Fixed Point Theory, Volume 4, No. 2, 2003, 143-158
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

SOME COMMON FIXED POINT THEOREMS FOR SEQUENCES OF NONSELF MULTIVALUED OPERATORS IN METRICALLY CONVEX METRIC SPACES

B. C. DHAGE[1], U. P. DOLHARE[2] AND ADRIAN PETRUŞEL[3]

[1]Kasubai, Gurukul Colony, Ahmedpur-413 515
Dist: Latur, Maharashtra, India
e-mail: bcd20012001yahoo.co.in

[2]Department of Mathematics
Dnyanopasak Mahavidyalaya, Jintur-431 509
Maharashtra, India

[3]Department of Applied Mathematics
Babeş-Bolyai University
3400, Cluj-Napoca, Romania
e-mail: petrusel@math.ubbcluj.ro

Abstract. In this paper some common fixed point theorems for sequences of nonself multivalued operators defined on a closed subset of a metrically convex metric space are proved. Our results extend some fixed point theorems of Dhage [4] to a sequence of nonself multimaps and include the fixed point result of Huang and Cho [6].

2000 Mathematics Subject Classification: 47H10, 54H25.

Key Words and Phrases: metrically convex metric space, multivalued operator, common fixed point.

1. Introduction

Fixed point theorems for nonself contraction multifunctions have been discussed in the literature, among many others, by Assad [1], Assad and Kirk [2], Ćirić and Ume [3]. Itoh [7] extended these results to a more general class of contraction multifunctions while Rhoades obtained a generalization of Itoh's fixed point theorem (see [7]) for the case of a multivalued operator F defined on a subset K of a metrically convex metric space X. Common fixed point theorems for a sequences $\{F_n\}$ of non-self multivalued operators in metrically convex metric space have been also proved by Huang and Cho [6]. All these results use a kind of boundary condition with respect to the multivalued operator F and the subset K of the metric space X, namely $F(\partial K) \subset K$, where ∂K denotes the boundary of K. In a recent paper [4], one of the present authors proved some fixed point theorems for the non-self multivalued operators on a metrically convex metric space, satisfying slightly stronger condition than Rhoades [8], but under a weaker boundary condition than that in the above mentioned papers.

The purpose of the present paper is to prove some common fixed point theorems for a sequence of non-self multivalued operators on a metrically convex metric space satisfying certain contraction type conditions and under a weaker boundary condition. Our results extend some theorems of Dhage [4] (to a sequence of multivalued operators) and include the result of Huang and Cho [6] under a slightly stronger contraction condition.

2. Main Results

Let (X, d) denote a metric space and let $CB(X)$ denote the class of all non-empty closed and bounded subsets of X.

Definition 2.1. *A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$, there is a $z \in X$, $x \neq z$, $y \neq z$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

We need the following lemma in the sequel.

Lemma 2.1. (Assad and Kirk [2]) *If K is a non-empty closed convex subset of a complete and metrically convex metric space (X, d), then for any $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

For any $A, B \in CB(X)$ denote:

$$\begin{aligned} D(A, B) &= \inf\{d(a, d) \mid a \in A, b \in B\}, \\ \delta(A, B) &= \sup\{d(a, b) \mid a \in A, b \in B\} \end{aligned}$$

and

$$H(A,B) = \max\Big\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(A,b)\Big\}.$$

The following properties of the functional δ are well-known (see for example Fisher [5] and Petruşel [11]) :

(i) $\delta(A,B) = 0$ if and only if $A = B = \{x^*\}$

(ii) $\delta(A,B) = \delta(B,A)$ and

(iii) $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$

for $A, B, C \in CB(X)$.

We need the following lemma in the sequel.

Lemma 2.2. Fisher [5] *Let $\{A_n\}$ and $\{B_n\}$ be two sequences in $CB(X)$ converging in $CB(X)$ to the sets A and respectively B. Then*

$$\lim_{n\to\infty} \delta(A_n, B_n) = \delta(A,B).$$

If $T : X \to CB(X)$ is a multivalued operator, then $F_T := \{x \in X |\, x \in T(x)\}$ denotes the fixed point set T, while $(SF)_T := \{x \in X |\ \{x\} = T(x)\}$ is the strict fixed point set of T.

Now we prove our first main result.

Theorem 2.1. *Let (X,d) be a complete and metrically convex metric space, K a non-empty, closed, convex and bounded subset of X. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of multivalued operators of K into $CB(X)$ satisfying for $i \neq j$,*

$$\begin{aligned}\delta(F_i(x), F_j(y)) \quad &\leq \quad \alpha \max\{d(x,y), D(x,F_i(x)), D(y,F_j(y))\} \\ &\qquad +\beta[D(x,F_j(y) + D(y,F_i(x))]\end{aligned} \tag{2.1}$$

for all $x, y \in K$, where $\alpha \geq 0$, $\beta \geq 0$ and $2\alpha + 3\beta < 1$.

