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TOWARDS COMPUTING LIFSHITS CONSTANT FOR HYPERSPACES

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Abstract. We show that the Lifshits constant for the hyperspace of compact convex subsets of the unit interval is equal to 1. Moreover we point out why is it also equal 1 for some other hyperspaces. This partially answers the question raised by J. Andres in the context of fractals for iterated function systems which are Lipschitz but noncontractive (see [1], comp. also [2], [3]).
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1. Lifshits constant

Let (X,d) be a complete metric space (with metric d). By $D(x,r)$ we denote the closed r-ball around x i.e.

$$D(x,r)=\{\, z\in X\,:\, d(z,x)\leqslant r\,\}.$$

We say that balls are *c-regular* ($c\geqslant 1$), iff for every $k<c$ there exist $\eta,\alpha\in(0,1)$ s.t. for any $x,y\in X$ and $r>0$ with $d(x,y)\geqslant(1-\eta)\,r$ the intersection $D(x;\,(1+\eta)\,r)\cap D(y;\,k(1+\eta)\,r)$ is contained in some closed ball with radius αr. Putting

$$\varkappa(X)=\sup\{\, c\geqslant 1\,:\,\text{balls are } c-\text{regular}\,\}.$$

defines the *Lifshits characteristic* of X. In the case of Banach space it is enough to consider the closed unit ball at 0. This constant is mainly connected with a generalization of the Banach Principle due to E.A. Lifshits, which allows one to cross-out the contractivity barrier (for the proof see [7], chapt.16).

Theorem 1 (Lifshits,1975)**.** *Let (X,d) be a bounded complete metric space and let $f : X \to X$ be a uniformly Lipschitz map:*

$$d[\, f^n(x), f^n(y)\,] \leqslant k\, d(x,y) \;\; \forall_{x,y \in X} \, \forall_{n \in \mathbb{N}},$$

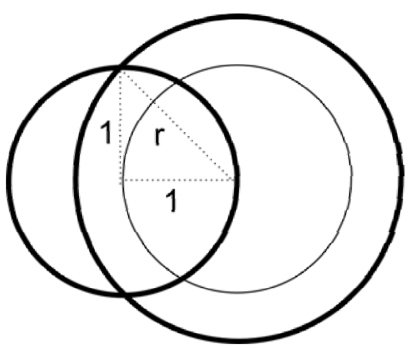
where $k < \varkappa(X)$ and f^n designates n-fold composition. Then f has a fixed point.

It also explains why nonexpansive selfmaps of bounded closed convex sets in a Hilbert space possess fixed points (comp. [6], chapt.I, par.2.1). The multivalued version of the Banach Principle was proved by S. Nadler (see [7], chapt.15). Until now however there exists (as far as is known to the author) only partial generalization of Lifshits' theorem for multifunctions given in [1] and [3]. The full and appropriate formulation in the multivalued case is still an open problem.

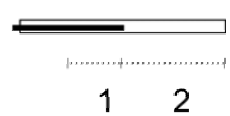
The following facts are known

a) $\varkappa(H) = \sqrt{2}$ for any Hilbert space H with $\dim H \geqslant 2$,
b) $\varkappa(\mathbb{R}) = 2$ for the real line $\mathbb{R}$,
c) $\varkappa(E) = 1$ for $E = \mathbb{R}^n$ furnished with either ℓ^1- or ℓ^∞-norm (i.e. max-norm) and $\dim E = n \geqslant 2$.

We illustrate a), b) and c) below in the planar situation (for unit ball).



$r = \sqrt{2}$ $\qquad$ ℓ^2



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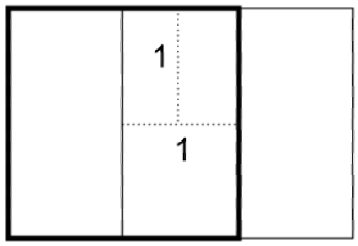
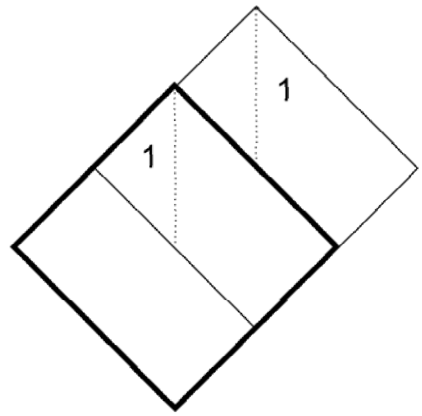


