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REMARKS ABOUT A BREZIS-BROWDER PRINCIPLE

MIHAI TURINICI

Seminarul Matematic "Al.Myller"
Universitatea "Al.I.Cuza"
11, Copou Boulevard, 6600 Iaşi, Romania
E-mail address: mturi@uaic.ro

Abstract. A lot of ordering principles comprising the Brezis-Browder one [6] is actually reductible to (hence equivalent with) the quoted principle. Likewise, the solvability results deduced via these statements are shown to be equivalent with some contributions in this area due to Altman [1,ch.5,Sect.2].
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1. Introduction

Let M be a nonempty set and $\leq$, some *quasi-ordering* (i.e., a reflexive and transitive relation) over it. Also, let $x \vdash \varphi(x)$ be a function from M to $[0, \infty[$. Call the point z in M, $(\leq, \varphi)$-*maximal*, in case
(1D1) $w \in M,\ z \leq w \Longrightarrow \varphi(z) = \varphi(w)$.
(This may be compared with the property of being (strongly) $(\leq)$-*maximal*
(1D2) $w \in M,\ z \leq w \Longrightarrow z = w$.
Note that, in general, these are distinct ones). Concerning the existence of such points, the following ordering principle in Brezis and Browder [6] is basic to considerations below.

Theorem 1. *Suppose that*
(1H1) *each ascending sequence in M has an upper bound*
(1H2) $x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)$ *(i.e.: φ is decreasing).*
Then, for each $x \in M$ there exists a $(\leq, \varphi)$-maximal element $z \in M$ with $x \leq z$. So, if in addition, the pair $(\leq, \varphi)$ satisfies

(1H3) $x, y \in M,\ x \leq y,\ \varphi(x) = \varphi(y) \Longrightarrow x = y,$
the $(\leq, \varphi)$*-maximal element* z *above is* $(\leq)$*-maximal too; hence* $(\leq)$ *is a Zorn quasi-order.*

A basic particular case of this result may be described as follows. Let (X, d) be a *complete* metric space; and $\psi : X \to [0, \infty[$, a *lower semicontinuous* (in short: lsc) function. The relation introduced as

(1D3) $x \leq_{(\psi)} y$ if and only if $d(x, y) \leq \psi(x) - \psi(y)$

is an *order* (i.e.: antisymmetric quasi-order) which also fulfils (1H1); and, moreover, (1H2)+(1H3) hold too (with ψ in place of φ). We thus derived

Theorem 2. *The following are valid*

(1.1) $\leq_{(\psi)}$ *is a Zorn ordering (cf. the convention above)*

(1.2) *if the selfmap* T *of* X *fulfils evaluations like*

(1H4) $d(x, Tx) \leq \psi(x) - \psi(Tx),$ *for each* $x \in X,$

then T *has at least one fixed point in* X.

Now, the first part of this is nothing but the variational Ekeland's principle [9]; while the second one is the Caristi-Kirk fixed point theorem [7]. Remember that, the statements above found some interesting applications to (nonlinear) *mapping theory* (cf. Kirk and Caristi [12]) or the *normal solvability theory* (as developed by Downing and Ray [8]). Therefore, an extension of these is not only useful from a theoretical perspective, but also from a practical one. For example, a lot of such enlargements of Theorem 1 was obtained in the papers by Altman [2] and Turinici [16]; see also Anisiu [4] or Kang and Park [11]. A recent contribution in this area is the one obtained in the paper by Bae, Cho and Yeom [5]. It is our aim in the following to show that, in the last analysis, this is *equivalent* with the Brezis-Browder ordering principle (subsumed to Theorem 1); details will be given in Section 2. Further, in Section 3, we show that the fixed point result (extending Theorem 2) established (via the quoted principle) by these authors is *equivalent* with the Caristi-Kirk fixed point theorem. Finally, in Section 4, the impact of these conclusions on the solvability results obtained by the same authors is analyzed. Some other aspects will be discussed elsewhere.