If $F_n(x) \cap K \neq \emptyset$ for each $x \in \partial K$ and each $n \in N$, then $F_{F_n} = (SF)_{F_n} = \{z\}$, for each $n \in N$. Moreover, for each $n \in \mathbb{N}$, F_n is continuous in z with respect to the Hausdorff-Pompeiu metric on X.

Proof. Let $x \in K$ be arbitrary and define a sequence $\{x_n\} \subset K$ as follows. Let $x_0 = x$ and take a point $x_1 \in F_1(x_0) \cap K$ if $F_1(x_0) \cap K \neq \emptyset$, otherwise choose a point $x_2 \in \partial K$ such that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1)$$

for some $y_1 \in Fx_0 \subset X \backslash K$.

Similarly choose a point $x_2 \in F_2(x_1) \cap K$ if $F_2(x_1) \cap K \neq 0$, otherwise choose a point $x_2 \in \partial K$, such that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2)$$

for some $y_2 \in F_2(x_1) \subset X \backslash K$.

Continuing in this way, choose $x_n \in F_n(x_{n-1}) \cap K$ if $F_n(x_{n-1}) \cap K \neq \emptyset$, otherwise select $x_n \in \partial K$ such that

$$d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n)$$

for some $y_n \in F_n(x_{n-1}) \subset X \backslash K$. Denote by

$$P = \{x_n \in \{x_n\} \mid x_n \in F_n(x_{n-1}), n \in N\}$$

and

$$Q = \{x_n \in \{x_n\} \mid x_n \in \partial K, x_n \in F_n(x_{n-1}), n \in N\}.$$

Clearly

$$\{x_n\} = P \cup Q \subset K.$$

Then for any two consecutive terms x_n, x_{n-1} of the sequence $\{x_n\}$, there are only the following three possibilities:

(i) $x_n, x_{n-1} \in P$,

(ii) $x_n \in P$, $x_{n+1} \in Q$,

(iii) $x_n \in Q$ and $x_{n+1} \in P$.

We will prove that $\{x_n\}$ is a Cauchy sequence in K.

Case I: Suppose that $x_n, x_{n+1} \in P$. Then by(2.1), we have,

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \delta(F_n(x_{n-1}), F_{n+1}(x_n)) \\
&\leq \alpha \max\{d(x_{n-1}, x_n), D(x_{n-1}, F_n(x_{n-1})), D(x_n, F_{n+1}(x_n))\} \\
&\quad +\beta[D(x_{n-1}, F_{n+1}(x_n)) + D(x_n, F_n(x_{n-1}))] \\
&\leq \alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&\quad +\beta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\
&\leq \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&\quad +\beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&= \max\{\alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \\
&\quad \alpha d(x_n, x_{n+1}) + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\} \\
&= \max\{(\alpha+\beta) d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}), \\
&\quad \beta d(x_{n-1}, x_n) + (\alpha+\beta) d(x_n, x_{n+1})\},
\end{aligned}$$

i.e.,

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n), \tag{2.2}$$

where

$$k = \max\left\{\frac{\alpha+\beta}{1-\beta}, \frac{\beta}{1-(\alpha+\beta)}\right\} < 1$$

since $2\alpha + 3\beta < 1$.

Case II: Let $x_n \in P$ and $x_{n+1} \in Q$. Then

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1})$$

for some $y_{n+1} \in F_{n+1}(x_n) \subset X \backslash K$.

Clearly

$$\begin{cases} d(x_n, x_{n+1}) \leq d(x_n, y_{n+1}), \\ d(x_n, y_{n+1)} \leq \delta(F_n(x_{n-1}), F_{n+1}(x_n)). \end{cases} \tag{2.3}$$

Now following the arguments similar to that in Case I,

$$d(x_n, y_{n+1}) \leq k d(x_{n-1}, x_n), \tag{2.4}$$

where again

$$k = \max\left\{\frac{\alpha+\beta}{1-\beta}, \frac{\beta}{1-(\alpha+\beta)}\right\} < 1.$$

