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ON KHAMSI'S FIXED POINT THEOREM

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Abstract. In the paper [4], Khamsi gives an abstract formulation to Sadowskii's fixed point theorem. In this paper we will present a general fixed point principle, which generalizes Khamsi's fixed point theorem.

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1. Introduction

May be one of the most interesting result in metric fixed point theory is Kirk's theorem [5]. The initial attempts to extend it to the nonlinear case were not very successful. In the paper [4], Khamsi considers a notion of convexity structure and discuss Sadowskii's fixed point theorem in this setting. Then he gives an interesting example of hyperconvex metric spaces.

In this paper we will present a general fixed point principle, which generalize Khamsi's fixed point theorem and we will give a general fixed point principle for the case of non self-mappings. Our approach is based on the retraction mapping principle.

2. Khamsi's fixed point theorem

Definition 2.1. [4] Let (X, d) be a metric space and F a family of bounded subsets of X. We will say:

(i) F has the *intersection property (IP)* if and only if $A \cap B \in F$ provided $A \in F$ and $B \in F$.

(ii) F has the *chain intersection property (CIP)* if and only if $\bigcap_{i\in I} A_i \in F$ provided $(A_i)_{i\in I}$ is a decreasing chain of elements in F.
In both cases, we may talk about the F-closure of $A \in P_b(X)$, which we will denote $co_F(A)$. Indeed, if F has *IP* , then we set

$$co_F(A) = \bigcap_{B\in F(A)} B,$$

where $F(A) = \{B \in F/A \subset B\}$.

Example 2.1. Let X be a normed linear space and $Y \subset X$ a closed bounded convex subset of X. Consider F to be the family of all the closed convex subsets of Y. Then F satisfies IP.

Definition 2.2. [4] Let (X, d) be a metric space and F a family of closed bounded subsets of X. We will say that F is α_K- *invariant* if and only if for any $A \in P_b(X)$, $co_F(A)$ exists and $\alpha_K(co_F(A)) = \alpha_K(A)$.

Definition 2.3. [1] A metric space X is said to be *hyperconvex* if and only if for any family $(x_i)_{i\in I}$ of points in X and any family$(r_i)_{i\in I}$ of positive numbers such that $d(x_i, x_j) \leq r_i + r_j$, $\forall i, j \in I$, then we have :

$$\bigcap_{i\in I} B(x_i, r_i) \neq \emptyset.$$

Remark 2.1. [2] If X is a hyperconvex metric space and (H_i) is a decreasing chain of bounded hyperconvex subsets of X, then $\bigcap_i H_i$ is not empty and is hyperconvex. Therefore, the family $H = \{H \subset P_b(X)/H \neq \emptyset, H$ is hyperconvex $\}$ satisfies CIP (but fails to satisfy IP, i.e. the intersection of two hyperconvex is not necessarily hyperconvex).

Proposition 2.1. (Khamsi) [4] Let X be a hyperconvex metric space and H an associated family to X. Then H is α_K- invariant.

Definition 2.4. [4] Let (X, d) be a metric space and F a family of bounded subsets of X. We will say that F satisfies the property *(S)* (for Schauder) if and only if for any $Y \in F$ nonempty compact set and any $f : Y \to Y$ continuous map, we have $F_f \neq \emptyset$.

Proposition 2.2. (Khamsi) [3] Let X be a hyperconvex metric space and H an associated family to X. Then the family H satisfies (S).

Theorem 2.2. (Khamsi) [4] Let (X, d) be a metric space and F a family of bounded subsets of X. We assume that:
(i) F satisfy IP (or CIP)
(ii) F satisfy the property *(S)*
(iii) F is α_K- invariant.
Then, for any nonempty $Y \in F$ and any continuous $f : Y \to Y$, which is condensing, we have $F_f \neq \emptyset$.

Corollary 2.1. (Kirk) [5] Let X be a bounded hyperconvex metric space and $f : X \to X$ a continuous condensing map. Then $F_f \neq \emptyset$.

