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EXISTENCE RESULTS FOR SYSTEM OF PERIODIC OPERATOR EQUATIONS

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Abstract. In this paper we present some existence results for a system of periodic operator equations. We extend these results to delays equations. Our results can be applied to high order equations by reducing them to first order systems.
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1. Introduction

In this paper, motivated by chapter 12 in [1], we present some existence results for the system of periodic operator equations:

$$\begin{cases} y^{'}(t) - A(t)y(t) = Ny(t) \quad for\ a.e.\ t \in [0,T] \\ y(0) = y(T) \end{cases} \tag{1.1}$$

Here $N : C([0,T],R^n) \to C([0,T],R^n)$, $N = (N_1, N_2, ..., N_n)$ is a continuous operator.
By a solution of (1.1) we mean an absolutely continuous function

$$y : [0,T] \to R^n, \quad y = (y_1, y_2, ..., y_n), \quad y \in AC([0,T],R^n)$$

which satisfies (1.1) almost everywhere on [0,T]. Any such a function is extended to R by periodicity.

First we present a general existence result, and next, we discuss the case when A is identically zero and N is the integro-differential operator with delays:

$$(1.2)\quad Ny(t) = r(t) + g(t, y(t-\theta_1))y(t-\theta_1) + h(t, y(t-\theta_2)) + \int_0^t k_1(t,s) f_1(s, y(s))ds + \int_0^T k_2(t,s) f_2(s, y(s))ds\ .$$

The particular case of a single equation without delays was discussed in [1]. Our results extend those in [1] in two directions: to systems of equations, and to delays equations. In addition, our results can be applied to high order equations (by reducing them to systems of first order).

2. A general existence principle

First we present a general existence principle for (1.1) which in particular, for $n = 1$, reduces to Theorem12.1.1 in [1].

Theorem 2.1. *Assume*

$$(2.1)\qquad N : C([0,T], R^n) \to L^1([0,T], R^n) \text{ is a continuous operator,}$$

$$(2.2)\qquad \begin{cases} \text{for each constant } B \geq 0 \text{ there exists } h_B \in L^1[0,T] \text{ such that} \\ \text{for any } y \in C([0,T], R^n) \text{ with } \|y\|_0 = \sup\limits_{t\in[0,T]} \|y(t)\|_{R^n} \leq B \\ \text{we have } \|Ny(t)\|_{R^n} \leq h_B(t) \text{ for a.e. } t \in [0,T], \end{cases}$$

and

$$(2.3)\qquad A \in L^1([0,T], M_{nn}(R)) \text{ with } I_n - e^{-\int_0^T A(s)ds} \text{ invertible.}$$

Here I_n is the unity matrix from $M_{nn}(R)$, and for a matrix $D \in M_{nn}(R)$ by e^D we mean $\sum\limits_{k=0}^{\infty} \frac{1}{k!} D^k$.

In addition assume that there is a constant M independent of λ with $\|y\|_0 \neq M$ for any solution $y \in AC([0,T], R^n)$ to

$$(2.4)\qquad \begin{cases} y'(t) - A(t)y(t) = \lambda Ny(t) \quad \text{a.e. } t \in [0,T] \\ y(0) = y(T) \end{cases}$$

for each $\lambda \in (0,1)$.

Then (1.1) has at least one solution $y \in AC([0,T], R^n)$ with $\|y\|_0 \leq M$.

Proof. We consider the operator

$$S : C([0,T],R^n) \to C([0,T],R^n)$$

given by

$$\begin{aligned} Sy(t) &= -b(T)\cdot[I_n - b(T)]^{-1}\cdot b^{-1}(t)\cdot\int_0^t b(s)Ny(s)ds \\ &\quad -[I_n - b(T)]^{-1}\cdot b^{-1}(t)\cdot\int_t^T b(s)Ny(s)ds\ , \end{aligned}$$

where

$$b(t) = e^{-\int_0^t A(s)ds}\ .$$

Then problem (2.4) is equivalent to the fixed point problem $\lambda S y = y$.

Let $U = \{y \in C([0,T],R^n) : \|y\|_0 < M\}$. We show that $S : \overline{U} \to C([0,T],R^n)$ is a completely continuous operator.

First note that since $A \in L^1([0,T],M_{nn}(R))$, there exists $m_A > 0$ such that

$$\int_0^t \|A(s)\|\,ds \le m_A \ for\ all\ t \in [0,T].$$

Here by $||A(t)||$ we mean $\sup\limits_{1\le i,j\le n} |a_{ij}(t)|$, where $A(t) = [a_{ij}(t)]_{1\le i,j\le n}$.

1) Now for any $y \in \overline{U}$ and any $t \in [0,T]$ we have

$$\begin{aligned} \|Sy(t)\|_{R^n} &\le \left\|[I_n - b(T)]^{-1}\right\|\cdot e^{2m_A}\cdot||b(T)||\cdot\int_0^t h_M(s)ds \\ &\quad + \left\|[I_n - b(T)]^{-1}\right\|\cdot e^{2m_A}\cdot\int_t^T h_M(s)ds \le m_T \end{aligned}$$

where

$$m_T = (1 + ||b(T)||)\cdot\left\|[I_n - b(T)]^{-1}\right\|\cdot e^{2m_A}\cdot\|h_M\|_{L^1}\ ,$$

and so $S(\overline{U})$ is bounded.