From (2.3) and(2.4), it follows that

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n).$$

Case III: Suppose that $x_n \in Q$ and $x_{n+1} \in P$. We note that then $x_{n-1} \in P$. By definition of $\{x_n\}$, there is a point $y_n \in F_n(x_{n-1})$ such that

$$d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n). \tag{2.5}$$

We have successively:
$d(x_n, x_{n+1}) \leq d(x_n, y_n) + d(y_n, x_{n+1}) \leq$
$\leq d(x_n, y_n) + \delta(F_n(x_{n-1}), F_{n+1}(x_n)) \leq d(x_n, y_n)+$
$+\alpha \max\{d(x_{n-1}, x_n), D(x_{n-1}, F_n(x_{n-1})), D(x_n, F_{n+1}(x_{n-1}))\}$
$+\beta[D(x_{n-1}, F_{n+1}(x_n) + D(x_n, F_n(x_{n-1}))] = d(x_n, y_n)+$
$+\alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, y_n), d(x_n, x_{n+1})\}+$
$+\beta[d(x_{n-1}, x_{n+1}) + d(x_n, y_n)] \leq d(x_n, y_n)+$
$+\alpha \max\{d(x_{n-1}, y_n), d(x_n, x_{n+1})\}+$
$+\beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, y_n)] = d(x_{n-1}, y_n)+$
$+\alpha \max\{d(x_{n-1}, y_n), d(x_n, x_{n+1})\} + \beta[d(x_{n-1}, y_n) + d(x_n, x_{n+1})]$.
From (2.4) of Case II applied to $n-1$, we have

$$d(x_{n-1}, y_n) \leq kd(x_{n-2}, x_{n-1}) \tag{2.6}$$

and hence

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_{n-2}, x_{n-1}) + \alpha \max\{kd(x_{n-2}, x_{n-1}), d(x_n, x_{n+1})\} \\ &\quad +\beta[kd(x_{n-2}, x_{n-1}) + d(x_n, x_{n+1})] \\ &= \max\{(1+\alpha+\beta)kd(x_{n-2}, x_{n-1}) + \beta d(x_n, x_{n+1}), \\ &\quad (1+\beta)kd(x_{n-2}, x_{n-1}) + (\alpha+\beta)d(x_n, x_{n+1})\}. \end{aligned}$$

This further implies that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \max\left\{\tfrac{(1+\alpha+\beta)k}{1-\beta}, \tfrac{(1+\beta)k}{1-(\alpha+\beta)}\right\} d(x_{n-2}, x_{n-1}) \\ &= qd(x_{n-2}, x_{n-1}), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} q &= \max\left\{\frac{(1+\alpha+\beta)k}{1-\beta}, \frac{(1+\beta)k}{1-(\alpha+\beta)}\right\} \\ &= k\max\left\{\frac{1+\alpha+\beta}{1-\beta}, \frac{1+\beta}{1-(\alpha+\beta)}\right\} \\ &= k\left(\frac{1+\beta}{1-(\alpha+\beta)}\right) < 1. \end{aligned}$$

Now for any $n \in N$, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq qd(x_{2n-2}, x_{2n-1}) \\ &\leq q^n d(x_0, x_1). \end{aligned}$$

Since n is arbitrary, one has

$$d(x_n.x_{n+1}) \leq q^n d(x_0, x_1).$$

Then for any positive integer p,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \sum_{i=1}^{n+p+1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=1}^{n+p+1} q^i d(x_0, x_1) \\ &= q^n \frac{(1-q^{n+p-1})}{1-q} d(x_0, x_1) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in K. Since K is closed, it is complete and there is a point $z \in K$ such that $\lim x_n = z$ exists. We show that z is a fixed point of F_n. Without loss of generality, we may assume that $x_{n+1} \in F_{n+1}(x_n)$ for some $n \in N$. Then,

$$\begin{aligned} \delta(z, F_j(z)) &\leq \delta(z, x_{n+1}) + \delta(x_{n+1}, F_j(z)) \\ &= \delta(z, x_{n+1}) + \delta(F_j(z), F_{n+1}(x_n)) \\ &= \delta(z, x_{n+1}) + \alpha \max\{d(z, x_n), D(z, F_j(z)), D(x_n, F_{n+1}(x_n))\} \\ &\quad +\beta[D(z, F_{n+1}(x_n)) + D(x_n, F_j(z))] \\ &\leq \delta(z, x_{n+1}) + \alpha \max\{d(z, x_n), \delta(z, F_j(z)), d(x_n, x_{n+1})\} \\ &\quad +\beta[d(z, x_{n+1}) + \delta(x_n, F_j(z))]. \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality yields that

$$\begin{aligned} \delta(z, F_j(z)) &\leq 0 + \alpha \max\{0, \delta(z, F_j(z), 0, \beta\delta(z, F_j(z))\} \\ &= \alpha\delta(z, F_j(z)), \end{aligned}$$

which implies that $\delta(z, F_j(z)) = 0$ since $\alpha < 1$, i.e., $F_j(z) = \{z\}$ for each $j \in N$.

