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NONDENSELY DEFINED EVOLUTION IMPULSIVE DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper, we shall establish sufficient conditions for the existence of integral solutions for some nondensely defined evolution impulsive differential equations in Banach spaces with nonlocal conditions. The main tool to get its the Leray–Schauder alternative.
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1. Introduction

In this paper, we shall prove existence results, for the following evolution equation with nonlocal conditions, of the form

$$y'(t) = Ay(t) + F(t, y(t)), \quad t \in J = [0, T], \ t \neq t_k, \ k = 1, \dots, m \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m \tag{2}$$

$$y(0) + g(y) = y_0, \tag{3}$$

where $A : D(A) \subset E \to E$ is nondensely defined closed linear operator, $F : J \times E \longrightarrow E$ is continuous, $g : C(J', E) \longrightarrow E$, $(J' = J \backslash \{t_1, \dots, t_m\})$, $I_k : E \to \overline{D(A)}$, $k = 1, \dots, m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $y_0 \in E$, and E is a Banach space with norm $|\cdot|$.

As indicated in [4], [5], [8] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [8], the author used

$$g(y) = \sum_{i=1}^{p} c_i y(t_i), \tag{4}$$

where $c_i,\ i = 1, \dots, p$ are given constants and $0 < t_1 < t_2 < \cdots \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, equation (4) allows the additional measurements at $t_i,\ i = 1, \dots, p$.

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years, because all the structure of its emergence has deep physical background and realistic mathematical model. The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade; see the monographs of Bainov and Simeonov [2], Lakshmikantham, et al. [11], and Samoilenko and Perestyuk [14] where numerous properties of their solutions are studied, and detailed bibliographies are given.

When operator A generates a C_0 semigroup, or equivalently, when a closed linear operator A satisfies

(i) $\overline{D(A)} = E$, (D means domain),

(ii) the Hille-Yosida condition that is, there exists $M \geq 0$ and $\tau \in \mathbb{R}$ such that $(\tau, \infty) \subset \rho(A)$, $\sup\{(\lambda I - \tau)^n |(\lambda I - A)^{-n}| : \lambda > \tau\ n \in \mathbb{N}\} \leq M$,

where $\rho(A)$ is the resolvent operator set of A and I is the identity operator, then the equation (1) with nonlocal conditions have been studied extensively. Existence, uniqueness, and regularity, among other thinks, are derived. See [3]–[5], [8], [12], [13].

However, as indicated in [7], we sometimes need to deal with nondensely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,1]$ and consider $A = \dfrac{\partial^2}{\partial x^2}$ in $C([0,1],\mathbb{R})$ in order to measure the solutions in the sup-norm, then domain

$$D(A) = \{\phi \in C^2([0,1],\mathbb{R}) : \phi(0) = \phi(1) = 0\}$$

is not dense in $C([0,1],\mathbb{R})$ with the sup-norm. See [7] for more examples and remarks concerning the nondensely defined operators.

Our purpose here is to extend the results of densely defined impulsive evolution equations with nonlocal conditions. We use the Schaefers fixed point theorem to derive the existence of integral solutions (when the operator is nondensely defined).

The plan of this paper is as follows: In section 2, we state some facts about integrated semigroups and integral solutions that will be used later. In section 3, we prove the existence of integral solutions for the problem (1)-(3) when A is not necessarily densely defined but satisfies the Hille-Yosida condition. Finally in section 4 we prove the existence of integral solutions for the problem (1), (2), (4)

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.

$C(J,E)$ is the Banach space of continuous functions from J to E normed by

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|Ny| : |y| = 1\}.$$

Definition 2.1. ([1]). *Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:*

(i) $S(0) = 0$;

(ii) *$t \to S(t)$ is strongly continuous;*

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr,$ *for all* $t, s \geq 0$.

Definition 2.2. ([10]). *An operator A is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ ($\rho(A)$, is the resolvent set of A) and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded operators such that $S(0) = 0$ and $R(\lambda, A) := (\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ exists for all λ with $\lambda > \omega$.*

Proposition 2.1. ([1]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A) \quad \text{and} \quad S(t)x = A\int_0^t S(s)xds + tx.$$

Definition 2.3. ([10]).