2. Main result

Let M be a nonempty set and $\triangleleft$, some *reflexive* relation over it. Also, let $\varphi : M \to [0, \infty[$ be a function. Call the point $z \in M$, $(\triangleleft, \varphi)$*-maximal*, if

(2D1) $w \in M, z \triangleleft w \Longrightarrow \varphi(z) = \varphi(w)$.

(This may be compared with the property of being (strongly) ($\triangleleft$)-*maximal*

(2D2) $w \in M, z \triangleleft w \Longrightarrow z = w$.

Note that, in general, these are distinct notions). To establish under which conditions such elements exist, we have to introduce some conventions. Call the sequence (x_n) in M, ($\triangleleft$)-*ascending* when

(2D3) $x_n \triangleleft x_{n+1}$, for all ranks n.

Also, we say that $y \in M$ is an *asymptotic* ($\triangleleft$)-*upper bound* of (x_n) provided

(2D4) for each n there exists $m \geq n$ with $x_m \triangleleft y$.

We are now in position to give an appropriate answer to the precised question.

Theorem 3. *Suppose that*

(2H1) $\begin{cases} \text{each } (\triangleleft)\text{-ascending sequence in } M \text{ has an} \\ \text{asymptotic } (\triangleleft)\text{-upper bound} \end{cases}$

(2H2) $x \triangleleft y \Longrightarrow \varphi(x) \geq \varphi(y)$ *(i.e.: φ is ($\triangleleft$)-decreasing).*

Then, for each $x \in M$ there exists a $(\triangleleft, \varphi)$-maximal element $z \in M$ with

(2.1) $x \triangleleft u_1 \triangleleft ... \triangleleft u_k \triangleleft z$, *for some $u_1, ..., u_k$ in M and some $k \geq 1$.*

So, if in addition, the pair $(\triangleleft, \varphi)$ satisfies

(2H3) $x, y \in M,\ x \triangleleft y,\ \varphi(x) = \varphi(y) \Longrightarrow x = y$,

the $(\triangleleft, \varphi)$-maximal element z above is ($\triangleleft$)-maximal too; and then, ($\triangleleft$) is a Zorn type relation.

The second part of this statement is just the relational principle in Bae, Cho and Yeom [5]. Moreover, as explicitly stated in that paper, their contribution extends Theorem 1; hence, so does Theorem 3. But, in this case, the question arises of which is the effectiveness of this extension. The answer is contained in

Proposition 1. *We have*

(2.2) *Theorem 1 $\Longrightarrow$ Theorem 3.*

Hence (cf. the above) these two statements are logically equivalent to each other.

Proof. Let $(\leq)$ be the quasi-order on M introduced by the convention

(2D5) $\begin{cases} x \leq y \text{ if and only if } x \triangleleft v_1 \triangleleft ... \triangleleft v_h \triangleleft y, \\ \text{for some } v_1, ..., v_h \in M \text{ and some } h \geq 1. \end{cases}$

From (2H1)+(2H2), it is not hard to see that conditions (1H1)+(1H2) are fulfilled for the pair $(\leq, \varphi)$. So, by the conclusion of Theorem 1 (the first half), it follows that for each $x \in M$ there exists a $(\leq, \varphi)$-maximal element $z \in M$ with $x \leq z$. And this, along with the generic inclusion

(2.3) $(\leq, \varphi)$- maximal $\Longrightarrow$ $(\triangleleft, \varphi)$-maximal

establishes the first half of the conclusion in Theorem 3. For the second half of the same, it will sufice noting that

(2.4) (2H3) $\Longrightarrow$ (1H3), in the context of (2H2).