To prove uniqueness, let z^* $(\neq z)$ be another common fixed point of $\{F_n\}$. Then by(2.1) we get

$$\begin{aligned} d(z, z^*) &\leq \delta(F_i(z), F_j(z^*)) \\ &\leq \alpha \max\{d(z, z^*), D(z, F_i(z), D(z^*, F_j(z^*))\} \\ &\quad +\beta[d(z, F_j(z^*)) + D(z^*, F_i(z))] \\ &= (\alpha + 2\beta)d(z, z^*), \end{aligned}$$

which is a contradiction since $\alpha + 2\beta \leq 2\alpha + 3\beta < 1$. Hence z = z^*.

Finally we prove the continuity of F_n for each $n \in N$. Let $\{z_n\}$ be any sequence in K converging to the unique common fixed point z of $\{F_n\}_{n=1}^{\infty}$. To conclude, it, is enough to prove that $\lim_n H(F_j(z_n), F_j z) = 1$ for each $i \in N$. We know that

$$(*) \qquad H(F_j(z_n), F_j(z)) \leq \delta(F_i(z_n), F_i(z)).$$

Now for any $i \neq j$,

$$\begin{aligned} \delta(F_i(z_n), F_i(z)) &= \delta(F_i(z_n), F_j(z)) \\ &\leq \alpha \max\{d(z_n, z), D(z_n, F_i(z_n), D(z, F_j(z))\} \\ &\quad +\beta[D(z_n, F_j(z)) + D(z, F_i(z_n))] \\ &\leq \alpha \max\{d(z_n, z), \delta(z_n, F_i(z_n)), 0\} \\ &\quad +\beta[\delta(z_n, F_j(z)) + \delta(z, F_i(z_n))]. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} \lim_n \delta(F_i(z_n), F_i(z)) &\leq \alpha \max\{0, \lim_n \delta(F_i(z_n), F_i(z)), 0\} \\ &\quad +\beta[0 + \lim_n \delta(F_i(z_n), F_i(z))] \\ &= (\alpha + \beta) \lim_n \delta(F_i(z_n), F_i(z)), \end{aligned}$$

which is possible only when $\lim_n \delta(F_i(z_n), F_i(z)) = 0$.

From $(*)$ it follows that $\lim\limits_{n\to\infty} H(F_i(z_n), F_i(z)) = 0$. This completes the proof. □

Remark 2.1. *With respect to condition (2.1), the following implications hold:*

i) (2.1) *and* $(x \in F_{F_i} \cap F_{F_j},\ i \neq j) \Rightarrow F_i(x) = F_j(x) = \{x\}$.

ii) (2.1) *and* $(x \in F_{F_i},\ y \in F_{F_j},\ i \neq j) \Rightarrow \delta(F_i(x), F_j(y)) \leq$
$\alpha \max\{d(x,y), D(x, F_i(x)), D(y, F_j(y))\} + \beta[D(x, F_j(y) + D(y, F_i(x)] \leq$
$\leq (\alpha + 2\beta) \cdot \delta(F_i(x), F_j(y))$.

Hence $\delta(F_i(x), F_j(y)) = 0$ *and so* $F_i(x) = F_i(y) = \{z\}$. *In conclusion* $z = x = y$.

iii) (2.1) *and* $(x \in F_{F_i}, y = x) \Rightarrow \delta(F_i(x), F_j(x)) \leq (\alpha + \beta) \cdot \delta(F_i(x), F_j(x))$.

Hence $\delta(F_i(x), F_j(x)) = 0$ *and so* $F_i(x) = F_j(x) = \{z\}$. *In conclusion* $z = x = y$.

Theorem 2.2. *Let* (X, d) *be a metrically convex complete space,* K *a non-empty, closed, convex and bounded subset of* X*. Let* $\{F_n\}_{n=1}^{\infty}$ *be a sequence of mappings from* K *into* $CB(X)$ *satisfying for* $i \neq j$,

$$\begin{aligned}\delta(F_i(x), F_j(y)) \quad \leq \quad & \alpha d(x,y) + \beta \max\Big\{\tfrac{1}{2}[D(x, F_i(x)) + D(y, F_j(y))], \\ & \tfrac{1}{2}[D(x, F_j(y)) + D(y, F_i(x))]\Big\}\end{aligned} \tag{2.8}$$

for all $x, y \in K$*, where* $\alpha \geq 0$ *and* $\beta \geq 0$ *satisfying* $\alpha^2 + \alpha + \alpha \cdot \beta + \frac{3\beta}{2} < 1$.