2) For any $0 < t_1 < t_2 \le T$ and $y \in \overline{U}$ we have

$$\|Sy(t_1) - Sy(t_2)\|_{R^n} \le \left\|b(T)\cdot[I_n - b(T)]^{-1}\right\|$$

$$\cdot\left[\left\|b^{-1}(t_1) - b^{-1}(t_2)\right\|\cdot\left\|\int_0^{t_1} b(s)Ny(s)ds\right\| + \left\|b^{-1}(t_2)\right\|\cdot\left\|\int_{t_1}^{t_2} b(s)Ny(s)ds\right\|\right]$$

$$+\left\|b(T)\cdot[I_n - b(T)]^{-1}\right\|$$

$$\cdot\left[\left\|b^{-1}(t_1)\right\|\cdot\left\|\int_{t_1}^{t_2} b(s)Ny(s)ds\right\|+\left\|b^{-1}(t_1)-b^{-1}(t_2)\right\|\cdot\left\|\int_{t_2}^{T} b(s)Ny(s)ds\right\|\right]$$

$$\leq M_1\left[\left\|b^{-1}(t_1)-b^{-1}(t_2)\right\|\cdot e^{m_A}\cdot\|h_M\|_{L^1}+e^{2m_A}\cdot\int_{t_1}^{t_2}\|Ny(s)\|\,ds\right]$$

$$+M_2\left[e^{2m_A}\cdot\int_{t_1}^{t_2}\|Ny(s)\|\,ds+\left\|b^{-1}(t_1)-b^{-1}(t_2)\right\|\cdot e^{m_A}\cdot\|h_M\|_{L^1}\right]$$

$$\leq M_1\left[\left\|b^{-1}(t_1)-b^{-1}(t_2)\right\|\cdot e^{m_A}\cdot\|h_M\|_{L^1}+e^{2m_A}\cdot\int_{t_1}^{t_2}h_M(s)ds\right]$$

$$+M_2\left[e^{2m_A}\cdot\int_{t_1}^{t_2}h_M(s)ds+\left\|b^{-1}(t_1)-b^{-1}(t_2)\right\|\cdot e^{m_A}\cdot\|h_M\|_{L^1}\right]$$

$$\leq e^{m_A}\left(M_1+M_2\right)\left[\left\|b^{-1}(t_1)-b^{-1}(t_2)\right\|\cdot\|h_M\|_{L^1}\right.$$

$$\left.+e^{m_A}\cdot\left[\int_0^{t_2}h_M(s)ds-\int_0^{t_1}h_M(s)ds\right]\right]$$

Since $b^{-1}(t)=e^{\int_0^t A(s)ds}$ is an uniformly continuous function on $[0,T]$, we have that for any $\varepsilon>0$ there exists $\delta_1(\varepsilon)>0$ such that $|t_1-t_2|<\delta_1(\varepsilon)$ implies

$$\left\|b^{-1}(t_1)-b^{-1}(t_2)\right\|<\frac{\varepsilon}{e^{m_A}\left(M_1+M_2\right)\left(e^{m_A}+\|h_M\|_{L^1}\right)}.$$

Similarly, the function $f(t)=\int_0^t h_M(s)ds$ is uniformly continuous on $[0,T]$, and there exists $\delta_2(\varepsilon)>0$ such that $|t_1-t_2|<\delta_2(\varepsilon)$ implies

$$\left\|\int_0^{t_2}h_M(s)ds-\int_0^{t_1}h_M(s)ds\right\|<\frac{\varepsilon}{e^{m_A}\left(M_1+M_2\right)\left(e^{m_A}+\|h_M\|_{L^1}\right)}.$$

Therefore, if $|t_1-t_2|<\min(\delta_1(\varepsilon),\delta_2(\varepsilon))$ we have

$$\|Sy(t_1)-Sy(t_2)\|_{R^n}\leq$$

$$\leq\frac{\varepsilon}{e^{m_A}\left(M_1+M_2\right)\left(e^{m_A}+\|h_M\|_{L^1}\right)}\cdot\left(M_1+M_2\right)\left[e^{m_A}\cdot\|h_M\|_{L^1}+e^{2m_A}\right]<\varepsilon$$

and so $S(\overline{U})$ is echicontinuous. Hence by the Ascoli-Arzela Theorem, S is completely continuous. Thus, from the Leray-Schauder Theorem (see [3]), we obtain that (1.1) has at least one solution y in $AC([0,T],R^n)$ with $\|y\|_0\leq M$.

3. Existence of nonnegative periodic solutions

Consider the problem

$$\begin{cases} y^{'}(t) = Ny(t) \quad for\ \ a.e.\ t \in [0,T] \\ y(0) = y(T) \end{cases} \tag{3.1}$$

Here we discuss the particular case when N is given by (1.2), where

$r : [0,T] \to R^n$,
$h, f_1, f_2 : [0,T] \times R^n \to R^n$,
$g : [0,T] \times R^n \to M_{nn}(R)$,
$k_1 : [0,T] \times [0,t] \to R$,
$k_2 : [0,T] \times [0,T] \to R$.