So, by the conclusion of Theorem 1 (the second half), the $(\leq, \varphi)$-maximal element z above is even $(\leq)$-maximal. And this, combined with the generic inclusion

(2.5) $(\leq)$-maximal $\Longrightarrow$ $(\triangleleft)$-maximal

ends the argument. The proof is complete. □

As a consequence, the statement in Bae, Cho and Yeom [5] is *logically equivalent* with the Brezis-Browder ordering principle. This may have a theoretical impact upon it; but, from a practical perspective, the situation is different. Finally, we must note that some other versions of Theorem 1 – distinct from the above one – were obtained by Altman [2] and Anisiu [4]; see also Kang and Park [11]. But, as precised in Turinici [17], these are *logically equivalent* with Theorem 1. For non-sequential extensions of this type we refer to Isac [10] and the references therein.

3. Caristi-Kirk statements

Let (X, d) be a complete metric space; and $x \vdash Tx$, some selfmap of X. The following fixed point result established in Bae, Cho and Yeom [5] is our starting point.

Theorem 4. *Suppose that, for some lsc function* $\varphi : X \to [0, \infty[$ *and some function* $c : [0, \infty[\to [0, \infty[$ *with*

(3H1) *c is upper semicontinuous (in short: usc) over its domain*

one has evaluations like

(3H2) $d(x, Tx) \leq \max\{c(\varphi(x)), c(\varphi(Tx))\}(\varphi(x) - \varphi(Tx)), \quad x \in X.$

Then, T has at least one fixed point in X.

The (rather involved) proof of this result is based on Theorem 3 above (the variant with (2H3) being accepted). In particular, when

(3D1) $c(t) = \gamma$, *for all* $t \geq 0$ *and some* $\gamma > 0$

this statement is nothing but the (standard) Caristi-Kirk fixed point theorem. Now, in the light of the developments made in Section 2, is is natural to ask *of whether or not is this extension effective. As we shall see below, the answer is negative; this will remain valid even for a certain counterpart of Theorem 4 which includes it as a particular case. Precisely, let* $H : [0,\infty[^2 \to [0,\infty[$ *be a function. (The objects of this type, with a practical relevance for us, correspond to the choices*

(3D2) $H(t,s) = \max(t,s), \quad H(t,s) = \min(t,s), \quad t,s \geq 0.$

But, these are not the only possible ones). Further, let θ *be arbitrary fixed in* $]0,\infty[$*; and* $h : [0,\theta] \to [0,\infty[$ *be a function with*

(3H3) h *is continuous and increasing on* $[0,\theta]$*.*

We say that the function $c : [0,\infty[\to [0,\infty[$ *is* (H,h)*-proper on* $[0,\theta]$*, if*

(3D3) $H(c(t),c(s)) \leq \dfrac{h(t)-h(s)}{t-s}$*, for all* (t,s) *with* $0 < s < t < \theta$*.*

When h *(taken as in (3H3)) is generic, we shall say that* c *is* H*-proper on* $[0,\theta]$*; note that this last property is hereditary with respect to* θ*.*

Having these precised, the following fixed point result is naturally coming into our discussion.

Theorem 5. *Suppose that, for some lsc function* $\varphi : X \to [0,\infty[$*, some function* $H : [0,\infty[^2 \to [0,\infty[$ *and some function* $c : [0,\infty[\to [0,\infty[$ *with*

(3H4) c *is* H*-proper on* $[0,\theta]$*, for all* $\theta > 0$

one has evaluations like

(3H5) $d(x,Tx) \leq H(c(\varphi(x)), H(c(\varphi(Tx)))(\varphi(x) - \varphi(Tx)), \quad x \in X.$

Then, T *has at least one fixed point in* X*.*

Concerning the relationships with the preceding statement, note that a condition like

(3H6) $\begin{cases} c \text{ is locally bounded on } [0,\infty[\\ (\mu(c,\theta) = \sup\{c(t); 0 \leq t \leq \theta\} < \infty, \text{ for each } \theta > 0) \end{cases}$ *implies*