If $F_n(K) \cap K \neq \emptyset$ *for each* $x \in \partial K$ *and* $n \in N$*, then* $F_{F_n} = (SF)_{F_n} = \{x^*\}$*, for each* $n \in N$*. Moreover, for each* $n \in \mathbb{N}$*,* F_n *is continuous in* x^* *with respect to the Hausdorff-Pompeiu metric.*

Proof. Let $x \in K$ be arbitrary and define a sequence $\{x_n\} \subset K$ as in the previous proof. So $\{x_n\} = P \cup Q$, where

$$P = \{x_n \in \{x_n\} \mid x_n \in F_n(x_{n-1}), n \in N\}$$

and

$$Q = \{x_n \in \{x_n\} \mid x_n \in \partial K, x_n \in F_n(x_{n-1}), n \in N\}.$$

We show that $\{x_n\}$ is a Cauchy sequence. Now for any two consecutive terms $x_n, x_{n+1} \in \{x_n\}$, there are following three cases:

Case I: Suppose that $x_n, x_{n+1} \in P$. Then by (2.9) we get

$$\begin{aligned}
& d(x_n, x_{n+1}) \\
\leq\ & \delta(F_n(x_{n-1})F_{n+1}(x_n)) \\
\leq\ & \alpha d(x_{n-1}, x_n) + \beta \max\Big\{\tfrac{1}{2}[D(x_{n-1}, F_n(x_{n-1})) + D(x_n, F_{n+1}(x_n))], \\
& \tfrac{1}{2}[D(x_{n-1}, F_{n+1}(x_{n-1})) + D(x_n, F_n(x_n))]\Big\} \\
\leq\ & \alpha d(x_{n-1}, x_n) + \beta \max\Big\{\tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
& \tfrac{1}{2}\max[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\Big\} \\
\leq\ & \alpha d(x_{n-1}, x_n) + \beta \max\Big\{\tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \\
& \tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\Big\} \\
\leq\ & \alpha d(x_{n-1}, x_n) + \tfrac{\beta}{2} d(x_{n-1}, x_n) + \tfrac{\beta}{2} d(x_n, x_{n+1}) \\
=\ & \left(\frac{\alpha+\frac{\beta}{2}}{1-\frac{\beta}{2}}\right) d(x_{n-1}, x_n).
\end{aligned} \tag{2.9}$$

Case II: Suppose that $x_n \in P$ and $x_{n+1} \in Q$. Then there is a point $y_{n+1} \in F_{n+1}(x_n) \subset X \backslash K$ such that

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}),$$

which further implies that

$$d(x_n, y_{n+1}) \leq d(x_n, y_{n+1})$$

and

$$d(x_n, y_{n+1}) \leq \delta(F_n(x_{n-1}), F_{n+1}(x_n)). \tag{2.10}$$

Now

$$\begin{aligned}
& d(x_n, y_{n+1}) \leq \delta(F_n(x_{n-1}), F_{n+1}(x_n)) \leq \alpha d(x_{n-1}, x_n) + \\
& \beta \max\Big\{\tfrac{1}{2}[D(x_{n-1}, F_n(x_{n-1})) + D(x_n, F_{n+1}(x_n))], \tfrac{1}{2}[D(x_{n-1}, F_{n+1}(x_n)) + \\
& + D(x_n, F_n(x_{n-1}))]\Big\} \leq \alpha d(x_{n-1}, x_n) + \\
& + \beta \max\Big\{\tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \tfrac{1}{2}[d(x_{n-1}, y_{n+1}) + d(x_n, x_n)]\Big\} \\
& = \alpha d(x_{n-1}, x_n) + \beta \max\Big\{\tfrac{1}{2}[d(x_{n-1}, x_n) + \\
& + d(x_n, x_{n+1})], \tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, y_{n+1})]\Big\} = \alpha d(x_{n-1}, x_n) + \\
& + \tfrac{\beta}{2} d(x_{n-1}, x_n) + \tfrac{\beta}{2} d(x_n, y_{n+1}) = k \cdot d(x_{n-1}, x_n),
\end{aligned}$$

where

$$k = \frac{\alpha + \frac{\beta}{2}}{1 - \frac{\beta}{2}} < 1.$$

From (3) it follows that

$$d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n).$$

Case III: Suppose that $x_n \in Q$ and $x_{n+1} \in P$. Note that $x_{n-1} \in P$. Then there is a point $y_n \in F_n(x_{n-1}) \subset X\backslash K$ such that

