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ON SOME OPEN PROBLEMS OF RADU

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Abstract. Answering two recent open problems of Radu, we give a class of t-norms for which a general contraction principle for probabilistic contractions holds.
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1. Introduction

In *Chapter 20* of [9], *V. Radu* presented three open problems in fixed point theory of probabilistic metric spaces. Let us introduce them.

Definition 1.1. ([11]) Let X be a nonempty set and F probabilistic distance on X. We say that the mapping $f : X \to X$ is a *probabilistic contraction* (or *B-contraction)* if there exists $k \in (0,1)$ such that

$$F_{f(x)f(y)}(kt) \geq F_{xy}(t), \forall x, y \in X, \forall t > 0.$$

Definition 1.2. ([8]) Let X be a nonempty set and F probabilistic distance on X. We say that the mapping $f : X \to X$ is a *probabilistic strict contraction* (or *strict B-contraction)* if there exists $k \in (0,1)$ such that

$$F_{f(x)f(y)}(kt) \geq \frac{F_{xy}(t)}{F_{xy}(t) + k(1 - F_{xy}(t))}, \forall x, y \in X, \forall t > 0.$$

Theorem 1.3. ([8]) *Let (X, F, T) be a complete generalized Menger space with $T \geq T_L$ and $f : X \to X$ be a strict B-contraction such that $F_{xf(x)}(u) > 0$ for some $x \in X$ and $u > 0$. Then f has a fixed point.*

The mappings from the above definitions satisfy contraction relations of the form

$$(PC\alpha\beta) : F_{f(x)f(y)}(\alpha(t)) \geq \beta(F_{xy}(t)), \forall x, y \in X, \forall t \geq 0$$

where $\alpha : [0,\infty) \to [0,\infty)$ and $\beta : [0,1] \to [0,1]$ have the properties:

$\alpha 1)$ α is increasing

$\alpha 2)$ $\alpha(t) \leq t, \forall t \geq 0$

$\alpha 3)$ $\lim_{n\to\infty} \alpha^n(t) = 0, \forall t \geq 0$

$\beta 1)$ β is increasng

$\beta 2)$ $\beta(u) \geq u, \forall u \in [0,1]$

$\gamma)$ the mapping $\gamma : [0,1] \to [0,1]$, $\gamma(s) := \max\{\alpha(s), 1-\beta(1-s)\}$ is continuous and verifies $\sum_n \lambda^n(s) < \infty$, $\forall s < 1$.

Theorem 2.3 leads to the following general contraction principle:

Theorem 1.4. ([9]) (*Contraction Principle) Let* (X, F, T) *be a complete generalized Menger space with* $T(a,b) \geq T_L(a,b) := Max(a+b-1, 0)$ *and consider (for* α *and* β *as above) a* $(PC\alpha\beta)$*-contraction* $A : X \to X$ *such that* $F_{xAx}(u) > 0$ *for some* $x \in X$ *and some* $u > 0$. *Then* A *has a fixed point.*

In this context, the following questions (Radu [9]) are quite natural:

Problem 14. Determine the class of triangular norms for which the Theorem 2.3. is still true.

Problem 15. Determine the class of triangular norms T *for which the Contraction Principle 2.4. holds on every complete generalized Menger space* (X, F, T).

Problem 16. Find appropriate conditions on α, β *and* γ *in general case, when the t-norm* T *is replaced by an arbitrary Archimedean one.*

In this paper we answer the first two of these questions.

2. Preliminaries

In this section we recall some classical notions from the probabilistic metric spaces theory. For more details concerning this problematic we refer the reader to the books [2], [10].

Definition 2.1. ([10]) A mapping $T : [0,1] \times [0,1] \to [0,1] = I$ is called a *t-norm (*shortly *t-norm)* if it satisfies the following conditions:

$N1)$ $T(a,b) = T(b,a) \quad \forall a, b \in I$

$N2)$ $a \leq c, b \leq d \Rightarrow T(a,b) \leq T(c,d)$

$N3)$ $T(a,1) = a, \ \forall a \in I.$

$N4)$ $T(a, T(b,c)) = T(T(a,b), c) \quad \forall a, b, c \in I.$

Among the important examples of t-norms we mention $T_L : I \times I \to I$, $T_L(a,b) = Max\{a+b-1, 0\}$ (*Lukasiewicz* t-norm), $T_P(a,b) = ab$ and $T_M(a,b) = Min\{a,b\}$.