Theorem 3.1. *Assume that (2.1) and (2.2) are satisfied for N given by (1.2). In addition assume:*

$$r(t) + h(t,0) \le 0 \quad for\ a.e.\ \ t \in [0,T] \tag{3.2}$$

$$\begin{cases} \|h(t,y)\|_{R^n} \le \Phi_1(t)\, \|y\|^{\alpha}_{R^n} + \Phi_2(t)\ \ for\ a.e.\ \ t \in [0,T]\ \ and\ y \ge 0, \\ \ where\ 0 \le \alpha < 1\ \ and\ \Phi_1, \Phi_2 \in L^1[0,T] \end{cases} \tag{3.3}$$

(3.4)
$$\begin{cases} there\ exists\ \beta \in L^1([0,T], R^n)\ \ and\ \tau \in L^1([0,T], R_+)\ \ with \\ \beta(t) \le g(t,y)y\ \ and\ \ \|g(t,y)y\|_{R^n} \le \tau(t)\, \|y\|_{R^n}\ \ for\ a.e.\ \ t \in [0,T] \\ and\ all\ y \ge 0;\ here\ \tau(t) > 0\ on\ a\ subset\ of\ [0,T]\ of\ positive\ measure. \end{cases}$$

(3.5)
$$there\ exists\ \rho \in L^1([0,T], R^n)\ with\ h(t,y) \ge \rho(t)\ a.e.\ t \in [0,T]\ and\ y \ge 0$$

$$\begin{cases} \displaystyle\int_0^t k_1(t,s) f_1(s,y(s))ds + \int_0^T k_2(t,s) f_2(s,y(s))ds \le 0 \\ for\ a.e.\ \ t \in [0,T]\ \ and\ all\ y \in C([0,T], R^n) \end{cases} \tag{3.6}$$

$$\begin{cases} there\ exists\ \rho_1 \in L^1[0,T]\ \ and\ \rho_2 \in L^1([0,T], R^n)\ \ with \\ k_1(t,s) f_1(s,y) \ge \rho_1(s)\rho_2(t)\ for\ a.e.\ \ t \in [0,T]\ \ and\ a.e.\ \ s \in [0,t] \\ and\ all\ y \ge 0 \end{cases} \tag{3.7}$$

$$\begin{cases} there\ exists\ \rho_3 \in L^1[0,T]\ \ and\ \rho_4 \in L^1([0,T], R^n)\ \ with \\ k_2(t,s) f_2(s,y) \ge \rho_3(s)\rho_4(t)\ for\ a.e.\ \ t \in [0,T]\ \ and\ a.e.\ \ s \in [0,T] \\ and\ all\ y \ge 0 \end{cases} \tag{3.8}$$

(3.9)
$$\begin{cases} \left\| \int_0^t k_1(t,s)f_1(s,y(s))ds \right\|_{R^n} \le \Phi_3(t)\,\|y\|_0^{\gamma} + \Phi_4(t) \;\; a.e. \;\; t \in [0,T]; \\ \textit{for any } y \in C([0,T],R_+^n); \textit{ where } \Phi_3, \Phi_4 \in L^1([0,T],R) \textit{ and } 0 \le \gamma < 1 \end{cases}$$

(3.10)
$$\begin{cases} \left\| \int_0^T k_2(t,s)f_2(s,y(s))ds \right\|_{R^n} \le \Phi_5(t)\,\|y\|_0^{\omega} + \Phi_6(t) \;\; a.e. \;\; t \in [0,T]; \\ \textit{for any } y \in C([0,T],R_+^n); \textit{ where } \Phi_5, \Phi_6 \in L^1([0,T],R) \textit{ and } 0 \le \omega < 1 \end{cases}$$

and

(3.11)
$$\int_0^T [-r(t)]dt < \int_0^T \liminf_{x\to\infty}[g(t,x)x]dt + \int_0^T \liminf_{x\to\infty}[h(t,x)]dt +$$
$$+ \int_0^T \int_0^t \liminf_{x\to\infty}[k_1(t,s)f_1(s,x)]dsdt + \int_0^T \int_0^T \liminf_{x\to\infty}[k_2(t,s)f_2(s,x)]dsdt$$

Then (3.1) has at least one solution $y \in AC([0,T],R^n)$ *with* $y(x) \ge 0$ *for all* $x \in [0,T]$.

