof. Let $p > 1$. By Lemma 8.1.3 in [26] (see also [21] or [27]) there exists a selection t of T such that $\delta(x, T(x)) \le pd(x, t(x))$. From the condition (v) it follows that

$$d(t(x), t(y)) \le \varphi(d(x,y), d(x,t(x)), d(y,t(y)), d(x,t(y)), d(y,t(x))).$$

From a fixed point theorem for φ-contraction (see [26]) the operator t has a unique fixed point x^*.

Let $x \in F_T$. If we take in (v), $y = x$, then

$$\delta(T(x)) = \delta(T(x), T(x)) \le u(0, \delta(x,T(x)), \delta(x,T(x)), 0, 0) \le$$

$$\le \varphi(\delta(T(x)), p\delta(T(x)), p\delta(T(x)), \delta(T(x)), \delta(T(x))).$$

From the condition (ii) we have $\delta(T(x)) = 0$, i.e., $T(x) = \{x\}$. So, $F_T = (SF)_T \neq \emptyset$.

Now the uniqueness follows from (v).

Remark 2.1. From the above theorem we have some results given by S. Reich [20], Lj. B. Ćirić [3], K. Iseki [12], I. A. Rus [21].

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to P_b(X)$ a multivalued operator. We suppose that*

(i) $x \in T(x)$, $\forall\, x \in X$;

(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a Picard sequence $x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$ such that

$$\delta(T(x_{n+1})) \leq \varphi(\delta(T(x_n))), \quad n \in \mathbb{N}.$$

Then $x_n \to x^*$ as $n \to \infty$ and $x^* \in (SF)_T$.

Proof. From the condition (ii) it follows that

$$\delta(T(x_n)) \leq \varphi^n(\delta(T(x_0))) \to 0 \text{ as } n \to \infty.$$

This implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. So $(x_n)_{n\in\mathbb{N}}$ is convergent. Let x^* be the limit of $(x_n)_{n\in\mathbb{N}}$. From (i) $x^* \in T(x^*)$ and from $\delta(T(x^*)) = 0$, it follows that $x^* \in (SF)_T$.

Remark 2.2. If we take $\varphi(t) = kt$, $0 \leq k < 1$, we have a result given by H. W. Corley in [4].

Remark 2.3. For others results for reflexive multivalued operators see S. Danes, M. Hegedüs and P. Medvegyev [6].

Theorem 2.3 ([24]). *Let (X, d) be a bounded complete metric space and $T : X \to P(X)$ a (δ, φ)-contraction, i.e., φ is a comparison function ([26]) and*

$$\delta(T(Y)) \leq \varphi(\delta(Y)),\ \forall\, Y \in P(X),\ T(Y) \subset Y.$$

Then

(i) $F_T = (SF)_T = \{x^*\}$;

(ii) There exists a selection $t : X \to X$ of T which is P.O., i.e., $F_t = \{x^*\}$ and

$$t^n(x_0) \to x^* \text{ as } n \to \infty, \text{ for each } x_0 \in X;$$

(iii) $T^n(x) \xrightarrow{H} \{x^*\}$ as $n \to \infty$, $\forall\, x \in X$.

Proof. (i). Let $X_1 := \overline{T(X)}, \ldots, X_{n+1} := \overline{T(X_n)}$, $n \in \mathbb{N}^*$.

We remark that

(a) $X \supset X_1 \supset \cdots \supset X_n \supset \ldots$;

(b) $X_n \in P_{b,cl}(X)$ and $T(X_n) \subset X_n$;

(c) $\delta(X_n) \leq \varphi^n(\delta(X)) \to 0$ as $n \to \infty$.

Let $X_\infty := \bigcap_{n\in\mathbb{N}} X_n$. From (a)+(b)+(c) we have that $T(X_\infty) \subset X_\infty$ and $\delta(X_\infty) = 0$. So, $X_\infty = \{x^*\}$, and $x^* \in (SF)_T$. On the other hand $F_T \subset \bigcap_{n\in\mathbb{N}} X_n$. This implies that $F_T = (SF)_T = \{x^*\}$.

(ii). Let $t(x) \in T(x)$, $\forall\ x \in X$. Let $x, y \in X$. Then $t(x), t(y) \in T(X)$ and $t^n(x), t^n(y) \in T^n(X)$.

From (i) we have that $t^n(x) \to x^*$ as $n \to \infty$.

(iii). We remark that $T^n(x) \subset T^n(X)$, $\forall\ x \in X$.

Now (i) $\Rightarrow$ (iii).

Remark 2.4. From the Theorem 2.3 we have some results by S. Reich [20], Lj. B. Ćirić [3], K. Iseki [12], I. A. Rus [21], [22], [23], [24], A. Petruşel and I. A. Rus [19],...

Remark 2.5. If (X, d) is a bounded complete metric space then Theorem 2.3 implies Theorem 2.1.

3. Strict fixed points and fractal operators

Let (X, d) be a metric space. An operator $t : X \to X$ is Picard operator if $F_t = \{x^*\}$ and $t^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$ ([26]).

Let $T : X \multimap X$. The operator $\widehat{T} : P(X) \to P(X)$ defined by $\widehat{T}(Y) := \bigcup_{x\in Y} T(x)$ is called the fractal operator generated by T ([5], [8], [9], [16], [18], [19],...).

We have

Theorem 3.1. *Let (X, d) be a complete metric space, $U \subset P_{cl}(X)$ and $T : X \to U$. We suppose that*

(i) $x \in X$ implies $\{x\} \in U$;

(ii) $Y \in U$ implies $T(Y) \in U$;

(iii) $T : (U, H) \to (U, H)$ is Picard operator.

Then

$$(SF)_T \neq \emptyset \text{ implies } F_T = (SF)_T = \{x^*\}.$$

Proof. Let Y^* be the unique fixed point of $\widehat{T}$. From (iii) it follows that

$$T^n(x) \stackrel{H}{\to} Y^* \text{ as } n \to \infty, \text{ for all } x \in X.$$

If we take $x = x^* \in (SF)_T$, we have $Y^* = \{x^*\}$.

On the other hand if $x \in F_T$, then $x \in T(x) \subset T^2(x) \subset \cdots \subset T^n(x) \to \{x^*\}$. So, $F_T \subset \{x^*\}$.

From Theorem 3.1 we have

Theorem 3.2. (Rus (1997); see [28], or [26]) *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ a contraction. Then*

$$(SF)_T \neq \emptyset \text{ implies } F_T = (SF)_T = \{x^*\}.$$

Remark 3.1. For others results of this type see A. Sîntămărian [28] and A. Muntean [14].

Remark 3.2. For some results on common strict fixed points see A. Ahmad and M. Imdad [1], M. Avram [2], T. L. Hicks [10], T. Hicks and B. E. Rhoades [11], K. Iseki [12], T. Kubiak [13], A. Muntean [14], A. S. Mureşan [15], N. Negoescu [17], I. A. Rus, A. Petruşel and G. Petruşel [27], A. Sîntămărian [28] and D. H. Tan and D. T. Nhan [30].

Remark 3.3. For a set-theoretic aspects of strict fixed point theory see I. A. Rus [25].

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