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SOME REMARKS ON KRASNOSELSKII'S FIXED POINT THEOREM

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Abstract. Let M be a closed convex non-empty set in a Banach space $(X, \|\cdot\|)$ and let $Px = Ax + Bx$ be a mapping such that: (i) $Ax + By \in M$ for each $x, y \in M$; (ii) A is continuous and AM compact; (iii) B is a contraction mapping. The theorem of Krasnoselskii asserts that in these conditions the operator P has a fixed point in M. In this paper some remarks about the hypothesis of this theorem are given. A variant for the cartesian product of two operators is also considered.

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1. Introduction

Two main results of fixed point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result (see [5], p. 31 or [6], p. 501).

Theorem K (Krasnoselskii). *Let M be a closed convex bounded non-empty subset of a Banach space $(X, \|\cdot\|)$. Suppose that A, B map M into X such that*

i) $Ax + By \in M$, for all $x, y \in M$;

ii) A is continuous and AM is contained in a compact set;

iii) B is a contraction mapping with constant $\alpha \in (0, 1)$.

Then there exists $x \in M$, with

$$x = Ax + Bx.$$

In [2] T.A. Burton remarks the difficulty to check the hypothesis i) and replaces it with the weaker condition:

i')

$$(x = Ax + By,\ y \in M) \Rightarrow (x \in M)\,.$$

The proof idea is the following: for every $y \in M$ the mapping $x \to Bx + Ay$ is a contraction. Therefore, there exists $\varphi : M \to M$ such that $\varphi(y) = B\varphi(y) + Ay$, for every $y \in M$. Then the problem is reduced to prove that φ admits a fixed point; to this aim Schauder's theorem is used (see [6], p. 57).

In [3] the authors show that instead of Schauder's theorem one can use Schaefer's fixed point theorem (see [5], p. 29) which yields in normed spaces or more generally in locally convex spaces.

In the present paper, stimulated by the ideas contained in [2] and [3] we shall continue the analysis of Krasnoselskii's result. We shall give in addition a variant of the result contained in [1] within we shall use Schaefer's fixed point theorem.

2. General results

Let $(X, \|\cdot\|)$ be a Banach space, $M \subset X$ be a convex closed (not necessary bounded) subset of X. Let in addition $A, B : M \to X$ be two operators; consider the equation

$$x = Ax + Bx. \tag{1}$$

A way to proof that equation (1) admits solutions in M is to write (1) under the equivalent form

$$x = Hx \tag{2}$$

and to apply a fixed point theorem to operator H. There exist two possibilities to build the operator H.

Case 1. The operator $I - B$ admits a continuous inverse; then

$$H = (I - B)^{-1} A. \tag{3}$$

Case 2. The operator B admits a continuous inverse; then

$$H = B^{-1} (I - A)\,. \tag{4}$$

If we want to apply Schauder's theorem to operator H, then in the Case 1 we must suppose that A fulfills hypothesis ii) from Theorem K and in the Case 2 we must suppose that $I-A$ fulfills this hypothesis.

If we would suppose $A, B : X \to X$, then it will exists another possibility i.e. to consider the equation

$$x = (A+T)\,x + (B-T)\,x,$$

where $T : X \to X$ is an arbitrary operator chosen such that the Case 1 or the Case 2 yields.

We state the following two general results.

Proposition 1. *Suppose that*

i) M is a closed convex set;

ii) $I-B : X \to X$ is an injective operator;

iii) $(I-B)^{-1}$ is continuous;

iv) $A : M \to X$ is a continuous operator and $A\,(M)$ is contained into a compact set;

v) the following inequalities hold:

$$A\,(M) \subset (I-B)\,(X) \tag{5}$$

$$(I-B)^{-1}\,A\,(M) \subset M. \tag{6}$$

Then the equation (1) *has solutions in M.*

Indeed, by hypotheses one can apply to operator H Schauder's fixed point theorem. Remark that the condition (5) which assures the existence of operator H is automatically fulfilled if $I-B$ is a surjective operator.