3. Main results

The main result of the paper is the following:

Theorem 3.1. Let $(X, S(X), M)$ be a fixed point structure, $\theta : Z \to \mathbf{R}_+$ and $\eta : P(X) \to P(X)$ a closure operator. Let $S(X) \subset S_1(X) \subset \eta(Z) \subset Z$, which satisfies the following condition: $A \in S_1(X)$ and $A \in F_\eta \cap Z_\theta$ implies $A \in S(X)$.
Let $Y \in S_1(X)$ and $f \in M(Y)$.
We suppose that:
(i) $\theta(\eta(A)) = \theta(A), \forall A \in Z$,
(ii) $A \in Z, x \in X \Rightarrow A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$,
(iii) $S_1(X)$ satisfy the following intersection property:
$A_i \in S_1(X), A_{i+1} \subset A_i, i \in \mathbf{N} \Rightarrow \bigcap_{i \in N} A_i \in S_1(X)$,
(iv) f is θ- condensing mapping.

Then:
(a) $\exists A \in I(f) \cap S(X)$
(b)$F_f \neq \emptyset$.
(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

Proof. Let $a \in Y$ and $A = \{a\} \subset Y$. Then by a lemma in [10] there exists $A_0 \subset Y$, which satisfies the following conditions:
$(c_1) A \subset A_0$
$(c_2) A_0 \in F_\eta$
$(c_3) A_0 \in I(f)$
$(c_4) \eta(f(A_0) \cup A) = A_0$.

Thus by (i), (ii) and (c_4) we have $\theta(\eta(f(A_0)\cup A)) = \theta(f(A_0)\cup A) = \theta(f(A_0)\cup \{a\}) = \theta(f(A_0)) = \theta(A_0)$. The mapping f is θ- condensing, this implies $\theta(A_0) = 0$, thus $A_0 \in Z_\theta$.
By (c_2) we have $A_0 \in F_\eta \cap Z_\theta$, this implies $A_0 \in S(X)$ and by (c_3) we have $A_0 \in I(f) \cap S(X)(a)$.
We consider $f/_{A_0} : A_0 \to A_0, A_0 \subset Y, f \in M(Y)$, we have $f/_{A_0} \in M(A_0)$. Since $(X, S(X), M)$ is a fixed point structure, we have $F_f \neq \emptyset(b)$.
If $F_f \in Z$, from $\theta(F_f) = \theta(f(F_f))$ we have $\theta(F_f) = 0(c)$.

Proof of the theorem 2.2. Let $S(X) := \{Y \in P_{cp}(X)/f \in C(Y) \Rightarrow F_f \neq \emptyset\}$ and $M = C$.
Thus $(X, S(X), M)$ is a fixed point structure. We consider the Kuratowski's mapping $\theta = \alpha_K : P_b(X) \to \mathbf{R}_+$, and $\eta : P(X) \to P(X)$ a closure operator, $\eta(A) = \overline{co}_F(A), \forall A \in P(X)$. Let $S_1(X) = F$ from hypothesis.
If $A \in S_1(X) = F$ and $A \in F_\eta \cap Z_\theta$, we have A bounded, $\overline{co}_F(A) = A$ and $\alpha_K(A) = 0$, this implies A is compact. For any $f \in C(A)$, from(ii) we have $F_f \neq \emptyset$, thus $A \in S(X)$.
$Y \in F = S_1(X)$, from hypothesis $f \in M(Y)$. F is α_K-invariant, for any $A \in P_b(X), x \in X$ we have $A \cup \{x\} \in P_b(X)$ and $\alpha_K(A \cup \{x\}) = \alpha_K(A)$.
$S_1(X) = F$ has the intersection property CIP, f is α_K -condensing mapping, and for any $f \in C(Y), Z \subset Y, f(Z) \subset Z \Rightarrow f/_Z \in C(Z)$. Then by the above theorem, we have $F_f \neq \emptyset$.

Further on we will present a general fixed point principle for the case of non-self mappings. Our approach is based on the retraction mapping principle. Then we will give as application, Khamsi's fixed point theorem for the case of non-self mappings.

Definition 3.1. Let X be a nonempty set and $Y \subset X$ a nonempty subset of X. A mapping $\rho : X \to Y$ is called a *retraction* of X onto Y, if $\rho/_Y = 1_Y$.

Definition 3.2. A mapping $f : Y \to X$ is called *retractible* onto Y by $\rho : X \to Y$, if $F_f = F_{\rho\circ f}$.

