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LOCAL FIXED POINT THEORY FOR THE SUM OF TWO OPERATORS IN BANACH SPACES

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Abstract. The present paper studies the local version of the well-known fixed point theorems of Krasnoselskii [5] and Nashed and Wong [5]. Some applications of newly developed local fixed point theorems to nonlinear functional integral equations of fixed type are also discussed.

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1. Introduction

Fixed point theory constitutes an important and the core part of the subject of nonlinear functional analysis and is useful for proving the existence theorems for nonlinear differential and integral equations. The local fixed point theory is useful for proving the existence of the local solution of the problems governed by nonlinear differential or integral equations. In the present paper we shall obtain the local versions of the well- known fixed point theorems of Krasnoselskii [5] and Nashed and Wong [7] and discuss some of their applications to functional integral equations.

Throughout this paper let X denote a Banach space with a norm $\|\cdot\|$. Let $a \in X$ and let r be a positive real number. Then by $B_r(a)$ and $\overline{B}_r(a)$ we respectively denote an open and a closed ball in X centered at the point $a \in X$ and of radius r. A mapping $T : X \to X$ is called a *contraction* if there exists a constant $0 \le \alpha < 1$ such that

$$\|Tx - Ty\| \le \alpha \|x - y\| \tag{1.1}$$

for all $x, y \in X$ and the constant α is called a *contraction constant* of T. The local version of the well-known Banach fixed point theorem is

Theorem 1.1. ([5, page 10-11]) *Let $T : \overline{B}_r(a) \to X$ be the contraction with contraction constant α. If T satisfies*

$$\|a - Ta\| \leq (1 - \alpha)r \tag{1.2}$$

for some $a \in X$ and $r > 0$, then T has a unique fixed point in $\overline{B}_r(a)$.

A mapping $T : X \to X$ is called *compact* if $\overline{T(X)}$ is a compact subset of X and *totally bounded* if $T(S)$ is a totally bounded subset of X for any bounded subset S of X. Again a map $T : X \to X$ is called completely continuous if it is continuous and totally bounded.

The local version of the famous Schauder fixed theorem may be given as follows.

Theorem 1.2. *Let $a \in X$ and let r be a positive real number. If $T : \overline{B}_r(a) \to \overline{B}_r(a)$ be a completely continuous operator, then T has a fixed point.*

Theorems 1.1 and 1.2 have been extensively used in the literature for proving the existence of the solution of nonlinear differential and integral equations in the neighborhood of a point in the function space in question.

The next important topological fixed point theorem in its original form is

Theorem 1.3. (Krasnoselskii [6]) *Let S be a closed convex and bounded subset of X and let $A, B : S \to X$ be two operators such that*

(a) *A is a contraction,*
(b) *B is completely continuous, and*
(c) *$Ax + By \in S$ for all $x, y \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution in S .

Theorem 1.3 is useful in the study of nonlinear integral equations of mixed type which arise as a inversion of the perturbed differential equations and so it has attracted the attention of the several authors. See Burton [1] and the references therein. Attempts have been made to improve or generalize Theorem 1.3 in the course of time by weakening of the hypothesis (a) or (b) or (c) of it. We focus our attention on the hypothesis (c) of Theorem 1.3.

The following reformulation of Theorem 1.3 is note-worthy and is proved in Reinermann [10].

Theorem 1.4. *Let S be a closed convex and bounded subset of a Banach space X and let $A, B : S \to X$ be two operators such that*

(a) *A is contraction,*

(b) *B is completely continuous, and*

(c) *$Ax + Bx \in S$ for all $x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution.

Remark 1.1. *Unlike Theorem 1.3, the operators A and B in Theorem 1.4 need not map S into itself and the hypothesis* (c) *is also considerably weakened.*

The following two more re-formulations of Theorem 1.4 have been recently obtained in the literature by Regan [9] and Burton [1] under some weaker hypothesis (c) thereof.