Proof. We use the notation $1_n = (1,1,...,1) \in R^n$.
For any $y \in C([0,T],R^n)$ let

$$K_1y(t) = \int_0^t k_1(t,s)f_1(s,y(s))ds$$

and

$$K_2y(t) = \int_0^T k_2(t,s)f_2(s,y(s))ds \; .$$

Consider the family of problems

$$(3.12) \qquad \begin{cases} y'(t) - \tau(t)y(t) = \lambda\left[f^\star(t,y) - \tau(t)y(t) + K_1y(t) + K_2y(t)\right] \\ \hfill \textit{for a.e. } t \in [0,T] \\ y(0) = y(T) \end{cases}$$

where $0 < \lambda < 1, \tau$ is as in (3.4), and $f^\star = (f_1^\star, f_2^\star, ..., f_n^\star)$, where

$$f_i^\star(t,y(t)) =$$

$$= \begin{cases} r_i(t) + h_i(t,0), \text{ if there exists } j \in \{0,1,...,n\} \text{ such that } y_j(t) < 0 \\ r_i(t) + y_i(t-\theta_1)g(t,y(t-\theta_1)) + h_i(t-\theta_2)), \\ \hfill \text{if } y_j(t) \ge 0 \text{ for all } j \in \{0,1,...,n\} \end{cases}$$

1) First will show that any solution y of (3.12) satisfies

$$y(t) \geq 0 \quad for \ all \quad t \in [0,T]. \tag{3.13}$$

Let y be a solution of (3.12). Suppose (3.12) does not hold. Then, there exists $i \in \{0,1,...,n\}$ and $t_0 \in [0,T]$ a point of negative global minimum for y_i. Because of the periodicity we may suppose $t_0 \in [0,T)$. Then there exists $t_1 < t_0$ with

$$y_i(t) < 0 \ \ on \ [t_0,t_1] \ \ and \ \ y_i(t) \geq y_i(t_0) \ \ on \ \ [t_0,t_1]\,.$$

Then, we have

$$0 \leq y_i(t_1) - y_i(t_0)$$

$$= \int_{t_0}^{t_1} \left[\lambda f_i^\star(t,y(t)) + (1-\lambda)\tau(t)y_i(t) + \lambda K_1^i y(t) + \lambda K_2^i y(t)\right] dt$$

$$= \int_{t_0}^{t_1} \left[\lambda r_i(t) + \lambda h_i(t,0) + \lambda y_i(t) + (1-\lambda)\tau(t)y_i(t) + \lambda K_1^i y(t) + \lambda K_2^i y(t)\right] dt$$

Using (3.2) and (3.6), since $y_i(t) < 0$ on $[t_0,t_1]$ we obtain

$$0 \leq y_i(t_1) - y_i(t_0) < 0$$

a contradiction. Thus (3.13) is true.

2) Next we show that there exists a positive constant M with

$$\|y\|_0 \leq M \text{ for any solution } y \text{ of (3.12).}$$

If this is not true, then there exist two sequences $(\lambda_n) \subset (0,1)$ and $(y_n) \subset AC([0,T],R^n)$ with

(3.14)

$$\begin{cases} y_n'(t) - \tau(t)y_n(t) = \lambda_n[r(t) + g(t,y_n(t-\theta_1))y_n(t-\theta_1) + h(t,y_n(t-\theta_2)) \\ \qquad\qquad -\tau(t)y_n(t) + K_1y_n(t) + K_2y_n(t)] \ \ for \ \ a.e. \ \ t \in [0,T] \\ y_n(0) = y_n(T) \\ \|y_n\|_0 \to \infty \end{cases}$$

Then, we easily see that

$$0 \geq -\frac{1-\lambda_n}{\lambda_n}\int_0^T \tau(t)y_n(t)dt = \int_0^T [r(t) + g(t,y_n(t-\theta_1))y_n(t-\theta_1) + \\ +h(t,y_n(t-\theta_2)) + K_1y_n(t) + K_2y_n(t)]dt \ ,$$

and so

$$\int_0^T [-r(t)]dt \geq \int_0^T [g(t, y_n(t-\theta_1))y_n(t-\theta_1)]\, dt$$
$$+\int_0^T [h(t, y_n(t-\theta_2)) + K_1 y_n(t) + K_2 y_n(t)]\, dt$$

Then

$$\int_0^T [-r(t)]dt \geq \liminf_{n\to\infty} \int_0^T [g(t, y_n(t-\theta_1))y_n(t-\theta_1)]\, dt$$

$$+\liminf_{n\to\infty} \int_0^T [h(t, y_n(t-\theta_2))]\, dt + \liminf_{n\to\infty} \int_0^T K_1 y_n(t)dt + \liminf_{n\to\infty} \int_0^T K_2 y_n(t)dt$$

where $n \to \infty$ in S_1(S_1 is a subsequence of {1,2,...,n}).

Now (3.4),(3.5), (3.7), (3.8) and Fatou's lemma implies

(3.15)

$$\int_0^T [-r(t)]dt \geq \int_0^T \liminf_{n\to\infty} [g(t, y_n(t-\theta_1))y_n(t-\theta_1)]\, dt$$
$$+\int_0^T \liminf_{n\to\infty} [h(t, y_n(t-\theta_2))]\, dt + \int_0^T \int_0^t \liminf_{n\to\infty} [k_1(t,s) f_1(s, y_n(s))]\, dsdt$$
$$+\int_0^T \int_0^T \liminf_{n\to\infty} [k_2(t,s) f_2(s, y_n(s))]\, dsdt$$

when $n \to \infty$ in S_1.

Let $v_n = \frac{1}{\|y_n\|_0} y_n$.