Theorem 3.2. Let $(X, S(X), M)$ be a fixed point structure, $\theta : Z \to \mathbf{R}_+$ and $\eta : P(X) \to P(X)$ a closure operator. Let $S(X) \subset S_1(X) \subset \eta(Z) \subset Z$, which satisfies the following condition:
$A \in S_1(X)$ and $A \in F_\eta \cap Z_\theta$ implies $A \in S(X)$.
Let $Y \in S_1(X)$ and $f : Y \to X$ a mapping and $\rho : X \to Y$ a retraction.
We suppose that:

(i) $\theta(\eta(A)) = \theta(A), \forall A \in Z$,
(ii) $A \in Z, x \in X \Rightarrow A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$,
(iii) $S_1(X)$ satisfy the following intersection property:
$A_i \in S_1(X), A_{i+1} \subset A_i, i \in \mathbf{N} \Rightarrow \bigcap_{i \in N} A_i \in S_1(X)$,
(iv) f is a strong θ- condensing mapping
(v) ρ is $(\theta, 1)$- contraction mapping
(vi) f is retractible onto Y by ρ and $\rho \circ f \in M(Y)$.
Then $F_f \neq \emptyset$ and if $F_f \in Z$, we have $\theta(F_f) = 0$.

Proof. Let $a \in Y$ and $A = \{a\} \subset Y$. The set $Y \in F_\eta$ and $\rho \circ f : Y \to Y$. Then by a lemma in [10] there exists $A_0 \subset Y$, which satisfies the following conditions:
$(c_1) A \subset A_0$
$(c_2) A_0 \in F_\eta$
$(c_3) A_0 \in I(\rho \circ f)$
$(c_4) \eta((\rho \circ f)(A_0) \cup A) = A_0$.
The mapping $\rho \circ f : Y \to Y$ is strong θ-condensing, because from conditions (v) and (iv) we have:

$$\theta((\rho \circ f)(A)) \leq \theta(f(A)) < \theta(A), \forall A \in P(Y) \cap Z, \theta(A) \neq 0$$

Thus by (i), (ii) and (c_4) we have $\theta(\eta((\rho \circ f)(A_0) \cup A)) = \theta((\rho \circ f)(A_0) \cup A) = \theta((\theta \circ f)(A_0) \cup \{a\}) = \theta((\rho \circ f)(A_0)) = \theta(A_0)$. The mapping $\rho \circ f : Y \to Y$ is strong θ- condensing, this implies $\theta(A_0) = 0$, thus $A_0 \in Z_\theta$.
By (c_2) we have $A_0 \in F_\eta \cap Z_\theta$, this implies $A_0 \in S(X)$ and by (c_3) we have $A_0 \in I(\rho \circ f) \cap S(X)$.
We consider $\rho \circ f/_{A_0} : A_0 \to A_0, A_0 \subset Y, \rho \circ f \in M(Y)$, $(X, S(X), M)$ is a fixed point structure, thus $F_{\rho \circ f} = F_f \neq \emptyset$. If $F_f \in Z$, from $\theta(F_f) = \theta((\rho \circ f)(F_f)) < \theta(F_f)$, $\rho \circ f$ is θ-condensing, we have $\theta(F_f) = 0$.

Theorem 3.3. Let (X, d) be a metric space and F a family of bounded subsets of X. We assume that:
(i) F satisfy CIP
(ii) F satisfy the property *(S)*
(iii) F is α_K- invariant.
Then, for any nonempty $Y \in F$ and any continuous $f : Y \to X$, which

is a strong α_K-condensing mapping, such that f is retractible onto Y by $\rho : X \to Y$, and ρ is a strong $(\alpha_K, 1)$-contraction mapping, we have $F_f \neq \emptyset$.

Proof. Let $S(X) := \{Y \in P_{cp}(X)/f \in C(Y) \Rightarrow F_f \neq \emptyset\}$ and $M = C$. Thus $(X, S(X), M)$ is a fixed point structure. We consider the Kuratowski's measure of noncompactness $\theta = \alpha_K : P_b(X) \to \mathbf{R}_+$, and $\eta : P(X) \to P(X)$ a closure operator, $\eta(A) = \overline{co}_F(A), \forall A \in P(X)$. Let $S_1(X) = F$ from hypothesis.

$Y \in F = S_1(X)$, $\rho \circ f \in M(Y)$. From theorem 3.2 we have $F_f \neq \emptyset$ and if $F_f \in Z$, thus $\alpha_K(F_f) = 0$.

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