(i) *An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous if, for all $\tau > 0$ there exists a constant L such that*

$$|S(t) - S(s)| \leq L|t - s|, \quad t, s \in [0, \tau].$$

(ii) *An integrated semigroup $(S(t))_{t\geq 0}$ is called non degenerate if $S(t)x = 0$, for all $t \geq 0$ implies that $x = 0$.*

Definition 2.4. *We say that the linear operator A satisfies the Hille-Yosida condition if there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and*

$$\sup\{(\lambda - \omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N}, \ \lambda > \omega\} \leq M.$$

Theorem 2.1. ([10]). *The following assertions are equivalent:*

(i) *A is the generator of a non degenerate,locally Lipschitz continuous integrated semigroup ;*
(ii) *A satisfies the Hille-Yosida condition.*

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [1], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$.

In order to define the solution of (1)–(3) we shall consider the following space

$$\Omega = \{y : [0,T] \to E : y_k \in C(J_k, E), k = 0, \dots, m \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k}, k = 0, \dots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \dots, m$.

Definition 2.5. *Given $F \in L^1(J \times E, E)$ and $y_0 \in E$ we say that $y : J \to E$ is an integral solution of (1)-(3) if*

(i) $y \in \Omega$,

(ii) $\int_0^t y(s)ds \in D(A)$ *for* $t \in J$,

(iii) $y(t) = y_0 - g(y) + A\int_0^t y(s)ds + \int_0^t F(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k^-)),$ $t \in J$.

From (ii) it follows that $y(t) \in \overline{D(A)},\ \forall\ t \geq 0$. Also from (iii) it follows that $y_0 - g(y) \in \overline{D(A)}$. So, if we assume that $y_0 \in \overline{D(A)}$, we conclude that $g(y) \in \overline{D(A)}$.

Here and hereafter we assume that

(H1) A satisfies the Hille-Yosida condition.

Let $B_\lambda = \lambda R(\lambda, A)$, then for all $x \in \overline{D(A)},\ B_\lambda x \to x$ as $\lambda \to \infty$.

Lemma 2.1. *If y is an integral solution of (1)-(3), then it is given by*

$$y(t) = S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)F(s, y(s))ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad \text{for } t \in J.$$

Proof. Let y be a solution of the problem (1)-(3). Define $w(s) = S(t-s)y(s)$. Then we have

$$\begin{aligned} w'(s) &= -S'(t-s)y(s) + S(t-s)y'(s) \\ &= -AS(t-s)y(s) - y(s) + S(t-s)y'(s) \\ &= S(t-s)[y'(s) - Ay(s)] - y(s) \\ &= S(t-s)F(s, y(s)) - y(s). \end{aligned} \tag{5}$$

Consider $t_k < t,\ k = 1, \dots, m$. Then integrating the previous equation we have

$$\int_0^t w'(s)ds = \int_0^t S(t-s)F(s, y(s))ds - \int_0^t y(s)ds.$$

For $k = 1$

$$w(t_1^-) - w(0) + w(t) - w(t_1^+) = \int_0^t S(t-s)F(s, y(s))ds - \int_0^t y(s)ds$$

or

$$\begin{aligned} \int_0^t y(s) &= S(t)y(0) + \int_0^t S(t-s)F(s, y(s))ds + w(t_1^+) - w(t_1^-) \\ &= S(t)(y_0 - g(y)) + \int_0^t S(t-s)F(s, y(s))ds + S(t-t_1)I_1(y(t_1^-)) \end{aligned}$$

Now for $k = 2, \dots, m$ we have that

$$\begin{aligned} &\int_0^{t_1} w'(s)ds + \int_{t_1}^{t_2} w'(s)ds + \ldots + \int_{t_k}^{t} w'(s)ds \\ &\quad = \int_0^t S(t-s)F(s, y(s))ds - \int_0^t y(s)ds \Leftrightarrow \\ &w(t_1^-) - w(0) + w(t_2^-) - w(t_1^+) + \ldots + w(t_k^+) - w(t) \\ &\quad = \int_0^t S(t-s)F(s, y(s))ds - \int_0^t y(s)ds \Leftrightarrow \end{aligned}$$

$$\begin{aligned} \int_0^t y(s)ds &= w(0) + \sum_{0<t_k<t} [w(t_k^+) - w(t_k^-)] + \int_0^t S(t-s)F(s, y(s))ds \\ &= S(t)(y_0 - g(y)) + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)) \\ &+ \int_0^t S(t-s)F(s, y(s))ds. \end{aligned}$$

By differentiating the above equation we have that

$$y(t) = S'(t)(y_0 - g(y)) + \sum_{0<t_k<t} S'(t-t_k)I(y(t_k^-)) + \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))ds,$$

which proves the lemma.