Definition 2.2. ([1], [2]) We say that the t-norm T is *of Hadžić-type* and we write $T \in \mathcal{H}$ if the family $\{T^n\}_{n\in N}$ of its iterates defined, for each x in $[0,1]$, by

$$T^0(x) = 1 \text{ and } T^{n+1}(x) = T(T^n(x), x), \forall n \geq 0$$

is equicontinuous at $x = 1$, that is,

$$\forall \varepsilon \in (0,1) \exists \delta \in (0,1) \text{ such that } x > 1-\delta \Longrightarrow T^n(x) > 1-\varepsilon,\ \forall n \geq 1.$$

If T is a t-norm and $(x_n)_{n\geq 1}$ is a given sequence of numbers in $[0,1]$, one can define recurrently $\mathbf{T}_{i=1}^n x_i$ by $\mathbf{T}_{i=1}^1 x_i = x_1$and $\mathbf{T}_{i=1}^n x_i = T(\mathbf{T}_{i=1}^{n-1} x_{i.}, x_n)$ $\forall n \geq 2$.

$T_{i=1}^\infty\ x_i$ is defined ([2]) by $\lim_{n\to\infty} T_{i=1}^n\ x_i$ and $T_{i=n}^\infty x_i$ by $T_{i=1}^\infty\ x_{n+i}$. More about this important notion of countable extension of t-norms and many examples of such t-norms can be found in [2].

Definition 2.3. ([10]) The class of (*generalized) distribution functions,* denoted by Δ_+, is the class of all functions $F : [0,\infty) \to [0,1]$ with the properties:

a) $F(0) = 0$;

b) F is nondecreasing;

c) F is left continuous on $(0,\infty)$.

D_+ is the subset of Δ_+ containing the functions F which also satisfy the condition $\lim\limits_{x\to\infty} F(x) = 1$.

A special element of D_+ is the function ε_0, defined by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } \ t = 0 \\ 1, & \text{if } \ t > 0 \end{cases}.$$

If X is a nonempty set, a mapping $F : X \times X \longrightarrow \Delta_+$ is called a *probabilistic distance* and $F(x,y)$ is usually denoted by F_{xy}.

Definition 2.4. ([3], [10]) If X is a nonempty set, F is a probabilistic distance and T is a t-norm, the triple (X, F, T) is called *a generalized Menger* *space* if the following axioms are satisfied:

$$(PM0) \quad : \quad F_{xy} = \varepsilon_0 \text{ iff } x = y$$

$$(PM1) \quad : \quad F_{xy} = F_{yx},\ \forall x, y \in X$$

$$(PM2_M) \quad : \quad F_{xy}(t+s) \geq T(F_{xz}(t), F_{zy}(s)),\ \forall x, y, z \in X, \forall t, s > 0.$$

Proposition 2.5. ([10]) Let (X, F, T) be a generalized Menger space. If $\sup_{a<1} T(a,a) = 1$ then the family $\mathcal{U}_F := \{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1)}$ where

$$U_{\varepsilon,\lambda} = \{(x,y) \in X \times X : F_{xy}(\varepsilon) > 1 - \lambda\}$$

is a base for a metrizable uniformity on X, called the *F-uniformity.*

The F-uniformity naturally determines a metrizable topology on X, called *the* (ε-λ) *topology* or *the F-topology :* a subset O *of* X *is F-open* iff for every $p \in O$ there exists $\varepsilon > 0, \lambda \in (0,1)$ such that $N_p(\varepsilon, \lambda) = \{q \in X : F_{pq}(\varepsilon) > 1 - \lambda\} \subset O$.

In the following all topological notions refer to the F-topology.

3. Main results

Answer to Problem 14.