Proposition 2. *Suppose that*

i) M is a closed convex set;

ii) $B : X \to X$ is an injective operator;

iii) B^{-1} is a continuous operator;

iv) $I-A : M \to X$ is a continuous operator and $(I-A)\,(M)$ is contained into a compact set;

v) the following inequalities hold:

$$(I-A)\,(M) \subset B\,(X) \tag{7}$$

$$B^{-1}(I-A)(M) \subset M. \tag{8}$$

Then the equation (1) *has solutions in* M.

The proof is like the proof of Proposition 1.

3. Particular results

Without loss of generality one can admit

$$A0 = 0, \tag{9}$$

(or $B0 = 0$); indeed, one can write (1) under the form

$$x = A_1 x + B_1 x,$$

where $A_1 x = Ax + B0$, $B_1 x = Bx - B0$.

Clearly, the translated operators A_1, B_1 keep many algebraic and topological properties of the operators A, B.

Suppose that $B : X \to X$ is a contraction mapping with constant $\alpha < 1$.

By inequalities

$$(1-\alpha)\|x-y\| \leq \|(I-B)x - (I-B)y\| \leq (1+\alpha)\|x-y\|, \ (\forall)\ x,\ y \in X.$$

it follows in the case $B0 = 0$

$$(1-\alpha)\|x\| \leq \|(I-B)x\| \leq (1+\alpha)\|x\|, \ (\forall)\ x \in X. \tag{10}$$

Admitting the hypotheses of Proposition 1 about A, let us set

$$h_\rho := \sup_{x \in \overline{B_\rho}} \{\|Ax\|\},$$

where

$$\overline{B_\rho} := \{x \in X,\ \|x\| \leq \rho\}.$$

One has the following

Corollary 1. *Suppose that* $B : X \to X$ *is a contraction mapping with constant* $\alpha \in (0,1)$, $B0 = 0$, $A : \overline{B_\rho} \to X$ *is continuous and* $A\left(\overline{B_\rho}\right)$ *is compact. If*

$$h_\rho \leq (1-\alpha)\rho, \tag{11}$$

then the equation (1) *has solutions in* $\overline{B_\rho}$.

Indeed, as is known, $I - B : X \to X$ is homeomorphism. The operator H given by (3) does exist and (11) assures the inclusion $HM \subset M$, since (10) implies

$$\left\| (I - B)^{-1} x \right\| \leq \frac{1}{1 - \alpha} \|x\| .$$

Consider now the case when B is **expansive**, i.e. B satisfies the condition

$$\|Bx - By\| \geq \beta \|x - y\| , \text{ for all } x, \ y \in X, \tag{12}$$

with $\beta > 1$.

Clearly, if B is expansive then B is injective and B^{-1} is continuous (it is well known that if $\dim X < \infty$, then every expansive mapping is surjective, so it is a homeomorphism; since B^{-1} maps X into X it is a contraction mapping having the constant $\frac{1}{\beta}$; hence every expansive mapping $B : \mathbb{R}^n \to \mathbb{R}^n$ admits an unique fixed point).

From the inequalities

$$\|(I - B) x - (I - B) y\| \geq \|Bx - By\| - \|x - y\| \geq (\beta - 1) \|x - y\| , \tag{13}$$

it follows also $I - B$ is injective and $(I - B)^{-1}$ is continuous, since

$$\left\| (I - B)^{-1} x - (I - B)^{-1} y \right\| \leq \frac{1}{\beta - 1} \|x - y\| . \tag{14}$$

Therefore, by Proposition 1 one gets the following corollary.

Corollary 2. *Suppose that* B *is an expansive mapping,* $B0 = 0$ *and* $I - B$ *is surjective. If* $A : \overline{B_\rho} \to X$ *is continuous with* $A\left(\overline{B_\rho}\right)$ *compact and*

$$h_\rho \leq \rho (\beta - 1) , \tag{15}$$

then the equation (1) *has solutions in* $\overline{B_\rho}$.