3. Existence result

In this section we are concerned with the existence of solutions for problem (1)–(3).

Let Ω' be the set of all functions that belong in Ω and have values in $\overline{D(A)}$.

Let us list the following hypotheses:

(H2) For each $t \in J$, the function $F(t,\cdot)$ is continuous and for each y, the function $F(\cdot,y)$ is measurable.

(H3) The operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.

(H4) There exists a continuous function $m : [0,T] \to \mathbb{R}^+$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$|F(t,x)| \le m(t)\psi(|x|), \quad t \in J, \quad x \in E.$$

(H5) $g : \Omega' \to \overline{D(A)}$ is completely continuous (i.e., continuous and takes a bounded set into a compact set) and there exists $G > 0$ such that $|g(y)| \le G$, for all $y \in \Omega$.

(H6) $I_k : E \to \overline{D(A)}$ and there exist constants $d_k,\ k = 1,\dots,m$ such that

$$\|I_k(y)\|_{\overline{D(A)}} \le d_k, \quad y \in E.$$

(H7) $y_0 \in \overline{D(A)}$ and

$$\int_0^T \max(\omega, Mm(s))ds < \int_c^\infty \frac{ds}{s+\psi(s)},$$

where $c = M\left(|y_0| + G + \sum_{k=1}^m e^{-\omega t_k} d_k\right)$ and M and ω are from the Hille-Yosida condition

The following Leray–Schauder alternative, known also as Schaefer's fixed point theorem is crucial in the proof of our main results.

Theorem 3.1. ([6]) *Let S be a Banach space and $K : S \to S$ be a completely continuous map. If the set*

$$\Phi = \{x \in S : \ x = \sigma N x, \ \text{for some } 0 < \sigma < 1\}.$$

is bounded then K has a fixed point.

Now, we are able to state and prove our main theorem in this section.

Theorem 3.2. *Assume that assumptions (H1)-(H7) hold. Then the problem (1)-(3) has at least one integral solution on J.*

Proof. Consider the operator $N : \Omega' \longrightarrow \Omega'$ defined by

$$\begin{aligned} N(y)(t) &= S'(t)[y_0 - g(y)] + \frac{d}{dt}\int_0^t S(t-s)F(s, y(s))ds \\ &\quad + \sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)), \ t \in J. \end{aligned}$$

Step 1. *N is continuous*

Let $\{y_n\}$ be a sequence in Ω' with $\lim_{n\to\infty} y_n = y$ in Ω'. By the continuity of F with respect to the second argument, we deduce that for each $s \in J$, $F(s, y_n(s))$ converges to $F(s, y(s))$ in E, and we have that

$$\begin{aligned} |N(y_n)(t) - Ny(t)| \le Me^{\omega T}\Bigg[&|g(y_n) - g(y)| + \int_0^T e^{-\omega s}|F(s, y_n(s)) \\ &- F(s, y(s))|ds + \sum_{k=1}^m e^{-\omega t_k}\|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|_{\overline{D(A)}}\Bigg] \end{aligned}$$

The sequence $\{y_n\}$ is bounded in Ω', then by assumption (H5), and using Lebesgue's dominated convergence theorem and the continuity of g we obtain that

$$\lim_{n\to\infty} N(y_n) = N(y) \ \text{ in } \Omega',$$

which implies that the mapping N is continuous on Ω'.

Step 2. *N maps bounded sets into compact sets*

Firstly we will prove that $\{Ny(t) : y \in B\}$ is relatively compact in E, where B be a bounded set in Ω' let $t \in J$ is fixed.

If $t = 0$, then $\{Ny(0) : y \in B\} = \{y_0 - g(y) : y \in B\}$ is relatively compact since we assumed that g is completely continuous.