Then $\|v_n\|_0 = 1$, $v_n(0) = v_n(T)$, and

(3.16)

$$v_n'(t) = (1-\lambda_n)\tau(t)v_n(t) + \lambda_n g(t, y_n(t-\theta_1))v_n(t-\theta_1)+$$

$$+\frac{\lambda_n [h(t, y_n(t-\theta_2)) + K_1 y_n(t) + K_2 y_n(t) + r(t)]}{\|y_n\|_0} \quad a.e. \ t \in [0, T].$$

Let

$$\mu_n(t) = (1-\lambda_n)\tau(t)v_n(t) + \lambda_n g(t, y_n(t-\theta_1))v_n(t-\theta_1).$$

From

$$(1-\lambda_n)\tau(t)v_n(t) \geq 0$$

and

$$\lambda_n g(t, y_n(t-\theta_1))v_n(t-\theta_1) \geq \frac{\lambda_n \beta(t)}{\|y_n\|_0}$$

we obtain

$$\mu_n(t) \geq \frac{\lambda_n \beta(t)}{\|y_n\|_0} \; a.e. \, t \in [0,T] \tag{3.17}$$

On the other hand, $\|v_n\|_0 = 1$ implies

$$\mu_n(t) \leq (1-\lambda_n)\tau(t)v_n(t) + \lambda_n \tau(t) v_n(t-\theta_1) \leq \|(1-\lambda_n)\tau(t)v_n(t)\|_{R^n} +$$

$$+ \|\lambda_n \tau(t) v_n(t-\theta_1)\|_{R^n} \leq (1-\lambda_n)\tau(t) + \lambda_n \tau(t) = \tau(t) \tag{3.18}$$

and so, from (3.17) and (3.18) we get

$$\|\mu_n(t)\|_{R^n} \leq \max\left\{ \frac{\|\beta(t)\|_{R^n}}{\|y_n\|_0}, \tau(t) \right\} \; a.e. \, t \in [0,T]$$

Since $\|y_n\|_0 \to \infty$ there exists an integer n_1 such that

$$\|\mu_n(t)\|_{R^n} \leq \max\{\|\beta(t)\|_{R^n}, \tau(t)\} \;\; for \;\; any \;\; n \geq n_1 \;\; and \;\; a.e. \;\; t \in [0,T].$$

This, together with (3.3), (3.9), (3.10) and (3.16) implies

$$\left\|v_n'(t)\right\|_{R^n} \leq \max\{\|\beta(t)\|_{R^n}, \tau(t)\} + \sum_{i=1}^{6} \Phi_i(t) + \|r(t)\|_{R^n} \tag{3.19}$$

a.e. $t \in [0,T]$, for any $n \geq n_1$.

Then, there exists a subsequence S_1 of $\{n_1, n_1+1, ...\}$ with

$$\begin{cases} v_n \to v \text{ in } C([0,T], R^n) \\ v_n' \to v' \text{ weakly in } L^1([0,T], R^n) \quad when \;\; n \to \infty \;\; in \;\; S_1. \\ \lambda_n \to \lambda \end{cases} \tag{3.20}$$

Next, we consider the equation

(3.21)

$$\begin{cases} v_n'(t) = (1-\lambda_n)\tau(t)v_n(t) + \lambda_n g(t, y_n(t-\theta_1))v_n(t-\theta_1) + \\ \qquad + \dfrac{\lambda_n \left[h(t, y_n(t-\theta_2)) + K_1 y_n(t) + K_2 y_n(t) + r(t)\right]}{\|y_n\|_0} \; a.e. \, t \in [0,T] \\ v_n(0) = v_n(T) \end{cases}$$

For $n \in S_1$ and $\psi \in L^\infty[0,T]$ we obtain

$$\int_0^T v_n'(t)\psi(t)dt = \int_0^T \left[(1-\lambda_n)\tau(t)v_n(t) + \lambda_n g(t, y_n(t-\theta_1))v_n(t-\theta_1)\right]\psi(t)dt +$$

$$+\lambda_n \int_0^T \frac{[h(t, y_n(t-\theta_2)) + K_1 y_n(t) + K_2 y_n(t) + r(t)]}{\|y_n\|_0}\psi(t)dt \tag{3.22}$$

From (3.3), (3.9), (3.10) and $\|y_n\| \to \infty$ we obtain

$$\lim_{\substack{n\to\infty \\ n\in S}} \lambda_n \int_0^T \frac{[h(t, y_n(t-\theta_2)) + K_1 y_n(t) + K_2 y_n(t) + r(t)]}{\|y_n\|_0}\psi(t)dt = 0 \tag{3.23}$$

In addition (3.20) yields

$$\lim_{\substack{n\to\infty \\ n\in S}} \int_0^T v_n'(t)\psi(t)dt = \int_0^T v'(t)\psi(t)dt\,. \tag{3.24}$$

Also

$$\mu_n(t) \le 1_n \cdot \|(1-\lambda_n)\tau(t)v_n(t)\|_{R^n} + 1_n \cdot \|\lambda_n g(t, y_n(t-\theta_1))v_n(t-\theta_1)\|_{R^n}$$

$$\le 1_n \cdot \tau(t)\left[\|v_n(t-\theta_1)\|_{R^n} + \|v_n(t)\|_{R^n}\right].$$

Then, from (3.17), we deduce

$$\frac{\beta(t)}{\|y_n\|_0} \le \mu_n(t) \le 1_n \cdot \tau(t)\left[\|v_n(t-\theta_1)\|_{R^n} + \|v_n(t)\|_{R^n}\right]$$

Since $v_n \overset{n\in S}{\longrightarrow} v$ in $C([0,T],R^n)$ and $\|y_n\|_0 \to \infty$, there exists an integer n_2 such that

$\|\mu_n(t)\|_{R^n} \le \max\{\tau(t)[2 + v(t-\theta_1) + v(t)], \|\beta(t)\|_{R^n}\}$ for $n \ge n_2$ and $n \in S_1$.