One can renounce to surjectivity hypothesis of $I - B$ replacing it by the condition

$$A\overline{B_\rho} \subset (I - B) X.$$

Consider now, as in Proposition 2 $M = \overline{B_\rho}$ and set

$$k_\rho := \sup_{x \in \overline{B_\rho}} \{ \|(I - A) x\| \} . \tag{16}$$

Corollary 3. *Let* $B : X \to X$ *be an expansive and surjective operator with* $B0 = 0$ *and* $I - A : M \to X$ *a continuous operator with* $(I - A) \overline{B_\rho}$ *compact.*

If

$$k_\rho \le \beta\rho, \tag{17}$$

then the equation (1) *has solutions in* $\overline{B_\rho}$.

Indeed, by (12) it follows for $y = 0$,

$$\left\|B^{-1}x\right\| \le \frac{1}{\beta}\left\|x\right\|.$$

Corollary 4. *Suppose that A satisfies the hypotheses of Corollary 3. Let $B : X \to X$ be a linear injective continuous Fredholm operator having null index. If*

$$\left\|B^{-1}\right\| \le \frac{\rho}{k_\rho}, \tag{18}$$

then the equation (1) *has solutions in* $\overline{B_\rho}$.

An interesting particular case of Corollary 3 is the following.

Corollary 5. *Suppose that*

i) $\dim X < \infty$;

ii) $A : \overline{B_\rho} \to X$ *is continuous;*

iii) $B : X \to X$ *is an expansive mapping with constant* $\beta > 1$.

Then, if the relation (17) *yields, the equation* (1) *admits solutions in* $\overline{B_\rho}$.

Indeed, as we remarked, by hypothesis i) it results B is homeomorphism. Applying to H given by (4) the fixed point theorem of Brouwer one obtains the result.

4. Results via Schaefer's theorem

In what follows we give a particular form of Schaefer's theorem which can be found in [5], p. 29.

Theorem S. (Schaefer) *Let* $(X, \|\cdot\|)$ *be a normed space,* H *be a continuous mapping of* X *into* X, *which maps bounded sets of* X *into compact sets. Then either*

I) the equation $x = \lambda Hx$ *has a solution for* $\lambda = 1$

or

II) the set of all such solutions x, *for* $0 < \lambda < 1$ *is unbounded.*

One can state now variants of Propositions 1, 2 by renouncing to completeness of X.

Proposition 3. *Let* $(X, \|\cdot\|)$ *be a normed space. Suppose that* *i)* $A : X \to X$ *is a continuous operator with* A *mapping bounded sets of* X *into compact sets*;

ii) $I - B$ *is a homeomorphism*;

iii) the set

$$\{x \in X,\ \ (\exists)\ \ \lambda \in (0,1)\,,\, x = \lambda H x\} \tag{19}$$

is bounded, where H *is given by* (3).

Then the equation (1) *admits solutions in* X.

Proposition 4. *Let* $(X, \|\cdot\|)$ *be a normed space. Suppose that*

i) $B : X \to X$ *is a homeomorphism;*

ii) $I - B$ *is a continuous operator with* $I - B$ *mapping bounded sets of* X *into compact sets*;

iii) the set (19), *where* H *is given by* (4), *is bounded.*

Then the equation (1) *admits solutions in* X.

5. Remarks

Reverting to Theorem K, let x be an arbitrary solution for (1) and y the unique fixed point of B.