If $t \in (0, T]$, choose ϵ such that $0 < \epsilon < t$. Then

$$\begin{aligned} N(y)(t) &= S'(t)[y_0 - g(y)] + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda F(s, y(s))ds \\ &\quad + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) \\ &= S'(t)[y_0 - g(y)] + S'(\epsilon)\lim_{\lambda\to\infty}\int_0^{t-\epsilon} S'(t-\epsilon-s)B_\lambda F(s, y(s))ds \\ &\quad + \lim_{\lambda\to\infty}\int_{t-\epsilon}^t S'(t-s)B_\lambda F(s, y(s))ds + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)). \end{aligned}$$

Since $S'(t)$ is compact, we deduce that there exists a compact set W_1 such that

$$S'(\epsilon)\lim_{\lambda\to\infty}\int_0^{t-\epsilon} S'(t-\epsilon-s)B_\lambda F(s, y(s))ds \in W_1,$$

for $y \in B$. Furthermore by (H4) there exists a positive constant b_1 such that

$$\left|\lim_{\lambda\to\infty}\int_{t-\epsilon}^t S'(t-s)B_\lambda F(s, y(s))ds\right| \le b_1\epsilon, \quad \text{for} \quad y \in B.$$

Moreover, by (H5) and since $S'(t)$ is compact the set

$$\left\{S'(t)[y_0 - g(y)] + \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)) : y \in B\right\}$$

is relatively compact. We conclude that $\{Ny(t) : y \in B\}$ is totally bounded and therefore, it is relatively compact in E.

Finally, let us show that NB is equicontinuous. For every $0 < \tau_0 < \tau \leq T$ and $y \in B$,

$$\begin{aligned}
|Ny(\tau) - Ny(\tau_0)| &= |(S'(\tau) - S'(\tau_0))[y_0 - g(y)]| \\
&\quad + \left| \lim_{\lambda\to\infty} \int_0^{\tau_0} [S'(\tau - s) - S'(\tau_0 - s)] B_\lambda F(s, y(s)) ds \right| \\
&\quad + \left| \lim_{\lambda\to\infty} \int_{\tau_0}^{\tau} S'(\tau - s) B_\lambda F(s, y(s)) ds \right| \\
&\quad + \left| \sum_{0<t_k<\tau_0} [S'(\tau - t_k) - S'(\tau_0 - t_k)] I_k(y(t_k^-)) \right| \\
&\quad + \left| \sum_{\tau_0 \leq t_k < \tau} S'(\tau - t_k) I_k(y(t_k^-)) \right| \\
&\leq \big| [S'(\tau) - S'(\tau_0)][y_0 - g(y)] \big| \\
&\quad + \left| [S'(\tau - \tau_0) - I] \lim_{\lambda\to\infty} \int_0^{\tau_0} S'(\tau_0 - s) B_\lambda F(s, y(s)) ds \right| \\
&\quad + M e^{\omega T} \int_{\tau_0}^{\tau} e^{-\omega s} m(s) \psi(|y(s)|) ds \\
&\quad + \sum_{0<t_k<\tau_0} \|S'(\tau - t_k) - S'(\tau_0 - t_k)\|_{B(E)} d_k \\
&\quad + M e^{\omega T} \sum_{\tau_0 \leq t_k < \tau} e^{-\omega t_k} d_k.
\end{aligned}$$

The right-hand side tends to zero as $\tau \to \tau_0$, since $S'(t)$ is strongly continuous and the compactness of $S'(t),\ t > 0$ implies the continuity in the uniform operator topology. Thus, NB is equicontinuous.

The equicontinuity for $\tau_0 = 0$ is obvious. As a consequence of the above steps and the Arzelá-Ascoli theorem we deduce that N maps K into precompact sets sets in $\overline{D(A)}$.

Step 3 *The set*

$$\Phi = \{x \in \Omega' : \ x = \sigma N x, \ \textit{for some } 0 < \sigma < 1\}$$

is bounded

For $y \in \Phi$, there exists $\sigma \in (0,1)$ such that $y = \sigma N y$, that is

$$y(t) = \sigma S'(t)[y_0 - g(y)] + \sigma \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))ds + \sigma \sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), \quad t \in J$$

Using assumptions (H5)-(H7), we get

$$e^{-\omega t}|y(t)| \leq M\left[|y_0| + G + \int_0^t e^{-\omega s}m(s)\psi(|y(s)|)ds + \sum_{k=1}^m e^{-\omega t_k}d_k\right]. \quad (6)$$