Let $S_2 = \{n \in S_1 : n \ge n_2\}$.

From the Dunford-Pettis Theorem (see [2]) the set

$$\{\mu_n \in L^1([0,T],R^n) | n \in S_2\}$$

is weakly sequential compact, and so there exists $S_3 \subset S_2$and $\mu \in L^1([0,T],R^n)$ such that

$$\mu_n \overset{n\in S}{\longrightarrow} \mu \; weakly \;\; in \; L^1([0,T],R^n). \tag{3.25}$$

Let $n \to \infty$ in S_3 in (3.22), and using (3.23), (3.24) and (3.25) we get

$$\int_0^T v^{'}(t)\psi(t)dt = \int_0^T \mu(t)\psi(t)dt\,. \tag{3.26}$$

Also $v(0) = v(T)$.
Next we claim that

$$\mu(t) \geq 0 \; for \; a.e.t \in [0,T] \tag{3.27}$$

Let m be an integer. Fix m and let $e = \dfrac{1}{m}$.
Then, from (3.4), $\|y_n\|_0 \to \infty$, and $v_n \overset{n\in S_2}{\longrightarrow} v$ in $C([0,T],R^n)$ there exists n_3 such that

$$-e \leq \mu_n^i(t) \leq (1-\lambda_n)\tau_i(t)v_n^i(t-\theta_1) + \lambda_n \left\| g(t, y_n(t-\theta_1))v_n(t-\theta_1)\right\|_{R^n}$$

$$< \lambda_n\tau_i(t)(1+e) + (1-\lambda_n)\tau_i(t)\left\|v_n(t-\theta_1)\right\|_{R^n} \cdot M < \tau_i(t)(1+e)$$

$$for\;\; every\; n \geq n_3 \; and \; n \in S_3.$$

Let

$$K = \{u \in L^1([0,T],R^n| - e \leq u^i(t) \leq (1+e)\tau_i(t) \;\; a.e. \;\, t \in [0,T]\}$$

Since K is convex and closed, $\mu_n \in K$ for $n \geq n_3, n \in S_3$, and (3.25) is true,we have that $\mu \in K$.
Then

$$-e \leq \mu_i(t) \leq (1+e)\tau_i(t) \; a.e. \; t \in [0,T], \; i \in \{1,2,...,n\}$$

We can do this for each $e = \dfrac{1}{m}$, $m \in \{1,2,...,n\}$ and so

$$0 \leq \mu^i(t) \leq \tau_i(t) \; a.e. \; t \in [0,T], \; i \in \{1,2,...,n\}.$$

Then, (3.27) is true.
This, together with (3.26) implies that v is nondecreasing on [0,T].
Since $v(0) = v(T)$, $\;v = c \geq 0$, where c is a constant. But $\|v\|_0 = 1$, and so $c > 0$.
Then, from $v_n(t) = \dfrac{y_n(t)}{\|y_n\|_0} \to v = c$ there exists $n_4 \in S_3$ with

$$\|v_n(t) - c\|_{R^n} \leq \frac{c}{2}.$$

This implies $y_n \to v$ in S_3 for any $t \in [0,T]$.

Then, from (3.15), we get

$$\int_0^T [-r(t)]dt \geq \int_0^T \liminf_{x\to\infty} [g(t,x)x]\, dt + \int_0^T \liminf_{x\to\infty} [h(t,x)]\, dt +$$

$$+\int_0^T \int_0^t \liminf_{x\to\infty} [k_1(t,s)f_1(s,x)]\, dsdt + \int_0^T \int_0^T \liminf_{x\to\infty} [k_2(t,s)f_2(s,x)]\, dsdt$$

This contradicts (3.11) and so there exists a positive constant M with $\|y\|_0 \leq M$ for any solution y of (3.12).

The existence of a solution is now guaranteed by Theorem2.1.

Example. The periodic integro-differential system

$$\text{(3.28)}\quad \begin{cases} y^{'}(t) = \|y(t-\theta)\|_{R^n}^{\omega} - t^{-v} - \int_0^y \sqrt{s^2+t^2} e^{-|y(s)|} ds \ a.e. t \in [0,T] \\ y(0) = y(T) \end{cases}$$

where $0 \leq \omega < 1$, and $0 \leq v < 1$.