Setting

$$a := \inf_{x \in M} \{\|Ax\|\}\,,\, b := \sup_{x \in M} \{\|Ax\|\}\,,$$

one has

$$\|x - y\| = \|Ax + Bx - By\| \le \|Ax\| + \alpha\,\|x - y\| \le b + \alpha\,\|x - y\|\,,$$

therefore

$$\|x - y\| \le \frac{b}{1 - \alpha}.$$

Similarly,

$$\|x - y\| \ge \|Ax\| - \alpha\,\|x - y\| \ge a - \alpha\,\|x - y\|\,,$$

hence

$$\|x - y\| \ge \frac{a}{1 + \alpha}.$$

Finally, between the unique fixed point of B and every fixed point of operator $A + B$ one has the relation

$$\frac{a}{1 + \alpha} \le \|x - y\| \le \frac{b}{1 - \alpha}. \tag{20}$$

If $B0 = 0$, then

$$\frac{a}{1+\alpha} \leq \|x\| \leq \frac{b}{1-\alpha},$$

relation true for each solution of equation (1).

6. Theorems of Krasnoselskii's type for a cartesian product of operators

In [1] the problem of the existence of solutions (x, y) for the system

$$\left\{ \begin{array}{l} x = F(x,y) \\ y = G(x,y) \end{array} \right. \tag{21}$$

One can make the same remarks like in previous sections. To obviate the repetition we deal only with the possibility of application Schaefer's theorem to the problem (21).

Let $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ be two Banach spaces and let $F : X_1 \times X_2 \to X_1$, $G : X_1 \times X_2 \to X_2$ be two operators.

We state and prove the following result.

Theorem A. *Suppose that*:

i) $F(x, y)$ *is continuous with respect to* y*, for every* $x \in X_1$ *fixed;*

ii)

$$\|F(x_1, y) - F(x_2, y)\|_1 \leq L \|x_1 - x_2\|_1, \text{ for all } x_1,\ x_2 \in X_1 \text{ and } y \in X_2,$$

with $L \in (0, 1)$;

iii) there exists a constant $C > 0$ *such that*

$$\|F(0, y)\|_1 \leq C \|y\|_2, \text{ for all } y \in X_2;$$

iv) $G(x, y)$ *is continuous on* $X_1 \times X_2$;

v) G *is a compact operator.*

Then either the system

$$\left\{ \begin{array}{l} x = F(x,y) \\ y = G(x,y) \end{array} \right.$$

admits a solution or the set of all such solutions for $\lambda \in (0, 1)$ *of the system*

$$\left\{ \begin{array}{l} x = \lambda F\left(\frac{x}{\lambda}, y\right) \\ y = \lambda G(x,y) \end{array} \right.$$

is unbounded.

Proof. Firstly we prove that if $\lambda \in (0,1]$ then $\lambda F\left(\frac{x}{\lambda}, y\right)$ is contraction mapping with respect to x, for every $y \in X_2$.

Indeed, if $x \in X_1$, then $\frac{x}{\lambda} \in X_1$.

Evaluate for x_1, $x_2 \in X_1$ and $y \in X_2$,

$$\begin{aligned} \left\|\lambda F\left(\frac{x_1}{\lambda}, y\right) - \lambda F\left(\frac{x_2}{\lambda}, y\right)\right\|_1 &= \lambda \left\|F\left(\frac{x_1}{\lambda}, y\right) - F\left(\frac{x_2}{\lambda}, y\right)\right\|_1 \leq \\ &\leq \lambda L \left\|\frac{x_1}{\lambda} - \frac{x_2}{\lambda}\right\|_1 = L\left\|x_1 - x_2\right\|_1. \end{aligned}$$

Consider $y \in X_2$ arbitrary. Denote by $g(y)$ the unique solution of equation

$$x = \lambda F\left(\frac{x}{\lambda}, y\right).$$

Therefore, for every $y \in X_2$ there exists an unique $g(y) \in X_1$ such that

$$g(y) = \lambda F\left(\frac{g(y)}{\lambda}, y\right).$$

Define $T : X_2 \to X_2$ by

$$Ty := K\left(g(y), y\right), \text{ for every } y \in X_2. \tag{22}$$

We show that the hypotheses of Schaefer's theorem are fulfilled.