(3H4) (with $H = \max$*); because, for each* $\theta > 0$*, (3D3) holds with*

(3D4) $h(t) = \mu(c,\theta)t, \quad t \in [0,\theta].$

This, along with (3H1) $\Longrightarrow$ *(3H6), shows that Theorem 4 is a particular case of this statement. On the other hand, a condition like (3H7)* c *is decreasing on* $]0,\infty[$ *and* $h(t) = \int_0^t c(s)ds < \infty, \quad \forall t \geq 0$

also gives (3H4) (with $H = \min$*); the proof being straightforward, we do not give details. It is worth remarking at this moment that the written condition is not reductible to (3H6) above. For, e.g., the function*

(3D5) $\begin{cases} c(t) &= \textit{arbitrary}, \quad \textit{for} \quad t = 0, \\ &= t^{-1/2}, \qquad \textit{for} \quad t > 0 \end{cases}$

fulfills evidently (3H7) (with $h(t) = 2t^{1/2}, t \geq 0$*) but not (3H6); hence the claim. So, technically speaking, Theorem 5 appears as a strict extension of Theorem 4. But, from a logical viewpoint, the situation is opposite. This is precised in*

Proposition 2. *We have*

(3.1) *Theorem 2* $\Longrightarrow$ *Theorem 5 (*$\Longrightarrow$ *Theorem 4).*

Hence, these atatements are logically equivalents to each other.

Proof. Clearly, all we have to do is to verify the former implication. Let the premises of Theorem 5 be realized. Fix some $\theta > \inf \varphi(X)$; note that (by definition)

(3.2) $\varphi(x_0) < \theta$, for at least one $x_0 \in X$.

Denote for simplicity

(3D5) $X(x_0) = \{x \in X; \varphi(x) \leq \varphi(x_0)\}$ (where x_0 is as above).

By the lsc assumption about φ, this subset is (nonempty and) closed (hence complete). Suppose (by contradiction) that

(3H8) T has no fixed point in $X(x_0)$.

As a direct consequence of this, we must have

(3.3) $H(c(\varphi(x)), c(\varphi(Tx))) > 0, \quad$ for all $x \in X(x_0)$.

(Because, any point with the opposite property is, by (3H5), fixed under T, contradiction). This, in turn, yields a relation like

(3.4) $\varphi(x) \geq \varphi(Tx)(\geq 0), \quad$ for each $x \in X(x_0)$;

or, in other words, T is a *selfmap* of $X(x_0)$. Moreover, either of the inequalities above must be strict; i.e.,

(3.5) $\varphi(x) > \varphi(Tx)(> 0), \quad$ for each $x \in X(x_0)$.

[For otherwise, if $\varphi(x_1) = \varphi(Tx_1)$, then x_1 is fixed under T, by (3H5); and, if $\varphi(Tx_2) = 0$, then Tx_2 is fixed under T by the selfmap property, (3.4) and (3H5). Hence the claim]. Let $t \vdash h(t)$ be the continuous increasing function from $[0, \theta]$ to $[0, \infty[$ given by (3H4) and put

(3D7) $\psi(x) = h(\varphi(x)), \quad x \in X(x_0)$.

This function is easily seen to be lsc over $X(x_0)$. Moreover, by (3H4)+(3H5) one has (via (3.5))

(3.6) $d(x, Tx) \leq \psi(x) - \psi(Tx), \quad$ for all $x \in X(x_0)$.

This shows that Theorem 2 is applicable to these data. And, from its conclusion, T has at least one fixed point in $X(x_0)$, in contradiction to (3H8). The proof is complete. □

In particular, this shows that the (subsumed to Theorem 4) fixed point result in Bae, Cho and Yeom [5] is but a reformulation of the Caristi-Kirk fixed point theorem. This fact, aside from simplifying its proof, gives us the exact place of Theorem 4 within this series of statements. For a number of related aspects we refer to the paper by Ray and Walker [15]; see also Park and Bae [13].