Theorem 1.5. (Regan [9]) *Let S be a closed convex and bounded subset of a Banach space X and let $A, B, : X \to X$ be two operators such that*

(a) *$A + B : S \to X$.*

(b) *$A + B$ is condensing, and*

(c) *If $\{(x_j, \lambda_j)\}$ is a sequence in $\partial S \times [0, 1]$ converging to (x, λ) with $x = \lambda(A+B)x$ and $0 < \lambda < 1$, where ∂S is the boundary of S, then $\lambda_j(A+B)x \in S$ for large j.*

The measures of noncompactness and condensing mappings require a high technicalities which a nonspecialist working in the field of nonlinear problems may find difficulty to tackle it with and therefore, Theorem 1.3 is generally used as a handy tool in applications to perturbed nonlinear equations. For the more details of condensing maps the readers are referred to Zeidler [13].

Theorem 1.6. (Burton [1]) *Let S be a closed convex and bounded subset of X and $A : X \to X$ and $B : S \to X$ such that*

(a) *A is a contraction.*

(b) *B is completely continuous, and*

(c) *$\{x = Ax + By \text{ for all } y \in S\} \Rightarrow x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution.

In this paper we shall prove another formulation of Theorem 1.3 again by modifying the hypothesis (c) in a different way which ultimately yields the local version of Theorem 1.3.

2. Local Fixed Point Theorems

Theorem 2.1. *Let $a \in X$, r a positive real number. Let $A : X \to X$ and $B : \overline{B}_r(a) \to X$ be two operators such that*

(a) *A is a contraction with a contraction constant α.*

(b) *B is completely continuous, and*

(c) *$\|a - (Aa + By)\| \leq (1-\alpha)r$ for all $y \in \overline{B}_r(a)$.*

Then the operator equation $Ax + Bx = x$ has a solution in $\overline{B}_r(a)$.

Proof. The proof follows by applying Theorem 1.2 to the operator T defined by

$$T = (I - A)^{-1}B.$$

First we claim that T is well defined and

$$T : \overline{B}_r(a) \to \overline{B}_r(a). \tag{2.1}$$

Notice that $(I - A)^{-1}$ exists and is continuous on X in view of hypothesis (a). Hence the mapping T in (2.1) is well defined. Let $y \in X$ and define a mapping A_y on $\overline{B}_r(a)$ by

$$A_y(x) = Ax + By.$$

We show that A_y is a contraction on $\overline{B}_r(a)$. For any $x_1, x_2 \in \overline{B}_r(a)$, we have

$$\|A_y(x_1) - A_y(x_2)\| = \|Ax_1 - Ax_2\| \leq \alpha\|x_1 - x_2\|,$$

where $0 < \alpha < 1$, and so A_y is a contraction on $\overline{B}_r(a)$. Again by hypothesis (c),

$$\|a - A_y(a)\| = \|a - (Aa + By)\| \leq (1-\alpha)r.$$

Hence an application of Theorem 1.1 yields that there is a unique point x^* in $\overline{B}_r(a)$ such that

$$A_y(x^*) = x^*, \quad i.e., x^* = Ax^* + By \quad \text{or} \quad (1-A)x^* - By \tag{2.2}$$

Now applying $(1 - A)^{-1}$ on both the sides of (2.3), we obtain

$$x^* = (1 - A)^{-1}By$$

or equivalently, $Ty = x^*$. This guarantees the claim (2.1). The operator T, which is the composition of a continuous and a completely continuous operator, is completely continuous. Now the desired conclusion follows by an application of Theorem 1.2. This completes the proof. $\square$

Taking $a = 0$, the origin of X, in Theorem 2.1 we obtain

Corollary 2.1. *Let $A : X \to X$ and $B : \overline{B}_r(0) \to X$ be two operators such that*

(a) *A is a contraction with the contraction constant α,*

(b) *B is a completely continuous, and*

(c) *$\|A0 + By\| \leq (1-\alpha)r$ for all $y \in \overline{B}_r(0)$.*

Then the operator equation $Ax + Bx = x$ has a solution in $\overline{B}_r(0)$.

Next we prove a local version of the following fixed point theorem of Nashed and Wong [7], which is also useful for applications in the theory of differential and integral equations.