Indeed, if $(y_n)_n$ is a sequence converging to y in X_2 as $n \to \infty$, then

$$\begin{aligned} \left\|g(y_n) - g(y)\right\|_1 &= \left\|\lambda F\left(\frac{g(y_n)}{\lambda}, y_n\right) - \lambda F\left(\frac{g(y)}{\lambda}, y\right)\right\|_1 \leq \\ &\leq \lambda \left\|F\left(\frac{g(y_n)}{\lambda}, y_n\right) - F\left(\frac{g(y)}{\lambda}, y_n\right)\right\|_1 + \\ &\quad + \lambda \left\|F\left(\frac{g(y)}{\lambda}, y_n\right) - F\left(\frac{g(y)}{\lambda}, y\right)\right\|_1 \\ &\leq \lambda L \left\|\frac{g(y_n)}{\lambda} - \frac{g(y)}{\lambda}\right\|_1 + \\ &\quad + \lambda \left\|F\left(\frac{g(y)}{\lambda}, y_n\right) - F\left(\frac{g(y)}{\lambda}, y\right)\right\|_1. \end{aligned}$$

Hence,

$$(1-L)\left\|g(y_n) - g(y)\right\|_1 \leq \lambda \left\|F\left(\frac{g(y)}{\lambda}, y_n\right) - F\left(\frac{g(y)}{\lambda}, y\right)\right\|_1$$

and $\lambda \left\|F\left(\frac{g(y)}{\lambda}, y_n\right) - F\left(\frac{g(y)}{\lambda}, y\right)\right\|_1 \to 0$ as $n \to \infty$ and the continuity of g follows immediately.

So, one has

$$\|Ty_n - Ty\|_2 = \|G(g(y_n), y_n) - G(g(y), y)\|_2$$

and since hypothesis iv) and $y = \lim_{n\to\infty} y_n$ it results the continuity of T.

Let $M_2 \in X_2$ be a bounded set. We prove that $TM_2 \subset X_2$ is compact.

Indeed, for $y \in M_2$ we have succesively

$$\begin{aligned} \|g(y)\|_1 &= \left\|\lambda F\left(\frac{g(y)}{\lambda}, y\right)\right\|_1 = \lambda \left\|F\left(\frac{g(y)}{\lambda}, y\right)\right\|_1 \leq \\ &\leq \lambda \left\|F\left(\frac{g(y)}{\lambda}, y\right) - F(0,y)\right\|_1 + \lambda \|F(0,y)\|_1 \leq \\ &\leq \lambda L \left\|\frac{g(y)}{\lambda} - 0\right\|_1 + \lambda \|F(0,y)\|_1 \leq \\ &\leq L\|g(y)\|_1 + \lambda C \|y\|_2 . \end{aligned}$$

It results

$$\|g(y)\|_1 \leq \frac{\lambda C}{1-L}\|y\|_2, \text{ for all } y \in M_2.$$

Therefore the set $g(M_2)$ is bounded in X_1. Since the set $g(M_2) \times M_2$ is bounded in $X_1 \times X_2$ and from hypothesis v) it follows that the set

$$T(M_2) = G(g(M_2) \times M_2)$$

is compact in X_2.

By applying Schaefer's theorem one gets either the equation

$$y = Ty$$

admits solutions in X_2 or the set of all solutions for $\lambda \in (0,1)$ of the equation

$$y = \lambda Ty$$

is unbounded.

Equivalently, either the system

$$\begin{cases} x = F(x,y) \\ y = G(x,y) \end{cases}$$

admits a solution $(g(y_0), y_0)$ or the set of all such solutions $(g(y_0), y_0)$ for $\lambda \in (0,1)$ of the system

$$\begin{cases} x = \lambda F\left(\frac{x}{\lambda}, y\right) \\ y = \lambda G(x,y) \end{cases}$$

is unbounded. □

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