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THE FIXED POINT ALTERNATIVE AND THE STABILITY OF FUNCTIONAL EQUATIONS

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Abstract. In this paper, we show that the theorems of Hyers, Rassias and Gajda concerning the stability of the Cauchy's functional equation in Banach spaces, are direct consequences of the alternative of fixed point.

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1. Introduction

The study of stability problems for functional equations originated from a question of S. M. Ulam concerning the stability of group homomorphisms:

Let G be a metric group with a metric d. Given $\varepsilon > 0$, does there exist a $k > 0$ such that ***if*** *a function $f : G \to G$ satisfies the inequality*

$$d(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon, \forall x, y \in G,$$

then *there exists an automorphism a* of *G with*

$$d(f(x), a(x)) < k\varepsilon, \forall x \in G \quad ?$$

D. H. Hyers gave an affirmative answer to the question of Ulam, for Banach spaces (see e.g. [6,11]). Th. M. Rassias in [10] and Gajda in [3] considered the stability problem with unbounded Cauchy differences. The unified form of these results is the following

Theorem (Hyers-Rassias-Gajda). *Suppose that E is a real normed space, F is a real Banach space and $f : E \to F$ is a given function, such that the following condition holds*

$$(\mathbf{1_p}) \qquad \|f(x+y) - f(x) - f(y)\|_F \leq \theta(\|x\|_E^p + \|y\|_E^p), \forall x, y \in E,$$

for some $p \in [0,\infty)\backslash\{1\}$. *Then there exists a unique additive function* $a : E \to F$ *such that*

$$(\mathbf{2_p}) \qquad \|f(x) - a(x)\|_F \leq \frac{2\theta}{|2-2^p|}\|x\|_E^p, \forall x \in E$$

This stability phenomenon is called *generalized Hyers-Ulam stability* and has been extensively investigated for different functional equations. It is to be noted that almost all proofs used the idea imaginated by D. H. Hyers. Namely, the additive function $a : E \to F$ is explicitly constructed, starting from the given function f, by the formulae

$$\begin{array}{llll} (\mathbf{2_{p<1}}) & a(x) & = & \lim\limits_{n\to\infty} \dfrac{1}{2^n} f(2^n x), \text{ if } p<1; \\ (\mathbf{2_{p>1}}) & a(x) & = & \lim\limits_{n\to\infty} 2^n f(\dfrac{x}{2^n}), \quad \text{if } p>1. \end{array}$$

This method is called a *direct method*. It is often used to construct a solution of a given functional equation and is seen to be a powerful tool for studying the stability of many functional equations (see e.g. [6]).

The aim of this note is to show that *the existence of the limit* $a(x)$ *and the estimation* $(\mathbf{2_p})$ can be simply obtained from the *alternative of fixed point.*

2. The contraction principle and the alternative of fixed point

For the reader's convenience and explicite later use, I will recall two fundamental results in fixed point theory.

2.1 Theorem (Banach's contraction principle). *Let* (X,d) *be a complete metric space, and consider a mapping* $A : X \to X$, *which is strictly contractive, that is*

$$(\mathbf{3}) \qquad d(Ax, Ay) \leq L d(x,y), \forall x, y \in X,$$

for some (Lipschitz constant) $L < 1$. *Then*

i) The mapping A *has one, and only one, fixed point* $x^* = A(x^*)$;

ii) The fixed point x^* *is globally attractive, that is*

$$(\mathbf{4}) \qquad \lim_{n\to\infty} A^n x = x^*,$$

for any starting point $x \in X$;

iii) One has the following estimation inequalities:

$$(\mathbf{5}) \qquad d(A^n x, x^*) \leq L^n d(x, x^*), \forall n \geq 0, \forall x \in X;$$

$$(\mathbf{6}) \quad d\left(A^n x, x^*\right) \leq \frac{1}{1-L} d(A^n x, A^{n+1} x), \forall n \geq 0, \forall x \in X;$$

$$(\mathbf{7}) \qquad d\left(x, x^*\right) \leq \frac{1}{1-L} d\left(x, Ax\right), \forall x \in X.$$

2.2 Theorem (The alternative of fixed point [9], see also[12, ch. 5]). *Suppose we are given a complete generalized metric space (X, d) -i.e. one for which d may assume infinite values- and a strictly contractive mapping $A : X \to X$, with the Lipschitz constant L. Then, for each given element $x \in X$,* **either**

(A_1) $d(A^n x, A^{n+1} x) = +\infty \quad , \ \forall n \geq 0$, **or**

(A_2) *There exists a natural number n_0 such that*

(A_{20}) $d(A^n x, A^{n+1} x) < +\infty, \forall n \geq n_0$ *;*

(A_{21}) *The sequence $(A^n x)$ is convergent to a fixed point y^* of A ;*

(A_{22}) *y^* is the unique fixed point of A in $Y = \{y \in X, \ d\left(A^{n_0} x, y\right) < +\infty\}$;*

(A_{23}) $\quad d\left(y, y^*\right) \leq \frac{1}{1-L} d\left(y, Ay\right), \forall y \in Y.$

2.3 Remark. (a) The fixed point y^*, if it exists, is not necessarily unique *in the whole space X*;

(b) Actually, if (A_2) holds, then (Y, d) is a complete *metric* space and $A\left(Y\right) \subset Y$. Therefore the properties $(A_{21}) - (A_{23})$ are easily seen to follow from **Theorem 2.1**.