We will show that for the class of triangular norms T *with* T *continuous in* $(a, 1)$ for every $a \in (0, 1)$, Theorem 1.3. remains true.

Theorem 3.1. *Every strict B-contraction in a complete generalized Menger space* (X, F, T) *with* T *continuous in* $(a, 1)$ for every $a \in (0, 1)$ *has a fixed point iff* $F_{xf(x)}(t) > 0$ *for some* $x \in X$ *and some* $t > 0$.

Actually, we will consider a more general class of contractions, introduced by us in [4] and *Theorem 3.1.* will be obtained as a corollary of *Theorem 3.5.* below.

In the following by Φ we will denote the class of all mappings $\varphi : (0,1) \to (0,1)$ with the properties:

i) φ is an increasing bijection;

ii) $\varphi(\lambda) < \lambda\ \forall \lambda \in (0,1)$.

Obviously, every such a comparison mapping is continuous and if $\varphi \in \Phi$ then $\lim_{n\to\infty} \varphi^n(\lambda) = 0\ \forall \lambda \in (0,1)$.

Definition 3.2. ([6]) Let X be a nonempty set and F be a probabilistic distance on X. Let also $\varphi \in \Phi$ and $k \in (0,1)$ be given. A mapping f :

$X \to X$ is called a (φ-k)-B *contraction* if the following contractivity condition holds:

$$x, y \in X, \varepsilon > 0, \lambda \in (0,1), F_{xy}(\varepsilon) > 1 - \lambda \Longrightarrow F_{f(x)f(y)}(k\varepsilon) > 1 - \varphi(\lambda).$$

Proposition 3.3. ([6]) *For a given $k \in (0,1)$, the mapping φ defined on $(0,1)$ by $\varphi(\lambda) = \frac{k\lambda}{1-\lambda+k\lambda}$ is in the class Φ and every strict k-B contraction is a $(\varphi$-$k)$-B contraction.*

In the proof of *Theorem 3.5.,* which is a slight modification of our proof [6, Theorem 3.11.] we will use the following

Lemma 3.4. *Let (X, U) be a separated sequentially complete uniform space and B be a base for the uniformity $\mathcal{U}$. If $f : X \to X$ is a mapping with the property that for every $U \in B$ there exists $K \in B$ such that*

$$(x, y) \in U \circ K \Rightarrow (fx, fy) \in U$$

and there exists $x \in X$ such that

$$\forall U \in \mathcal{B}\ \exists n = n(U, x) \in \mathbf{N}\ :\ (f^n(x), f^{n+1}(x)) \in U$$

then f has a fixed point.

For the proof of the lemma see e.g. [7].

A mapping with the contractivity condition from the above lemma is called *$\mathcal{B}$-contraction.*

Theorem 3.5. *Let (S, F, T) be a sequentially complete generalized Menger space with T continuous in $(a, 1)$* for every $a \in (0,1)$ *and $f : S \to S$ be a $(\varphi$-$k)$-B contraction. If there exist $p \in S$ and $\delta > 0$ such that $F_{pf(p)}(\delta) > 0$, then f has a fixed point.*

Proof. Let (S, F, T), $f : S \to S$, $p \in S$ *and* $\delta > 0$ be as in the statement of the theorem.

We will show that all the conditions of *Lemma 3.4.* are fulfilled.

a) f is a $\mathcal{U}_F$-contraction

Indeed, let $U = U_{\varepsilon,\lambda} \in \mathcal{U}_F$ be given. Then there exists $\varepsilon_1 > 0$ such that $\varepsilon + \varepsilon_1 = \frac{\varepsilon}{k}$.

Since $1 - \varphi^{-1}(\lambda) < 1 - \lambda = T(1 - \lambda, 1)$, by the continuity of T in $(1 - \lambda, 1)$ we deduce that there exists $\lambda_1 \in (0,1)$ such that:

$$T(1 - \lambda, 1 - \lambda_1) > 1 - \varphi^{-1}(\lambda)$$

where φ^{-1} is the inverse of φ.

