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A NOTE ON PEROV'S FIXED POINT THEOREM

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Abstract. Perov's theorem states that if $(X,\, d)$ is a generalized complete metric space (the metric is $d : X \times X \to R^n$) and the operator $T : X \to X$ satisfies the inequallity $d(T(x),\, T(y)) \leq A \cdot d(x,\, y)$ for all $x,\, y \in X$, where A is a matrix convergent to zero (it's eigenvalues are in the interior of the unit disc), then the operator T is a Picard operator. In [1] we have generalized the Banach fixed point theorem replacing the Lipschitz condition by the following more general (so called convex contraction condition, see [4]) metric condition

$$d(T^p x, T^p y) \leq \sum_{j=0}^{p-1} \alpha_j \cdot d(T^j x, T^j y) \text{ for all } x, y \in X$$

where $(a_j)_{j=1,n}$ are fixed numbers and $\sum_{j=0}^{p-1} \alpha_j < 1$. In this short note we answer the problem 1 from [1] by proving that this convex contraction condition guaranties the Picard quality of the operator in a generalized metric space too if $(a_j)_{j=1,n}$ are fixed matrices and $\sum_{j=0}^{p-1} \|\alpha_j\| < 1$ with an arbitrary matrix norm.

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1. Perov's fixed point theorem

Let (X, d) be a generalized complete metric space and $T : X \to X$ an operator. The metric $d : X \times X \to R^n$ has the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (the inequalities are defined by components in R^n).

The Perov fixed point theorem states that if $d(Tx, Ty) \leq A \cdot d(x, y)$ for all $x, y \in X$, where A is a matrix convergent to zero, then T has an unique fixed point and this can be obtained by succesive approximation (so T is a Picard operator).

In [1] we've introduced the following notions:

1. The sequence $(a_n)_{n\geq 1}$ is subconvex of order p if $a_{n+p} \leq \sum\limits_{i=0}^{p-1} \alpha_i \cdot a_{n+i}$ for all $n \geq 1$ where $\alpha_i \in (0,1)$ for $i = \overline{0, p-1}$ and $\sum\limits_{i=0}^{p-1} \alpha_i \leq 1$.

2. A sequence $(a_n)_{n\geq 1}$is subconvex if there exists $p \geq 1$ such that the sequence is subconvex of order p.

3.The sequence $(a_n)_{n\geq 1}$ is a convex sequence if there exist a natural number $p \geq 1$ such that $a_{n+p} = \sum\limits_{j=0}^{p-1} \alpha_j \cdot a_{n+j} \quad \forall n \geq 1$ where $\alpha_i \in (0,1)$ for $i = \overline{0, p-1}$ and $\sum\limits_{i=0}^{p-1} \alpha_i = 1$.

and we've proved the following theorem:

Theorem 1. *a) Every positive subconvex sequence is convergent.*

b) If $\sum\limits_{i=0}^{p-1} \alpha_i < 1$ and the sequence $(a_n)_{n\geq 1}$ satisfies the relations $a_{n+p} \leq \sum\limits_{j=0}^{p-1} \alpha_j \cdot a_{n+j}$ for all $n \geq 1$ where $\alpha_i \in (0,1)$ for $i = \overline{0, p-1}$, then $(a_n)_{n\geq 1}$ is convergent to 0 and $\sum\limits_{k=1}^{\infty} a_i$ is convergent.

The next theorem is a generalization of Perov's fixed point theorem.

Theorem 2. *If (X, d) is a generalized complete metric space and $T : X \to X$ an operator which satisfies the condition $d(T^p x, T^p y) \leq \sum\limits_{j=0}^{p-1} \Lambda_j \cdot d(T^j x, T^j y)$ for all $x, y \in X$ and $\sum\limits_{j=0}^{p-1} \|\Lambda_j\|_m < 1$ with an arbitrary matrix norm $\|\cdot\|_m$ (which is subordinated to a vector norm $\|\cdot\|_v$ on R^n), then*

a) T has an unique fixed point x^*

b) the sequence $(x_n)_{n\geq 1}$defined with $x_{n+1} = T(x_n)$ is convergent to x^* for all $x_0 \in X$.

c)$\|d(x^*, x_n)\|_v \leq \sum\limits_{j=0}^{\infty} c_j$, where $c_{n+p} = \sum\limits_{j=0}^{p-1} \|\Lambda_j\|_m \cdot c_{n+j}$ for all n≥ 1 and $c_j = \left\|d\left(T^{j+1}x, T^j x\right)\right\|_v$ for $0 \leq j \leq p-1$.