Theorem 2.2. *Let S be a closed convex and bounded subset of X and let $A, B : S \to X$ be two operators such that*

(a) *A is linear and bounded and there is a positive integer p such that A^p is a contraction,*

(b) *B is a completely continuous, and*

(c) *$Ax + By \in S$ for all $x, y \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution.

Theorem 2.3. *Let $A : X \to X$ and $B : \overline{B}_r(a) \to X$ be two operators such that*

(a) *A is linear and bounded and there exists a $p \in N$ such that A^p is a contraction with contraction constant α,*

(b) *B is completely continuous, and*

(c) *$\|a - A^p a\| + (\frac{1-\|A\|^p}{1-\|A\|})\|By\| \leq (1-\alpha)r$ for all $y \in \overline{B}_r(a)$.*

Then operator equation $Ax + Bx = x$ has a solution in $\overline{B}_r(a)$.

Proof. The proof involves applying Theorem 1.2 to the operator T defined by

$$T = (l - A)^{-1}B. \tag{2.3}$$

We claim that T is well defined and

$$T : \overline{B}_r(a) \to \overline{B}_r(a). \tag{2.4}$$

Now

$$\begin{aligned}(1-A)^{-1} &= 1 + A + A2 + \cdots \\ &= (1-A)^{-1}\left(\sum_{j=0}^{p-1} A^j\right)\end{aligned}$$

Clearly $(1 - A)^-1$ exists since A^p is a contraction. Also, the operator $(\sum_{j=0}^{p-1} A^j)$ is bounded and so the composition $(1-A)^{-1}(\sum_{j=0}^{p-1} A^j)$ and consequently the operator T is well defined. Now we shall prove the claim (2.4). Let $y \in \overline{B}_r(a)$ be fixed and define a mapping A_y on $B_r(a)$ by

$$A_y(x) = Ax + By.$$

We shall show that A_y^p is a contraction. Let $x_1, x_2 \in \overline{B}_r(a)$. Then by hypothesis (a),

$$\|A_y(x_1) - A_y(x_2)\| = \|Ax_1 - Ax_2\|.$$

Again

$$\begin{aligned}\|A_y^2(x_2) - A_y^2(x_2)\| &= \|A_y(A_y(x_1)) - A_y(A_y(x_2))\| \\ &= \|A^2x_1 - A^2x_2\|\end{aligned}$$

Similarly,

$$\|A_y^p(x_1) - A_y^p(x_2)\| = \|A^px_1 - A^px_2\| \le \alpha\|x_1 - x_2\|,$$

where $0 \le \alpha < 1$. As a result, A_y is a contraction on $\overline{B}_r(a)$.

Now

$$\begin{aligned}A_y(x) &= Ax + By \\ A_y^2(x) &= A_y(A_y(x)) \\ &= A_y(A_x + B_y) + By \\ &= A(Ax + By) + By \\ &= A^2x + ABy + By\end{aligned}$$

Similarly

$$\begin{aligned} A_y^3(x) &= A_y(A_y^2 x) \\ &= A(A^2x + ABy + By) \\ &= A^3x + A^2By + ABy + By \end{aligned}$$

By induction,

$$A_y^p(x) = A^p x + A^{p-1}By + A^{p-2}By + \cdots + By.$$

Therefore,

$$\begin{aligned} \|a - A^p y(a)\| &= \left\| a - A^p a - \left(\sum_{j=0}^{p-1} A^j \right) By \right\| \\ &\leq \|a - A^p a\| + \left\| \sum_{j=0}^{p-1} A^j \right\| \|By\| \\ &\leq \|a - Aa\| + \left(\sum_{j=0}^{p-1} \|A^j\| \right) \|By\| \\ &\leq \|a - A^p a\| + \left(\frac{1 - \|A\|^p}{1 - \|A\|} \right) \|By\| \\ &\leq (1 - \alpha) r. \end{aligned}$$

An application of Theorem 1.1 yields that there is a unique point $x^* = \overline{B}_r(a)$ such that

$$A_y(x^*) = x^*, \quad Ax^* + By = x^*$$

or, equivalently $(1 - A)^{-1}By = x^*$, i.e., $Ty = x^*$.