$$d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n),$$

which further implies that

$$d(x_n, y_n) \le d(x_{n-1}, y_n). \tag{2.11}$$

Now

$$\begin{aligned}
& d(x_n, x_{n+1}) \\
\le\ & d(x_n, y_n) + d(y_n, x_{n+1}) \\
=\ & d(x_n, y_n) + \delta(F_n(x_{n-1}), F_{n+1}(x_n)) \\
\le\ & d(x_n, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\Big\{ \tfrac{1}{2}[D(x_{n-1}, F_n(x_{n-1})) \\
& +D(x_n, F_{n+1}(x_n))], \tfrac{1}{2}[D(x_{n-1}, F_{n+1}(x_n)) + D(x_n, F_n(x_{n-1}))] \Big\} \\
=\ & d(x_n, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\Big\{ \tfrac{1}{2}[d(x_{n-1}, y_n) + d(x_n, x_{n+1})], \\
& \tfrac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, y_n)] \Big\} \\
\le\ & d(x_{n-1}, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\Big\{ \tfrac{1}{2}[d(x_{n-1}, y_n) + d(x_n, x_{n+1})], \\
& \tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, y_n)] \Big\} \\
=\ & d(x_{n-1}, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\Big\{ \tfrac{1}{2}[d(x_{n-1}, y_n) + d(x_n, x_{n+1})], \\
& \tfrac{1}{2}[d(x_{n-1}, y_n) + d(x_n, x_{n+1})] \Big\} \\
=\ & d(x_{n-1}, y_n) + \alpha d(x_{n-1}, x_n) + \tfrac{\beta}{2} d(x_{n-1}, y_n) + \tfrac{\beta}{2} d(x_n, x_{n+1}) \\
=\ & kd(x_{n-2}, x_{n-1}) + k\alpha d(x_{n-2}, x_{n-1}) + \tfrac{\beta}{2} kd(x_{n-2}, x_{n-1}) \\
& +\tfrac{\beta}{2} d(x_n, x_{n+1}).
\end{aligned} \tag{2.12}$$

It follows that $d(x_n, x_{n+1}) \le \left(\frac{k + k\alpha + k\frac{\beta}{2}}{1 - \frac{\beta}{2}} \right) d(x_{n-2}, x_{n-1})$
$= k \left(\frac{1 + \alpha + \frac{\beta}{2}}{1 - \frac{\beta}{2}} \right) d(x_{n-2}, x_{n-1})$

$= qd(x_{n-2}, x_{n-1})$, where

$$q = k\left(\frac{1+\alpha+\frac{\beta}{2}}{1-\frac{\beta}{2}}\right) = \left(\frac{\alpha+\frac{\beta}{2}}{1-\frac{\beta}{2}}\right)\left(\frac{1+\alpha+\frac{\beta}{2}}{1-\frac{\beta}{2}}\right) < 1.$$

Thus in all three cases one has

$$d(x_n, x_{n+1}) \leq qd(x_{n-2}, x_{n+1}).$$

Therefore

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq qd(x_{2n-2}, x_{2n-1}) \\ &\leq q^n d(x_0, x_1) \end{aligned}$$

for all $n \in N$. Since n is arbitrary, we have

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1).$$

This shows that $\{x_n\}$ is a Cauchy sequence in K. As K is closed, it is complete and there is a point $z \in K$ such that $\lim_n x_n = z$.

We show that z is a fixed point of F_n. Without loss of generality, we may assume that $x_{n+1} \in F_{n+1}(x_n)$. Then

$$\begin{aligned} & \delta(z_n, F_j(z)) \\ \leq\ & \delta(z, x_{n+1}) + \delta(x_{n+1}, F_j(z)) \\ =\ & \delta(z, x_{n+1}) + \delta(F_{n+1}(x_n), F_j(z)) \\ \leq\ & d(z, x_{n+1}) + \alpha d(x_n, z) + \beta \max\Big\{\frac{1}{2}[D(x_n, F_{n+1}(x_n)) + D(z, F_j(z))], \\ & \frac{1}{2}[D(x_n, F_j(z)) + D(z, F_{n+1}(x_n))]\Big\} \\ =\ & d(z, x_{n+1}) + \alpha d(x_n, z) + \beta \max\Big\{\frac{1}{2}[d(x_n, x_{n+1}) + \delta(z, F_j(z))], \\ & \frac{1}{2}[\delta(x_n, F_j(z)) + d(z, x_{n+1}))]\Big\}. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} \delta(z, F_j(z)) &\leq 0 + \beta \max\Big\{\frac{1}{2}[0 + \delta(z, F_j(z))], \frac{1}{2}[\delta(z, F_j(z)) + 0]\Big\} \\ &= \frac{\beta}{2}\delta(z, F_j(z)), \end{aligned}$$

which is possible only when $F_j(z) = \{z\}$ for $j \in N$. Again the uniqueness of z follows from the condition (2.9). Finally we prove the continuity of F_n for each $n \in N$. Let $\{z_n\}$ be any sequence in K converging to the unique common fixed point z of $\{F_n\}_{n=1}^{\infty}$.