Let $v(t)$ denote the right hand side of the above inequality, then

$$v'(t) = Me^{-\omega t}m(t)\psi(|y(t)|), \quad \text{for} \quad t \in J$$

and

$$v(0) = M\left(y_0 + G + \sum_{k=1}^m e^{-\omega t_k}d_k\right).$$

From (6), we have that $|y(t)| \leq e^{\omega t}v(t)$. Then

$$v'(t) \leq Me^{-\omega t}m(t)\psi(e^{\omega t}v(t)), \quad t \in J.$$

Accordingly, we have that

$$(e^{\omega t}v(t))' \leq \max\{\omega, Mm(t)\}(e^{\omega t}v(t) + \psi(e^{\omega t}v(t))), \quad t \in J,$$

which implies that

$$\int_c^{e^{\omega t}v(t)} \frac{ds}{s+\psi(s)} \leq \int_0^T \max(\omega, Mm(s))ds < \int_c^\infty \frac{ds}{s+\psi(s)}, \quad t \in J$$

Using (H7) we deduce that there exists a positive constant α which depends on T and the functions m, ψ such that $y(t) \leq \alpha$ for all $y \in \Phi$, which implies that Φ is bounded.

Consequently, the mapping N is completely continuous and Theorem 3.1 implies that N has at least one fixed point, which gives rise to an integral solution of the problem (1)-(3).

4. A special case

In this section, we suppose that the nonlocal condition is given by

$$g(y) = \sum_{k=1}^{m+1} c_k y(\eta_k) \tag{4*}$$

where $c_k,\ k = 1, \ldots, m+1$ are nonnegative constants and $0 \leq \eta_1 < t_1 < \eta_2 < t_2 < \cdots < t_m < \eta_{m+1} \leq T$.

Lemma 4.1. *Assume that*

(H8) *There exists a bounded operator $B : E \to E$ such that*

$$B = \left(I + \sum_{k=1}^{m+1} c_k S'(\eta_k) \right)^{-1}.$$

If y is an integral solution of (1), (2), (4) then it is given by*

$$\begin{aligned} y(t) &= S'(t)B \left[y_0 - \sum_{k=2}^{m+1} c_k \sum_{\lambda=1}^{k-1} S'(\eta_k - t_j) I_j(y(t_j^-)) \right. \\ &- \left. \sum_{k=1}^{m+1} c_k \int_0^{\eta_k} S'(\eta_k - s) F(s, y(s)) \right] \\ &+ \frac{d}{dt} \int_0^t S(t-s) F(s, y(s)) ds + \sum_{0<t_k<t} S'(t - t_k) I_k(y(t_k^-)),\ t \in J. \end{aligned}$$

Proof. Let y be a solution of the problem (1), (2), (4*). As in Lemma 2.1 we conclude that

$$\int_0^t y(s)ds = w(0) + \sum_{0<t_k<t} S(t - t_k) I_k(y(t_k^-)) + \int_0^t S(t-s) F(s, y(s)) ds \tag{7}$$

where $w(0) = S(t)y(0) = S(t) \left[y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k) \right]$.

It remains to find $y(\eta_k)$. For that reason we use equation (5) and we integrate it from 0 to $\eta_k,\ k = 1, \ldots, m+1$.

For $k = 1$

$$\int_0^{\eta_1} w'(s)ds = \int_0^{\eta_1} S(t-s)F(s,y(s))ds - \int_0^{\eta_1} y(s)ds \Leftrightarrow$$

$$w(\eta_1) - w(0) = \int_0^{\eta_1} S(t-s)F(s,y(s))ds - \int_0^{\eta_1} y(s)ds \Leftrightarrow$$

$$S(t-\eta_1)y(\eta_1) = S(t)y(0) + \int_0^{\eta_1} S(t-s)F(s,y(s))ds - \int_0^{\eta_1} y(s)ds.$$

For $k = 2, \dots, m+1$

$$\int_0^{\eta_k} w'(s)ds = \int_0^{\eta_k} S(t-s)F(s,y(s))ds - \int_0^{\eta_k} y(s)ds \Leftrightarrow$$

$$\int_0^{t_1} w'(s)ds + \int_{t_1}^{t_2} w'(s)ds + \dots + \int_{t_{k-1}}^{\eta_k} w'(s)ds$$
$$= \int_0^{\eta_k} S(t-s)F(s,y(s))ds - \int_0^{\eta_k} y(s)ds \quad \Leftrightarrow$$

$$w(t_1^-) - w(0) + w(t_2^-) - w(t_1^+) + \dots + w(\eta_k) - w(t_{k-1}^+)$$
$$= \int_0^{\eta_k} S(t-s)F(s,y(s))ds - \int_0^{\eta_k} y(s)ds.$$