If we consider the set $K = U_{\varepsilon_1,\lambda_1}$ then the following implications hold: $(p,q) \in U \circ K \Rightarrow \exists r \in X : (p,r) \in U,\ (r,q) \in K \Rightarrow F_{pr}(\varepsilon) > 1-\lambda$, $F_{rq}(\varepsilon_1) > 1-\lambda_1 \Rightarrow$

$F_{pq}(\varepsilon+\varepsilon_1) \geq T(1-\lambda, 1-\lambda_1) > 1-\varphi^{-1}(\lambda) \Rightarrow F_{pq}(\frac{\varepsilon}{k}) > 1-\varphi^{-1}(\lambda)$.

From the definition of $(\varphi$-$k)$-B contraction we deduce that $F_{f(p)f(q)}(\varepsilon) > 1-\lambda$, that is $(p,q) \in U \circ K \Rightarrow (f(p), f(q)) \in U$.

b) $\forall U \in \mathcal{U}_F\ \exists n \in N :\ (f^n(p), f^{n+1}(p)) \in U$.

Indeed, let $U = U_{\varepsilon,\lambda} \in \mathcal{U}_M$ be given. Since $F_{pf(p)}(\delta) > 0$, we can find $\delta_1 > 0$ such that $F_{pf(p)}(\delta) > 1-\delta_1$ and then $F_{f^r(p)f^{r+1}(p)}(k^r\delta) > 1-\varphi^r(\delta_1)$ for all $r \in N$. By choosing n such that $k^n\delta < \varepsilon$ and $\varphi^n(\delta_1) < \lambda$ we obtain $F_{f^r(p)f^{r+1}(p)}(\varepsilon) > 1-\lambda$, i.e. $(f^n(p), f^{n+1}(p)) \in U$.

The theorem (and *Theorem 3.1.* as well) is proved.

In order to answer to the other question, we begin by recalling some results from [5].

Definition 3.6. ([5]) Let φ be a mapping from $(0,1)$ to $(0,1)$ and T be a t-norm. We say that T *is φ-convergent* if

$$\forall\delta \in (0,1)\forall\lambda \in (0,1)\exists s\ (=s(\delta,\lambda)\ ) \in N : T_{i=1}^{n}(1-\varphi^{s+i}(\delta)) > 1-\lambda, \forall n \geq 1.$$

Proposition 3.7. *If $lim_{n\to\infty} T_{i=1}^{\infty}(1-\varphi^{n+i}(\delta)) = 1\ \forall\delta \in (0,1)$ then T is φ-convergent.*

Proof. If $\delta \in (0,1)$, $\lambda \in (0,1)$ are given, from $\lim_{n\to\infty} T_{i=1}^{\infty}(1-\varphi^{n+i}(\delta)) = 1$ it follows that there exists $s_0 \in N$ such that

$$T_{i=1}^{\infty}(1-\varphi^{s+i}(\delta)) > 1-\lambda, \forall s \geq s_0.$$

Since the sequence $(\mathbf{T}_{i=1}^{n}(1-\varphi^{n+i}(\delta)))_{n\geq 1}$ is nonincreasing, we have $T_{i=1}^{n}(1-\varphi^{s+i}(\delta)) \geq T_{i=1}^{\infty}(1-\varphi^{s+i}(\delta)) > 1-\lambda\ \forall s \geq s_0, \forall n \geq 1$, wherefrom it follows that T is φ-convergent.

In [2, page 39] it is proved that for $T \geq T_L$ the following implication holds:

$$\lim\ _{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

Using this result we can obtain the following

Example 3.8. Let $\varphi(t) = kt$ for all t in $(0,1)$. Then the t-norm T_L is φ-convergent.