ℓ^∞



ℓ^1

2. Hyperspace of the unit interval and the plane

Denote by $\mathcal{C}(X)$ the hyperspace of nonempty compact connected subsets (*continua*) of the metric space (X, d), respectively by $\mathcal{K}(X)$ the hyperspace of all nonempty compact subsets. Let $x \in X$, A, $B \subset X$. Recall standard notions

— *distance* of point x to set $B \subset X$: $d(x, B) = \inf_{b \in B} d(x, b)$,

— *excess* of set A over B: $e(A, B) = \sup_{a \in A} d(a, B)$,

— *Hausdorff semimetric*: $h(A, B) = \max\{ e(A, B), e(B, A)\}$.

We remark that the Hausdorff distance is a usual metric on the family of nonempty closed bounded subsets (in particular on $\mathcal{C}(X)$). For the theory of hyperspaces we refer to [5] and [8] (the second monograph contains a lot of geometrical models for hyperspaces). There is also an interesting paper [4] on metric structure of hyperspaces.

Firstly we shall restrict ourselves to the standardly metrized unit interval $I = [0, 1]$ in place of a general metric space X. Thus $\mathcal{C}(I)$ consists of closed intervals and points („degenerated intervals”), or equally of compact convex subsets.

Let $A = [a_1, a_2]$, $B = [b_1, b_2]$ be in $\mathcal{C}(I)$. Then one easily sees the following formula

$$h[A, B] = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

The hyperspace $\mathcal{C}(I)$ can be described as symmetric product $I\#I = I^2/\sim$, where $(a_1, a_2) \sim (a_1', a_2') \Leftrightarrow \{a_1, a_2\} = \{a_1', a_2'\}$. The above formula leads to a metric realization of this product (comp. Exerc. 2.16 p.15 and Example 5.1 p.33 in [8]):

Proposition 1. *The hyperspace $(\mathcal{C}(I), h)$ is isometric to the triangle $\Delta = \{ (a_1, a_2) \in I^2 \,/\, a_1 \leqslant a_2 \}$ equipped with metric induced by max-norm.*

Hence we infer

Theorem 2. *The Lifshits characteristic $\varkappa(\mathcal{C}(I))$ for the hyperspace of subcontinua in the unit interval I is equal to 1.*

One may expect that for other hyperspaces the situation is the same, although it seems that nobody has proved this in a rigorous way. This was suggested by K. Goebel and supports some of expectations from [9] that Lifshits-like theorem can be useless for the theory of iterated function systems and fractals.

Consider the hyperspace $\mathcal{K}(\mathbb{R}^2)$ of compacta in the standardly metrized plane $\mathbb{R}^2$ (i.e. with ℓ^2-norm). Denote by $\mathcal{D}^{(h)}(A, r) = \{ B \in \mathcal{K}(\mathbb{R}^2) \,/\, h[B, A] \leqslant r \}$ the closed r-ball in this hyperspace. It is easy to observe that for the unit sphere $S = \{ x \,/\, \|x\| = 1 \} \in \mathcal{K}(\mathbb{R}^2)$ and the unit ball $D = D(0,1) \in \mathcal{K}(\mathbb{R}^2)$ we have $\mathcal{D}^{(h)}(D, 1) \subset \mathcal{D}^{(h)}(S, 1)$, so the intersection cannot be small, namely $\mathcal{D}^{(h)}(D, 1) \cap \mathcal{D}^{(h)}(S, 1) = \mathcal{D}^{(h)}(D, 1)$. Therefore

Theorem 3. *The Lifshits characteristic $\varkappa(\mathcal{K}(\mathbb{R}^2))$ for the hyperspace of compact in the plane $\mathbb{R}^2$ is equal to 1.*

The same holds true for the hyperspace of any Euclidean space $\mathbb{R}^n$ i.e. $\varkappa(\mathcal{K}(\mathbb{R}^n)) = 1$.

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