After calculation we conclude that

$$\begin{aligned} S(t-\eta_k)y(\eta_k) &= S(t)y(0) + \sum_{0<t_j<\eta_k} S(t-t_j)I_j(y(t_j^-)) \\ &\quad + \int_0^{\eta_k} S(t-s)F(s,y(s))ds - \int_0^{\eta_k} y(s)ds. \end{aligned}$$

If we differentiate the above equation we have that

$$\begin{aligned} S'(t-\eta_k)y(\eta_k) &= S'(t)y(0) + \sum_{0<t_j<\eta_k} S'(t-t_j)I_j(y(t_j^-)) \\ &\quad + \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(t-s)B_\lambda F(s,y(s))ds. \end{aligned}$$

We rewrite the last relation as

$$\begin{aligned} S'(t-\eta_k)y(\eta_k) &= S'(t-\eta_k)S'(\eta_k)y(0) \\ &+ S'(t-\eta_k)\sum_{0<t_j<\eta_k} S'(\eta_k-t_j)I_j(y(t_j^-)) \\ &+ S'(t-\eta_k)\lim_{\lambda\to\infty}\int_0^{\eta_k} S'(\eta_k-s)B_\lambda F(s,y(s))ds. \end{aligned}$$

From this we conclude that

$$\begin{aligned} y(\eta_k) &= S'(\eta_k)y(0) + \sum_{0<t_j<\eta_k} S'(\eta_k - t_j)I_j(y(t_j^-)) \\ &\quad + \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s)B_\lambda F(s, y(s))ds. \end{aligned} \tag{8}$$

Equation (4*), using (8), becomes

$$\begin{aligned} y(0) &+ \sum_{k=1}^{m+1} c_k \Bigg[S'(\eta_k)y(0) + \sum_{0<t_j<\eta_k} S'(\eta_k - t_j)I_j(y(t_j^-)) \\ &\quad + \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s)B_\lambda F(s, y(s))ds \Bigg] = y_0 \end{aligned}$$

or

$$\begin{aligned} y(0)\left(I + \sum_{k=1}^{m+1} c_k S'(\eta_k)\right) &= y_0 - \sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu)I_\mu(y(t_\mu^-)) \\ &\quad - \sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s)B_\lambda F(s, y(s))ds. \end{aligned}$$

So

$$\begin{aligned} y(0) &= By_0 - B\sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu)I_\mu(y(t_\mu^-)) \\ &\quad - B\sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s)B_\lambda F(s, y(s))ds. \end{aligned} \tag{9}$$

Now, equation (7) with the help of (9) becomes

$$\begin{aligned} \int_0^t y(s)ds &= S(t)\Bigg[By_0 - B\sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu)I_\mu(y(t_\mu^-)) \\ &\quad - B\sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s)B_\lambda F(s, y(s))ds \Bigg] \\ &\quad + \sum_{0<t_k<t} S(t - t_k)I_k(y(t_k^-)) + \int_0^t S(t - s)F(s, y(s))ds, \end{aligned}$$

and if we differentiate the above equation we conclude that

$$\begin{aligned} y(t) &= S'(t)\Bigg[By_0 - B\sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu) I_\mu(y(t_\mu^-)) \\ &\quad - B\sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty}\int_0^{\eta_k} S'(\eta_k - s) B_\lambda F(s, y(s))ds\Bigg] \\ &\quad + \sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)) + \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))ds. \end{aligned}$$

The proof is completed.

Now, we are able to state and prove our main theorem in this section.

Theorem 4.1. *Assume that assumptions (H1)-(H4), (H6), (H8) hold. Also assume that*

(H9) $y_0 \in \overline{D(A)}$ *and*

$$\int_1^\infty \frac{ds}{s+\psi(s)} = \infty.$$

(H10) *The set* $\left\{ y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k) \right\}$ *is relatively compact.*

Then the problem (1), (2), (4) has at least one integral solution on J.*

Proof. Consider the operator $\overline{N} : \Omega' \longrightarrow \Omega'$ defined by

$$\begin{aligned} \overline{N}(y) &= S'(t)B\Bigg[y_0 - \sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu) I_\mu(y(t_\mu^-)) \\ &\quad - \sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty}\int_0^{\eta_k} S'(\eta_k - s) B_\lambda F(s, y(s))\Bigg] \\ &\quad + \frac{d}{dt}\int_0^t S(t-s)F(s,y(s))ds \\ &\quad + \sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)),\ t \in J. \end{aligned}$$