Now for any $i \neq j$,

$$\begin{aligned}
& \delta(F_j(z_n), F_i(z)) \\
= \ & \delta(F_i(z_n), F_j(z)) \\
\leq \ & \alpha d(z_n, z) + \beta \max\Big\{\frac{1}{2}[D(z_n, F_i(z_n)) + D(z, F_j(z))], \\
& \frac{1}{2}[D(z_n, F_j(z)) + D(z, F_i(z_n))]\Big\} \\
\leq \ & \alpha d(z_n, z) + \beta \max\Big\{\frac{1}{2}[\delta(z_n, F_i(z_n)) + 0], \frac{1}{2}[\delta(z_n, F_j(z) + \delta(z, F_i(z_n))]\Big\}.
\end{aligned}$$

Taking limit as $n \to \infty$,

$$\lim_n \delta(F_i(z_n), F_j(z)) \leq \frac{\beta}{2} \lim_n \delta(F_i(z_n), F_j(z)),$$

which is possible only when $\lim_n \delta(F_i(z_n), F_i(z)) = 0$. Since $H(F_i(z_n), F_i(z)) \leq \delta(F_i(z_n), F_i(z))$, we have $\lim_n H(F_i(z_n), F_i(z)) = 0$ and so F_i is continuous at z for each $i \in N$. This completes the proof. □

Now we will prove two results concerning the fixed point of sequence of non-self maps on the subsets of a metrically convex metric space satisfying a contraction condition more general than (2.1) and (2.9) and under certain compactness type condition.

Theorem 2.3. *Let (X, d) be a metrically convex metric space, K a compact convex subset of X. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of continuous multivalued operators from K into $CB(X)$ satisfying for $i \neq j$,*

$$\begin{aligned}
\delta(F_i(x), F_j(y)) \ < \ & \alpha \max\{d(x, y), D(x, F_i(x)), D(y, F_j(y))\} \\
& + \beta[D(x, F_j(y)) + D(y, F_i(x))]
\end{aligned} \tag{2.13}$$

for all $x, y \in K$ with right hand side not zero, where $\alpha > 0$, $\beta > 0$ and $2\alpha + 3\beta \leq 1$.

If $F_n(x_n) \cap K \neq \emptyset$ for each $x \in \partial K$ and $n \in N$, then $F_{F_n} = (SF)_{F_n} = \{x^*\}$, for each $n \in N$.

Proof. First we note that if the sequence $\{F_n\}$ of multivalued operators have a common fixed point, then from (2.14) it follows that the common fixed point is unique.

As F is continuous and K is compact so both sides of the inequality (2.14) are bounded on K. Now there are two cases:

Case I: Suppose that there exist some $x, y \in K$ such that the right hand side of (2.14) is zero. Then $z = x = y$ is a common fixed point of $\{F_n\}$ and so it is unique.

Case II: Now we assume that the right hand of the inequality (2.14) is positive on K. If $2\alpha + 3\beta < 1$, the desired conclusion follows from Theorem 2.1. Therefore we prove the result only in the case when $2\alpha + 3\beta = 1$.

Denote by

$$\begin{aligned} M(x,y) &= \alpha \max\{d(x,y), D(x,F_i(x)), D(y,F_j(y))\} \\ &+ \beta[D(x,F_j(y)) + D(y,F_i(x))]. \end{aligned}$$

Define a function $T : K \times K \to R^+$ by

$$T(x,y) = \frac{\delta(F_i(x), F_j(y))}{M(x,y)}, \quad x, y \in K. \tag{2.14}$$

Clearly the function T is well defined, since $M(x,y) \neq 0$ for all $x, y \in K$. Since each F_n is continuous, T is continuous and from compactness of K, it follows that T attains its maximum on $K \times K$ at some point say $(u,v) \in K^2$. Call the value c. From (2.14), we get $0 < c < 1$. By the definition of T we obtain

$$\frac{\delta(F_i(x), F_j(y))}{M(x,y)} \leq T(u,v) = c,$$

i.e.,

$$\begin{aligned} \delta(F_i(x), F_j(y)) &\leq cM(x,y) \\ &= \alpha' \max\{d(x,y), D(x,F_i(x)), D(y,F_j(y))\}, \\ &\quad +\beta'[D(x,F_j(y)) + D(y,F_i(x))] \end{aligned} \tag{2.15}$$

for all $x, y \in K$, where $\alpha' \geq 0$, $\beta' \geq 0$ satisfying $2\alpha' + 3\beta' = c(2\alpha + 3\beta) < 1$. Since K is compact, it is closed and bounded. Thus all the conditions of Theorem 2.1 are satisfied and hence an application of it yields the desired result. □