Here, we take:

$$r(t) = -(t^{-v_1}, t^{-v_2}, ..., t^{-v_n})$$
$$g = 0 \ in \ R^n$$
$$h(t,y) = (\|y\|_{R^n}^{\omega_1}, \|y\|_{R^n}^{\omega_2}, ..., \|y\|_{R^n}^{\omega_n})$$
$$f_1(s,y) = (e^{-|y_1|}, e^{-|y_2|}, ..., e^{-|y_n|})$$
$$f_2 = 0 \ in \ R^n$$
$$k_1(t,s) = -\sqrt{s^2+t^2}$$
$$k_2 = 0 \ in \ R^n$$

It is easy to see that (2.1), (2.2) and (3.2)-(3.11) are satisfied, and so, by Theorem 3.1, problem (3.28) has a nonnegative solution.

4. Some particular cases

1) If $Ny(t) = h(t, y(t))$ we obtain the following existence result

Theorem 4.1. *Assume that*

$$\text{(4.1)}\qquad h : [0,T] \times R^n \to R^n \ is \ a \ continuous \ function.$$

(4.2)

$$\begin{cases} for\ each\ constant\ A \geq 0\ there\ exists\ h_A \in L^1[0,T]\ such\ that\ for\ any \\ y \in C([0,T], R^n)\ with \|y\|_{R^n} \leq A\ we\ have\ \|h(t,y)\|_{R^n} \leq h_A(t) \\ for\ a.e.\ t \in [0,T] \end{cases}$$

$$h(t,0) \le 0 \ a.e.\, t \in [0,T] \tag{4.3}$$

$$\begin{cases} \|h(t,y)\|_{R^n} \le \Phi_1(t)\,\|y\|^{\alpha}_{R^n} + \Phi_2(t) \ \text{for a.e. } t \in [0,T] \text{ and } y \ge 0, \\ \text{where } 0 \le \alpha < 1 \text{ and } \Phi_1, \Phi_2 \in L^1[0,T] \end{cases} \tag{4.4}$$

(4.5)

$$\text{there exists } \rho \in L^1([0,T], R^n) \text{ with } h\,(t,y) \ge \rho(t) \ a.e.t \in [0,T] \text{ and } y \ge 0$$

$$0 < \int_0^T \liminf_{\|x\|\to\infty}[h(t,x)]dt \tag{4.6}$$

Then the problem

$$\begin{cases} y^{'}(t) = h(t,y(t)) \ \ a.e.\, t \in [0,T] \\ y(0) = y(T) \end{cases}$$

has a nonnegative solution.

2) If $Ny(t) = y(t)g(t,y(t))$ we obtain the following existence result

Theorem 4.2. *Assume that*

$$g : [0,T] \times R^n \to M_{nn}(R) \text{ is a continuous function} \tag{4.7}$$

(4.8)

$$\begin{cases} \text{for each constant } A \ge 0 \text{ there exists } h_A \in L^1[0,T] \text{ such that for any} \\ y \in C([0,T], R^n) \text{ with } \|y\|_{R^n} \le A \text{ we have } \|g(t,y)y\|_{R^n} \le h_A(t) \\ \text{for a.e. } t \in [0,T] \end{cases}$$

(4.9)

$$\begin{cases} \text{there exists } \beta \in L^1([0,T], R^n) \text{ and } \tau \in L^1([0,T], R_+) \text{ with} \\ \beta(t) \le g(t,y)y \text{ and } \|g(t,y)y\|_{R^n} \le \tau(t)\,\|y\|_{R^n} \text{ for a.e. } t \in [0,T] \\ \text{and all } y \ge 0\text{; here } \tau > 0 \text{ on a subset of [0,T] of positive measure.} \end{cases}$$

$$0 < \int_0^T \liminf_{\|x\|\to\infty}[g(t,x)x]dt \tag{4.10}$$

Then the problem

$$\begin{cases} y^{'}(t) = y(t)g(t,y(t)) \ \ a.e.\, t \in [0,T] \\ y(0) = y(T) \end{cases}$$

has a nonnegative solution.

3) If $Ny(t) = \int_0^t k(t,s)f(s,y(s))ds$ we obtain

Theorem 4.3. *Assume that*

(4.11)

$$f : [0,T] \times R^n \to R^n \text{ and } k : [0,T] \times [0,t] \to R \text{ satisfies } N \text{ is continuous}$$

(4.12)

$$\begin{cases} \text{for each constant } A \geq 0 \text{ there exists } h_A \in L^1[0,T] \text{ such that for any} \\ y \in C([0,T], R^n) \text{ with } \|y\|_{R^n} \leq A \text{ we have } \|Ny(t)\|_{R^n} \leq h_A(t) \\ \text{for a.e. } t \in [0,T] \end{cases}$$

(4.13)

$$\int_0^t k(t,s)f(s,y(s))ds \leq 0 (\in R^n) \text{ for a.e. } t \in [0,T] \text{ and all } y \in C([0,T], R^n)$$

$$(4.14) \qquad \begin{cases} \text{there exists } \rho \in L^1[0,T] \text{ and } \rho \in L^1([0,T], R^n) \text{ with} \\ k(t,s)f(s,y) \geq \rho(s)\rho(t) \text{ for a.e. } t \in [0,T] \text{ and a.e. } s \in [0,t] \\ \text{and all } y \geq 0 \end{cases}$$

(4.15)

$$\begin{cases} \left\| \int_0^t k(t,s)f(s,y(s))ds \right\|_{R^n} \leq \Phi(t) \|y\|_0^\gamma + \Phi(t) \text{ a.e. } t \in [0,T]; \\ \text{for any } y \in C([0,T], R_+^n), \text{ where } \Phi, \Phi \in L^1([0,T], R) \text{ and } 0 \leq \gamma < 1 \end{cases}$$

and

$$(4.16) \qquad 0 < \int_0^T \int_0^t \liminf_{\|x\| \to \infty} [k(t,s)f(s,x)] ds dt$$

Then the problem

$$\begin{cases} y'(t) = \int_0^t k(t,s)f(s,y(s))ds \text{ a.e.} t \in [0,T] \\ y(0) = y(T) \end{cases}$$

has a nonnegative solution.