4. Applications to mapping theory

Let X be an abstract set and $(Y, ||.||)$ a Banach space. Take a mapping $x \vdash P(x)$ from X to Y. Loosely speaking, the objective of the (nonlinear) *mapping theory* is to determine those (infinitesimal type) conditions upon our data so that a certain $y \in Y$ be in the *range* of P; or, equivalently (by a suitable translation) that the operator equation

(OE) $P(x) = 0$ (=the null element of Y)

should have a solution in X. Some basic results in this direction were obtained under the regularity assumption

(4H1) P has a *closed range* ($P(X)$ is closed in Y).

[The *closed graph* case - which necessitates a topological structure upon X - will be not discussed here]. For example, in the Altman's monograph [1,ch.5,Sect.2], the following answer to the posed question is given (by means of a transfinite induction argument).

Theorem 6. *Suppose that a constant* γ *in* $[0, 1[$ *may be found so as: for each* $x \in X$ *there exist* $x' \in X$ *and* ε *in* $]0, 1]$ *with*

(4H2) $||P(x') - (1 - \varepsilon)P(x)|| \leq \varepsilon\gamma||P(x)||$.

Then, (OE) *has at least on solution in* X.

A different way of proving this result is the one described by Theorem 2; see, for instance, Kirk and Caristi [12]. Further extensions of it were given by Altman [3], Turinici [18] and in the references therein. Here, we shall concentrate on the following version of the statement above. (The regularity condition (4H1) prevails).

Theorem 7. *Suppose that, a function* $c : [0, \infty[\to [0, 1[$ *with*

(4H3) $\mu(c, \theta) < 1$*, for each* $\theta > 0$ *(cf.* (3H6))

may be found so as: for each $x \in X$, there exists $x' \in X$ and ε in $]0,1]$ with
(4H4) $||P(x') - (1-\varepsilon)P(x)|| \le \varepsilon \max\{c(||P(x)||), c(||P(x')||)\}||P(x)||$.
Then, conclusion of Theorem 6 is retainable.

In particular, (4H3) holds under (3H1). Hence, this result includes the contribution in this area due to Bae, Cho and Yeom [5], obtained via Theorem 4. On the other hand, if the function $t \vdash c(t)$ is taken as in (3D1), Theorem 7 reduces to the Altman's result (subsumed to Theorem 6). Now, by the developments in the preceding section, one may conjecture that the converse implication must be true as well. It is our objective to show that this is indeed the case.

Proposition 3. *We have*
(4.1) *Theorem 6 $\Longrightarrow$ Theorem 7.*
Hence, these results are logically equivalent to each other.

Proof. Fix some x_0 in X and put
(4D1) $X_0 = \{x \in X; ||P(x)|| \le \tau\}$, where $\tau = ||P(x_0)||$.
Clearly, the closed range assumption (4H1) remains true with X_0 in place of X. Denote also
(4D2) $\gamma = \mu(c, \tau)$ (where τ is the above one).
Let x be arbitrary fixed in X_0. By the (accepted) premises of Theorem 7, there must be some x' in X and ε in $]0,1]$ such that (4H4) be valid. As a consequence,
(4.2) $||P(x') - (1-\varepsilon)P(x)|| \le \varepsilon||P(x)||$ (if we take (4H3) into account);
and this, in turn, yields
(4.3) $||P(x')|| \le (1-\varepsilon)||P(x)|| + \varepsilon||P(x)|| = ||P(x)||$ $(\le \tau)$.
Hence, the point x' given by (4H4) belongs to X_0. This, again by the quoted relation gives (cf. (4D2) above)
(4.4) $||P(x') - (1-\varepsilon)P(x)|| \le \varepsilon\gamma||P(x)||$.
Summing up, Theorem 6 applies to our data; and, from this, conclusion is clear. □

In other words, the (nonlinear) mapping result in Bae, Cho and Yeom [op.cit.] is equivalent with Altman's. For a number of related aspects we refer to Ray [14] and the references therein.

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