Theorem 2.4. *Let (X,d) be a metrically convex metric space, K a compact convex subset of X. Let $\{F_n\}$ be a sequence of continuous multivalued operator* *from K into $CB(X)$ satisfying for each $i \neq j$ the following condition:*

$$\begin{aligned}\delta(F_i(x), F_j(y)) \quad < \quad & \alpha d(x,y) + \beta \max\Big\{\tfrac{1}{2}[D(x, F_i(x) + D(y, F_j(y))], \\ & \tfrac{1}{2}[D(x, F_j(y) + D(y, F_i(x))]\Big\}\end{aligned} \tag{2.16}$$

for all $x, y \in K$ with right hand side not zero, where $\alpha > 0$, $\beta > 0$ and $2\alpha + \alpha\beta + \beta \leq 1$.

If $F_n(x) \cap K \neq \emptyset$ for each $x \in \partial K$ and $n \in N$, then $F_{F_n} = (SF)_{F_n} = \{x^*\}$, for each $n \in N$.

Proof. The proof is similar to the Theorem 2.3 with appropriate modifications. The result follows by an application of Theorem 2.2. The proof is complete. □

Theorem 2.5. *Let (X, d) be a metrically convex metric space, K a non-empty compact convex subset of X. Let $\{F_n\}$ be a sequence of continuous multivalued operator of K into $CB(X)$ satisfying for all $i \neq j$,*

$$\begin{aligned}H(F_i(x), F_j(y)) \quad < \quad & \alpha d(x,y) + h \max\Big\{\tfrac{1}{2}[D(x, F_i(x)) + D(y, F_j(y))], \\ & \tfrac{1}{2}[D(x, F_j(y)) + D(y, F_i(x))]\Big\}\end{aligned} \tag{2.17}$$

for all $x, y \in K$ with right hand side not zero, where $\alpha \geq 0$, $h \geq 0$ and $\alpha + \frac{3}{2}h + \frac{\alpha h}{2} < 1$,

If $F_n(x) \subset K$ for each $x \in \partial K$ and $n \in N$, then $\{F_n\}$ have a common fixed point $z \in K$.

Proof. The proof is similar to the Theorem 2.3 and now the desire conclusion follows by an application of Theorem 3.1 of Hung and Cho [6]. □

References

[1] N. A. Assad, *Fixed point theorems for set-valued transformations on compact sets*, Boll. U. M. I., **17**(4), (1973), 1-7.

[2] N. A. Assad and W. A. Kirk, *Fixed point theorem for set-valued mappings of contractive type*, Pacific J. Math., **43**(1972), 553-562.

[3] Lj. B. Ćirić, J. S. Ume, *Common fixed point theorems for multi-valued non-self mappings*, Publ. Math. Debrecen, **60**(2002), 359-371.

[4] N. J. Huang and Y. J. Cho, *Common fixed point theorems for a sequence of set-valued mappings*, Korean J. Math. Sci., **4**(1997), 1-10.

[5] S. Itoh, *Multi-valued generalized contractions and fixed point theorems*, Comment. Math. Univ. Carolin., **18**(1977), 247-248.

[6] B. Fisher, *Common fixed point and contraction mappings satisfying a rational inequality*, Math. Sem. Notes, **6**(1978), 29-35.

[7] B. C. Dhage, *A fixed point theorem for non-self multimaps in metrically convex metric space*, Comment. Math. Univ. Carolin., **40**(1999), 251-158.

[8] A. Muntean, *Fixed point principles and applications to mathematical economics*, Cluj University Press, Cluj-Napoca, 2002.

[9] N. Negoescu, *Common fixed point problems for pairs of φ-contractive mappings*, "Gh. Asachi" Publ. House, Iaşi, 1999, (in Romanian).

[10] B. E. Rhoades, *A fixed point theorem for a multi-valued non-self mappings*, Comment. Math. Univ. Carolin., **37**(1996), 401-404.

[11] Adrian Petruşel, *Operatorial inclusions*, House of the Book of Science, Cluj-Napoca, 2002.

[12] I. A. Rus, *Generalized contractions and applications*, Cluj Univ. Press, 2001.

[13] I. A. Rus, A. Petruşel, G. Petruşel, *Fixed point theory 1950-2000 : Romanian contributions*, House of the Book of Science, Cluj-Napoca, 2002.

[14] A. Sîntămărian, *Common fixed point theorems for multivalued mappings*, Semin. Fixed Point Theory Cluj-Napoca, **1**(2000), 93-102.