4) If $Ny(t) = \int_0^T k(t,s)f(s,y(s))ds$ we obtain:

Theorem 4.4. *Assume that*

(4.17)
$$f : [0,T] \times R^n \to R^n and\, k : [0,T] \times [0,T] \to R \; satisfies\; N \; is\; continuous$$

(4.18)
$$\begin{cases} \textit{for each constant } A \geq 0 \textit{ there exists } h_A \in L^1[0,T] \textit{ such that for any} \\ y \in C([0,T],R^n) \textit{ with } \|y\|_0 \leq A \textit{ we have } \|Ny(t)\|_{R^n} \leq h_A(t) \\ \textit{for a.e. } t \in [0,T] \end{cases}$$

(4.19)
$$\int_0^T k(t,s)f(s,y(s))ds \leq 0 \; for \;\; a.e. \;\; t \in [0,T] \;\; and \;\; all \;\; y \in C([0,T],R^n)$$

$$\begin{cases} \textit{there exists } \rho \in L^1[0,T] \textit{ and } \rho \in L^1([0,T],R^n) \textit{ with} \\ k(t,s)f(s,y) \geq \rho(s)\rho(t) \textit{ for a.e. } t \in [0,T] \textit{ and a.e. } s \in [0,T] \\ \textit{and all } y \geq 0 \end{cases} \tag{4.20}$$

(4.21)
$$\begin{cases} \left\| \int_0^T k(t,s)f(s,y(s))ds \right\|_{R^n} \leq \Phi(t) \, \|y\|_0^\gamma + \Phi(t) \; a.e. \; t \in [0,T]; \\ \textit{for any } y \in C([0,T],R^n_+) \textit{; where } \Phi, \Phi \in L^1([0,T],R) \textit{ and } 0 \leq \gamma < 1 \end{cases}$$

and

$$0 < \int_0^T \int_0^T \liminf_{x \to \infty} [k(t,s)f(s,x)]dsdt \tag{4.22}$$

Then the problem

$$\begin{cases} y^{'}(t) = \int_0^T k(t,s)f(s,y(s))ds \;\; a.e. \, t \in [0,T] \\ y(0) = y(T) \end{cases}$$

has a nonnegative solution.

5) Consider the problem:

$$\begin{cases} y^{''}(t) = R(t) + H_1(t,y(t)) + H_2(t,y^{'}(t)) \\ y(0) = y(T) \\ y^{'}(0) = y^{'}(T) \end{cases} \tag{4.23}$$

We can easily show that problem (4.23) is equivalent to the following problem:

$$\begin{cases} z^{'}(t) = r(t) + h(t, z(t) + g(t, z(t))z(t) \\ z(0) = z(T) \end{cases}$$

where

$$\begin{cases} z(t) = (y(t), y^{'}(t)) \\ r(t) = (0, R(t)) \\ h(t, z(t)) = (0, H_1(t, z_1(t)) + H_2(t, z_2(t)) \\ g(t, z(t)) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{cases}$$

From Theorem 3.1 we obtain

Theorem 4.5. *Assume that*

$$R \in L^1[0,T] \ \text{and} \ H_1, H_2 \in L^1([0,T] \times R, R) \tag{4.24}$$

$$R(t) + H_1(t,0) + H_2(t,0) \leq 0 \ \text{a.e.} \ \ t \in [0,T] \tag{4.25}$$

$$\begin{cases} |H_1(t,x) + H_2(t,y)| \leq \Phi_1(t) \left\|(x,y)\right\|_{R^2}^{\alpha} + \Phi_2(t) \, \text{for a.e.} \ t \in [0,T] \\ \text{and} \ x, y \geq 0 \ \text{where} \ 0 \leq \alpha < 1 \ \text{and} \ \Phi_1, \Phi_2 \in L^1[0,T] \end{cases} \tag{4.26}$$

$$\begin{cases} \text{there exists} \ \rho \in L^1[0,T] \ \ \text{such that} \\ H_1(t,x) + H_2(t,x) \geq \rho(t) \ \text{a.e.} \ t \in [0,T] \ \text{and} \ x, y \geq 0 \end{cases} \tag{4.27}$$

$$\int_0^T [-R(t)] \, dt \leq \int_0^T \liminf_{x \to \infty} [H_1(t,x)] dt + \int_0^T \liminf_{x \to \infty} [H_2(t,x)] dt \tag{4.28}$$

Then problem (4.23) has at least one nonnegative solution.

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