Definition 3.9. ([4, Definition 2.1]) Let Φ be the class of all mappings $\varphi : (0,1) \to (0,1)$. If X is a nonempty set, F is a probabilistic distance on X and $\varphi \in \Phi$, we say that the self-mapping f of X is a φ-H *contraction* if

$$(\varphi\text{-}H): \; x, y \in X, \; t \in (0,1), \; F_{xy}(t) > 1 - t \Longrightarrow \; F_{f(x)f(y)}(\varphi t) > 1 - \varphi(t).$$

Theorem 3.10. *([5, Theorem 2.3. and Remark 2.3.]) Let (X, F, T) be a complete generalized Menger space and $\varphi \in \Phi$ be such that $\varphi(t) < t \; \forall t \in (0,1)$ and the series $\sum_1^\infty \varphi^n(\lambda)$ is convergent for every $\lambda \in (0,1)$. If T is φ-convergent then every φ-H contraction f on X with the property $F_{xf(x)}(1) > 0$ for some $x \in X$ has a fixed point.*

An answer to Problem 15.

Theorem 3.11. *Let (X, F, T) be a complete generalized Menger space with $\sup\limits_{a<1} T(a,a) = 1$ and $f : X \to X$ be a mapping with the property*

$$F_{f(x)f(y)}(\alpha(t)) \geq \beta(F_{xy}(t)), \; \forall x, y \in X, \forall t > 0$$

where $\alpha : [0,\infty) \to [0,\infty)$ is increasing, $\alpha(s) < s \; \forall s \in (0,1)$, $\beta : [0,1] \to [0,1]$ is increasing, $\beta(u) > u, \forall u \in (0,1)$ and the mapping $\varphi : [0,1] \to [0,1]$, $\varphi(s) := \max\{\alpha(s), 1 - \beta(1-s)\}$ satisfies $\sum_n \varphi^n(s) < \infty$, $\forall s < 1$. If T is φ-convergent and $F_{xf(x)}(1) > 0$ for some $x \in X$, then f has a fixed point.

Proof. We will show that the contractions considered in *Problem 15.* are actually *φ-H contractions* and all the other conditions of *Theorem 3.10.* are satisfied.

We have: $F_{xy}(t) > 1 - t \Longrightarrow \beta(F_{xy}(t)) > \beta(1-t) \Longrightarrow F_{f(x)f(y)}(\alpha(t)) > \beta(1-t)$. Since $\varphi(s) \geq \alpha(s)$ and $\beta(1-s) \geq 1 - \varphi(s)$ for all $s \in (0,1)$, we deduce that $F_{f(x)f(y)}(\varphi(t)) > 1 - \varphi(t)$. Therefore, $F_{xy}(t) > 1 - t \Longrightarrow F_{f(x)f(y)}(\varphi(t)) > 1 - \varphi(t)$.

Moreover, from $\alpha(s) < s$ and $\beta(s) > s$ for all $s \in (0,1)$ it follows $\varphi(s) < s, \forall s \in (0,1)$.

Remark 3.12. In the proof of *Theorem 3.10.* from [5] the condition $\varphi(t) < t \; \forall t \in (0,1)$ is used only for the proof of continuity of f. As a matter of fact, this condition can be replaced with $\lim_{t \searrow 0} \varphi(t) = 0$ so, in the statement of *Theorem 3.11.* we can replace the condition: "$\alpha(s) < s \; \forall s \in (0,1)$ and $\beta(u) > u, \forall u \in (0,1)$ " with "$\lim_{t \searrow 0} \alpha(t) = 0$ and $\lim_{t \nearrow 1} \beta(t) = 1$".

In the following we will show that there exist t-norms T, different from T_L, and mappings φ such that $\varphi(s) < s$, $\sum_n \varphi^n(s) < \infty$, $\forall s < 1$ and T is φ-convergent.

Lemma 3.13. ([2, Prop. 1.70.]) *Let* $(x_n)_{n\in N}$ *be a sequence in* $[0,1]$ *such that* $\lim_{n\to\infty} x_n = 1$. *If* T *is a t-norm of Hadžić-type, then* $lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1$.

From the above lemma and *Proposition 3.7.* it follows that if $\varphi(t) = kt, \forall t \in (0,1)$ (for a given $k \in (0,1)$) *and* $T \in \mathcal{H}$ *then* $\sum_1^\infty \varphi^n(t)$ *is convergent for every* $t \in (0,1)$ *and* T *is* φ*-convergent.*

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