This proves the claim (2.4). Since A is linear bounded, it is continuous, and as a result $(1 - A)^{-1}$ is continuous on X. Now the operator T, which is a composition of a continuous and a completely continuous operator, is completely continuous on $\overline{B}_r(a)$. Hence the conclusion follows by an application of Theorem 1.2. □

Taking $a = 0$, the origin of X, in Theorem 2.3 we obtain

Corollary 2.2. *Let $A : X \to X$ and $B : \overline{B}_r(0) \to X$ be two operators such that*

(a) *A is linear, bounded, and there exist a $p \in N$ such that A^p is a contraction with contraction constant α,*

(b) *B is completely continuous, and*

(c) $\|By\| \leq (\frac{1-\|A\|}{1-\|A\|^p})(1-\alpha)r$ *for all $y \in \overline{B}_r(0)$.*

Then the operator equation $Ax + Bx = x$ has a solution in $\overline{B}_r(0)$.

3. Nonlinear Functional Integral Equations

Given a closed and boundary interval $J = [0, 1]$ of the real line R, consider the nonlinear functional integral equation (in short FIE)

$$x(t) = q(t) + \int_0^{\mu(t)} f(s, x(\theta(s))ds + \int_0^{\sigma(t)} g(s, x(\eta(s))ds \tag{3.1}$$

for $t \in J$, where $q : J \to R$, $\mu, \theta, \sigma, \eta : J \to J$ and $f, g : J \times R \to R$.

The special cases of FIE (3.1) have been studied in the literature extensively via different fixed point methods. We shall prove the existence of the local solution of FIE (3.1) by the application of our newly developed local fixed point theorem of the previous section.

We shall seek the solution of FIE (3.1) in the space $BM(J, R)$ of all bounded and measurable real-valued functions on J. Define a norm $\|\cdot\|_{BM}$ in $BM(J, R)$ by

$$\|x\|_{BM} = \max_{t \in J} |x(t)|.$$

Clearly $BM(J, R)$ becomes a Banach space with this maximum norm. By $L^1(J, R)$ we denote the space of all Lebesgue integrably real-valued functions on J with usual norm $\|\cdot\|_{L^1}$. We need the following definition in this sequel.

Definition 3.1. *A function $\beta : J \times R \to R$ is said to satisfy a condition of L^1-Caratheodory or simply L^1-Caratheodory if*

(i) *$t \to \beta(t, x)$is measurable for each $x \in R$,*

(ii) *$x \to \beta(t, x)$ is almost everywhere continuous for $t \in J$ and*

(iii) *for each real number $k > 0$ these exists a function $h_k \in L^1(J, R)$ such that*

$$|\beta(t, x)| \leq h_k(t), \quad a.e.\ t \in J$$

for all $x \in R$ with $|x| \leq k$.

We consider the following set of assumptions.

(A_0) The functions $\mu, \theta, \sigma, \eta : J \to J$ are continuous,

$(A_1\ )$ The function $q : J \to R$ is continuous,

(A_2) The function $f : J \times R \to R$ is continuous and there exists a function $\alpha \in L^1(J, R)$ such that

$$|f(t,x) - f(t,y)| \le \alpha(t)|x - y|, \quad \text{a.e. } t \in J$$

for all $x, y \in R$.

(A_3) The function $g(t, x)$ is L^1-Caratheodory.

Theorem 3.1. *Suppose that the assumption* (A_0)*-*(A_3) *hold. Further if there exists a real number* $r > 0$ *such that*

$$r \ge \frac{\|q\|_{BM} + F + \|h_r\|_{L^1}}{1 - \|\alpha\|_{L^1}}, \quad \|\alpha\|_{L^1} < 1, \tag{3.2}$$

where $F = \sup_{s \in J} |f(s, 0)|ds$*, then the FIE*(3.1) *has a solution* u *on* J *with* $\|u\| \le r$*.*