We claim that the set Φ is bounded. For $y \in \Phi$ there exists $\sigma \in (0,1)$ such that $y = \sigma \overline{N} y$, that is

$$\begin{aligned} y(t) &= \sigma S'(t)B\left[y_0 - \sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu) I_\mu(y(t_\mu^-))\right. \\ &- \left.\sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s) B_\lambda F(s, y(s))\right] \\ &+ \sigma \frac{d}{dt} \int_0^t S(t-s) F(s, y(s)) ds \\ &+ \sigma \sum_{0<t_k<t} S'(t-t_k) I_k(y(t_k^-)), \quad t \in J. \end{aligned}$$

Using assumptions (H3), (H5) we get

$$\begin{aligned} e^{-\omega t}|y(t)| \;\le\; & M\|B\|_{B(E)}\left[|y_0| + M\sum_{k=2}^{m+1}|c_k|\sum_{\mu=1}^{k-1} e^{\omega(\eta_k - t_\mu)} d_\mu\right. \\ & \left. + M\sum_{k=1}^{m+1}|c_k| \int_0^{\eta_k} e^{\omega(\eta_k - s)} m(s)\psi(|y(s)|)ds\right] \\ & + M\int_0^t e^{-\omega s} m(s)\psi(|y(s)|)ds + M\sum_{k=1}^{m} e^{-\omega t_k} d_k. \end{aligned} \tag{10}$$

Let $v(t)$ denote the right hand side of the above inequality, then

$$v'(t) = Me^{-\omega t} m(t)\psi(|y(t)|), \quad \text{for} \quad t \in J,$$

and

$$\begin{aligned} v(0) \;=\; & M\|B\|_{B(E)}\left[|y_0| + M\sum_{k=2}^{m+1}|c_k|\sum_{\mu=1}^{k-1} e^{\omega(\eta_k - t_\mu)} d_\mu\right. \\ & \left. + M\sum_{k=1}^{m+1}|c_k| \int_0^{\eta_k} e^{\omega(\eta_k - s)} m(s)\psi(|y(s)|)ds\right] + M\sum_{k=1}^{m} e^{-\omega t_k} d_k. \end{aligned}$$

From (10), we have that $|y(t)| \le e^{\omega t} v(t)$. Then

$$v'(t) \le Me^{-\omega t} m(t)\psi(e^{\omega t} v(t)), \quad t \in J.$$

Accordingly, we have that

$$(e^{\omega t}v(t))' \leq \max\{\omega, Mm(t)\}(e^{\omega t}v(t) + \psi(e^{\omega t}v(t))), \quad t \in J,$$

which implies that

$$\int_{v(0)}^{e^{\omega t}v(t)} \frac{ds}{s + \psi(s)} \leq \int_0^T \max(\omega, Mm(s))ds < +\infty, \quad t \in J.$$

Using (H9) we deduce that there exists a positive constant α which depends on T and the functions m, ψ such that $y(t) \leq \alpha$ for all $y \in \Phi$, which implies that Φ is bounded.

It remains to prove that $\overline{N}$ is compact. Let $\{y_n\}$ be a sequence in Ω' with $\lim_{n\to\infty} y_n = y$ in Ω'. By the continuity of F with respect to the second argument, we deduce that for each $s \in J$, $F(s, y_n(s))$ converges to $F(s, y(s))$ in E, and we have that

$$\begin{aligned}
&|\overline{N}(y_n)(t) - \overline{N}(y)(t)| \\
\leq\ & M^2\|B\|_{B(E)} \left[\sum_{k=2}^{m+1} |c_k| \sum_{\mu=1}^{k-1} e^{\omega(\eta_k - t_\mu)} \|I_\mu(y_n(t_\mu^-)) - I_\mu(y(t_\mu^-))\|_{\overline{D(A)}}\right. \\
+\ & \left.\sum_{k=1}^{m+1} |c_k| e^{\omega\eta_k} \int_0^{\eta_k} e^{-\omega s}|F(s, y_n(s)) - F(s, y(s))|ds\right] \\
+\ & Me^{\omega T}\left[\int_0^T e^{-\omega s}|F(s, y_n(s)) - F(s, y(s))|ds\right. \\
+\ & \left.\sum_{k=1}^{m} e^{-\omega t_k}\|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|_{\overline{D(A)}}\right].
\end{aligned}$$

The sequence $\{y_n\}$ is bounded in Ω', then by using Lebesgue dominated convergence theorem we obtain that

$$\lim_{n\to\infty} \overline{N}(y_n) = \overline{N}(y) \ \text{ in } \Omega',$$

which implies that the mapping $\overline{N}$ is continuous on Ω'.