Proof of theorem 2

The sequence $a_n = \left\|d\left(T^{n+1}x, T^n x\right)\right\|_v = \|d\left(x_n, x_{n+1}\right)\|_v$ is strictly subconvex because $a_{n+p} \leq \sum\limits_{j=0}^{p-1} \|\Lambda_j\|_m \cdot a_{n+j}$ with $\sum\limits_{j=0}^{p-1} \|\Lambda_j\|_m < 1$. Due to theorem 1, $\lim\limits_{n\to\infty} a_n = 0$ and $\sum\limits_{n=0}^{\infty} a_n$ is convergent. From this we deduce $\sum\limits_{n=0}^{\infty} d\left(x_n, x_{n+1}\right)$ is convergent and so there exist n_ε such that

$$d(x_{n+m}, x_m) = d\left(T^{m+n}x, T^m x\right) \leq \sum_{j=0}^{n-1} d\left(T^{m+j}x, T^{m+j+1}x\right) =$$
$$\sum_{j=0}^{n-1} d\left(x_{m+j}, x_{m+j+1}\right) \leq \varepsilon \text{ if } m \geq n_\varepsilon.$$

Hence the sequence $(x_n)_{n\geq 1}$ is Cauchy sequence. X is a complete metric space, so there exist x^* such that $\lim\limits_{n\to\infty} x_n = x^*$. Using $\lim\limits_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0$ we deduce that x^* is a fixed point for T. Taking m fixed in the above inequality and $n \to \infty$ we deduce that

$\|d(x^*, x_m)\|_v \leq \sum\limits_{j=0}^{\infty} c_j$, where $c_{n+p} = \sum\limits_{j=0}^{p-1} \|\Lambda_j\|_m \cdot c_{n+j} \quad \forall n \geq 1$ and $c_j = \left\|d\left(T^{j+1}x, T^j x\right)\right\|_v$ for $0 \leq j \leq p-1$.

x^* is a fixed point for T. By the other hand the given relation implies that T can't have more than one fixed point, so the theorem is proved.

2. An application

In iterative numerical solutions of a linear algebraic system it is often used the Banach fixed point theorem. Using theorem 2 we have the following conditions on the convergence of an iterative method:

Theorem 3. *If $Q \in M_n(R)$ is a matrix, α is a positive number such that $\left\|Q^2 - \alpha Q\right\|_m < 1 - \alpha$ then the sequence $x_{n+1} = b + Q \cdot x_n$ with $x_0 \in R^n$ is convergent to the unique solution of the system $(I_n - Q)x = b$.*

Proof. Let's consider the operator $T : R^n \to R^n$ defined by $T(x) = b + Q \cdot x$. $T(T(x)) = b + Q \cdot b + Q^2 \cdot x$, so $T^2(x) - T^2(y) = Q^2(x - y)$ and

$$\left\|T^2(x) - T^2(y)\right\|_v = \left\|Q^2(x-y)\right\|_v \leq \left\|(Q^2 - \alpha Q)(x-y)\right\|_v + \|\alpha Q(x-y)\|_v \leq$$
$$\leq \left\|Q^2 - \alpha Q\right\|_m \|x-y\|_v + \alpha \cdot \|T(x) - T(y)\|_v <$$
$$(1-\alpha)\|x-y\|_v + \alpha \cdot \|T(x) - T(y)\|_v$$

From this we deduce that the operator T is a Picard operator, so theorem 3 is true.

Remark. If $Q = \begin{bmatrix} 1/2 & -2/3 \\ 2/3 & 1/2 \end{bmatrix}$ and $\alpha = 1/8$ we have

$$Q^2 - \alpha Q = \begin{bmatrix} -37/144 & -7/12 \\ 7/12 & -37/144 \end{bmatrix}$$

so our theorem can be applied using the Minkovski matrix-norm. Indeed $\left\|Q^2 - \alpha Q\right\| = 121/144 < 7/8$. Using the same norm we have $\|Q\| = 7/6 > 1$ so the Banach fixed point theorem can't be applied and neither the Perov fixed point theorem, if we use this norm. In most applications it isn't used the euclidian matrix metric (because it needs the eigenvalues).

References

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