3. A proof of Hyers-Rassias-Gajda stability theorem

Let us consider the set

$$X := \{g : E \to F, \ p \cdot g\left(0\right) = 0\}$$

One can introduce a *generalized metric* $d = d_p : X \times X \to [0, +\infty]$ by the formula

$$d_p\left(g, h\right) = \sup_{x \neq 0} \frac{\left\|g\left(x\right) - h\left(x\right)\right\|_F}{\left\|x\right\|_E^p}$$

It is well-known and easy to prove that (X, d) *is complete.*

Now we will consider the (linear) mapping

$$A : X \to X, \ Ag\left(x\right) := \frac{1}{q} g\left(qx\right)$$

where $q = 2$ if $p < 1$, and $q = 2^{-1}$ if $p > 1$. We can write, for any $g, h \in X$:

$$\begin{aligned}\frac{\|Ag(x)-Ah(x)\|_F}{\|x\|_E^p} &= \frac{1}{q}\frac{\|g(qx)-h(qx)\|_F}{\|x\|_E^p} = \\ q^{p-1}\frac{\|g(qx)-h(qx)\|_F}{\|qx\|_E^p} &\leq q^{p-1}d(g,h).\end{aligned}$$

Therefore

$$d(Ag,Ah)\leq q^{p-1}d(g,h), \forall g,h\in X,$$

that is A is a *strictly contractive* selfmapping of X, with the Lipschitz constant $L=q^{p-1}$. If we set $y=x$ in the hypothesis (1_p), then we see that

$$\|2f(x)-f(2x)\|\leq 2\theta\|x\|^p, \forall x\in E,$$

which says that

$$\begin{aligned}d(f,Af) &\leq \theta<\infty, \text{ for } p<1, \text{ and}\\ d(f,Af) &\leq \frac{\theta}{2^{p-1}}<\infty, \text{ for } p>1\end{aligned}$$

From the alternative of fixed point, we see the existence of a mapping $a:X\to X$ such that:

1^0 a is a fixed point of A, that is

$$(8) \qquad a(2x)=2a(x), \forall x\in E$$

The mapping a is the unique fixed point of A in the set

$$Y=\{g\in X,\ d(f,g)<+\infty\}.$$

This says that a is the unique mapping $g:E\to F$ with *both* the properties $(8)-(9)$, where

$$(9) \quad \exists c\in(0,\infty) \text{ such that } \|g(x)-f(x)\|_F\leq c\|x\|_E^p,\ \forall x\in E.$$

2^0 $d(A^nf,a)\to 0$, which implies that

$$(10) \qquad \lim_{n\to\infty}\frac{f(q^nx)}{q^n}=a(x), \quad \forall x\in X.$$

Therefore at least one of the statements $(\mathbf{2_{p<1}})$ or $(\mathbf{2_{p>1}})$ is seen to be true.

3^0 $d(f,a)\leq\frac{1}{1-q^{p-1}}d(f,Af)\leq\frac{2\theta}{|2-2^p|}$, that is the inequality $(\mathbf{2_p})$ takes place.

Now, as it is well-known, the additivity of a follows immediately from $(\mathbf{1_p})$ and (10) : If in $(\mathbf{1_p})$ we replace x by $q^n x$ and y by $q^n y$, then we obtain

$$\left\| \frac{f(q^n (x+y))}{q^n} - \frac{f(q^n x)}{q^n} - \frac{f(q^n y)}{q^n} \right\|_F \leq L^n \theta(\|x\|_E^p + \|y\|_E^p), \forall x, y \in E,$$

and, letting $n \to \infty$, we get

$$a(x+y) = a\,(x) + a\,(y)\,, \forall x, y \in E.$$

3.1 Note If we apply the HRG-theorem to the functions

$$f_x : R \to F, f_x\,(t) := f\,(tx)\,,$$

then we obtain the following

Corollary. *If, in addition to the hypotheses of the Hyers-Rassias-Gajda theorem, we assume that the mappings $t \to f\,(tx)$ are continuous, then a is linear.*

3.2 Remark (see[1]**).** The same lines of the above proof can lead to the stability for more general equations, e.g. Jensen's equation, and weaker conditions on f, E, or F.

Our method can be used to obtain the following generalization of HRG-theorem:

3.3 Theorem ([7], or [6,Theorem 2.4]). *Let E and F be a real normed space and a real Banach space, respectively. Let a function $\psi : [0;\infty) \to [0;\infty)$ satisfy the following three conditions*

(α) $\lim_{t\to\infty} \frac{\psi(t)}{t} = 0$;

(β) $\psi(ts) \leq \psi(t)\psi(s)$, *for all* $t, s \in [0;\infty)$;

(γ) $\psi(t) < t$ *for all* $t > 1$.

If a function $f : E \to F$ satisfies the inequality

$$(1_\psi) \quad \|f(x+y) - f(x) - f(y)\|_F \leq \theta(\psi(\|x\|_E) + \psi(\|y\|_E)$$

for some $\theta > 0$ and for all $x, y \in E$, then there exists a unique additive function $a : E \to F$, such that

$$(2_\psi) \quad \|f(x) - a(x)\|_F \leq \frac{2\theta}{2 - \psi(2)} \psi(\|x\|_E), \forall x \in E.$$

The *proof* can be obtained as above: Consider the generalized metric

$$d_\psi\,(g, h) := \inf\{c \in [0, \infty], \text{ such that } \|g\,(x) - h\,(x)\|_F \leq c\psi(\|x\|_E), \forall x \in E\}$$

and observe that the mapping A, given by

$$Ag(x) := \frac{1}{2}g(2x),$$

has the Lipschitz constant at most $\frac{\psi(2)}{2}$, and that

$$\left\| \frac{f(2x)}{2} - f(x) \right\|_F \leq \theta\psi(\|x\|_E), \forall x \in E.$$

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