Next, we use Arzelá-Ascoli's Theorem to prove that $\overline{N}$ maps every bounded set into a compact set. Let B be a bounded set of Ω' and let $t \in J$ be fixed, then we need to prove that $\{\overline{N}(y)(t) : y \in B\}$ is relatively compact in $\overline{D(A)}$. If $t = 0$, then from hypothesis (H10) we have that $\{\overline{N}(y)(0) : y \in B\} =$

$\left\{ y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k) : y \in B \right\}$ is relatively compact. If $t \in (0, T]$, choose ϵ such that $0 < \epsilon < t$. Then

$$\begin{aligned}
\overline{N}(y)(t) \quad = \quad & S'(t)B\Bigg[y_0 - \sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu) I_\mu(y(t_\mu^-)) \\
& - \sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s) B_\lambda F(s, y(s))\Bigg] \\
& + S'(\epsilon) \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t - \epsilon - s) B_\lambda F(s, y(s)) ds \\
& + \lim_{\lambda\to\infty} \int_{t-\epsilon}^{t} S'(t - s) B_\lambda F(s, y(s)) ds + \sum_{0<t_k<t} S'(t - t_k) I_k(y(t_k^-)).
\end{aligned}$$

Since $S'(t)$ is compact, we deduce that there exists a compact set D_1 such that

$$S'(\epsilon) \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t - \epsilon - s) B_\lambda F(s, y(s)) ds \in D_1,$$

for $y \in B$. Furthermore by (H3) there exists a positive constant b_1 such that

$$\left| \lim_{\lambda\to\infty} \int_{t-\epsilon}^{t} S'(t - s) B_\lambda F(s, y(s)) ds \right| \le b_1 \epsilon, \quad \text{for} \quad y \in B.$$

We conclude that $\{\overline{N}(y)(t) : y \in B\}$ is totally bounded and therefore, it is relatively compact in $\overline{D(A)}$.

Finally, let us show that $\overline{N}B$ is equicontinuous. For every $0 < \tau_0 < \tau \leq T$ and $y \in B$,

$$\begin{aligned}
& \left|\overline{N}(y)(\tau) - \overline{N}(y)(\tau_0)\right| \\
\leq & \left| [S'(\tau) - S'(\tau_0)] \left[By_0 - B \sum_{k=2}^{m+1} c_k \sum_{\mu=1}^{k-1} S'(\eta_k - t_\mu) I_\mu(y(t_\mu^-)) \right.\right. \\
- & \left.\left. B \sum_{k=1}^{m+1} c_k \lim_{\lambda\to\infty} \int_0^{\eta_k} S'(\eta_k - s) B_\lambda F(s, y(s)) ds \right] \right| \\
+ & \left| [S'(\tau - \tau_0) - I] \lim_{\lambda\to\infty} \int_0^{\tau_0} S'(\tau_0 - s) B_\lambda F(s, y(s)) ds \right| \\
+ & Me^{\omega T} \lim_{\lambda\to\infty} \int_{\tau_0}^{\tau} e^{-\omega s} m(s) \psi(|y(s)|) ds \\
+ & \sum_{0 < t_k < \tau_0} \|S'(\tau - t_k) - S'(\tau_0 - t_k)\|_{B(E)} d_k \\
+ & Me^{\omega T} \sum_{\tau_0 \leq t_k < \tau} e^{-\omega t_k} d_k.
\end{aligned}$$

The right-hand side tends to zero as $\tau \to \tau_0$, since $S'(t)$ is strongly continuous and the compactness of $S'(t),\ t > 0$ implies the continuity in the uniform operator topology. Thus, NB is equicontinuous.

As a consequence of the above steps and the Arzelá-Ascoli Theorem we deduce that $\overline{N}$ maps K into precompact sets in $\overline{D(A)}$.

Consequently, the mapping $\overline{N}$ is compact and Theorem 3.1 implies that $\overline{N}$ has at least one fixed point, which gives rise to an integral solution of the problem (1), (2), (4*).

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