Proof. Consider the ball $\overline{B}_r(0)$ in the Banach space $BM(J, R)$, where the real number r satisfies the inequality (3.2). Define two operators A and B on $BM(J, R)$ by

$$Ax(t) = \int_0^{\mu(t)} g(s, x(\theta(s)))ds, \quad t \in J \tag{3.3}$$

and

$$Bx(t) = q(t) + \int_0^{\mu(t)} g(s, x(\eta(s)))ds, \quad t \in J. \tag{3.4}$$

Then the FIE (3.1) is equivalent to the fixed point equation $Ax + Bx = x$ on J. Hence the problem of the existence of the solution to FIE (3.1) is just reduced to finding the solution of the operator equation $Ax + Bx = x$. We shall show that the operators A and B satisfy all the conditions of Corollary 2.1. First we show that A is a contraction on $BM(J, R)$. Let $x, y \in BM(J, R)$.

Then by (A_2)

$$\begin{aligned} |Ax(t)-Ay(t)| &\leq \int_0^{\mu(t)} |f(s,x(\theta(s)))ds \\ &\leq \int_0^{\mu(t)} \alpha(s)|x(\theta(s))-y(\theta(s))ds \\ &\leq \int_0^1 \alpha(s)\|x-y\|_{BM}ds \end{aligned}$$

Taking the maximum over t, we obtain

$$\|Ax-Ay\|_{BM} \leq \|\alpha\|_{L^1}\|x-y\|_{BM}$$

where $\|\alpha\|_{L^1} < 1$, and so A is a contraction on $BM(J,R)$.

Notice that (A_0), (A_3) and the Lebegue dominated convergence theorem guarantees that $B : \overline{B}_r(0) \to X$ is continuous. Let $\{x_n\}$ be a sequence in $\overline{B}_r(0)$. Notice (A_3) implies

$$\begin{aligned} \|Bx_n\| &\leq \|q\|_{BM} + \sup_{t\in J}\int_0^{\mu(t)} g(s,x_n(\eta(s)))ds, \\ &\leq \|q\|_{BM} + \|h_r\|_{L^1}, \end{aligned}$$

and so $\{Bx_n\}$ is uniformly bounded. Also for $t_1, t_2 \in J$, notice

$$\begin{aligned} |Bx_n(t_1)-Bx_n(t_2)| &\leq |q(t_1)-q(t_2)| + \left|\int_{\sigma(t_1)}^{\sigma(t_2)} h_r(s)ds\right| \\ &= |q(t_1)-q(t_2)| + |p(t_1)-p(t_2)|, \end{aligned}$$

where $p(t) = \int_0^{\sigma(t)} h_r(s)ds$. Since p and q as uniformly continuous functions on J, we conclude that $\{Bx_n\}$ is an equi-continuous set in $BM(J,R)$. Hence $B : \overline{B}_r(a) \to BM(J,R)$ is compact in view of Arzela-Aseoli theorem.

Finally for any $y \in \overline{B}_r(0)$, we have

$$\begin{aligned} \|A0+By\|_{BM} &\leq \|A0\|_{BM} + \|By\|_{BM} \\ &\leq \|q\|_{BM} + \sup_{t\in J}\int_0^{\mu(t)} |f(s,0)|ds + \|h_r\|_{L^1} \\ &\leq (1-\|\alpha\|_{L^1})r \end{aligned} \tag{3.5}$$

Now an easy application of Corollary 2.1 implies that the FIE (3.1) has a solution in $\overline{B}_r(0)$. The proof is complete. □

Below we show that Theorem 3.1 could also be used to discuss the existence result for a certain differential equation. Consider the initial value problem of nonlinear perturbed first order functional differential equation (in short FDE)

$$\left\{\begin{array}{lcl} x'(t) & = & f(t, x(\theta(t)) + g(t, x(\eta(t))), \quad \text{a.e,} \quad t \in J, \\ x(0) & = & x_0 \in R, \end{array}\right. \tag{3.6}$$

where $\theta, \eta, J \to J$ are continuous and $f, g : J \times R \to R$.

By the solution of the FDE (3.6), we mean a function $x \in AC(J, R)$ that satisfies FDE(3.6) on J, where $AC(J, R)$ is a space of all absolutely real-valued functions on J.

Notice that $AC(J, R) \subset BM(J, R)$.

Theorem 3.2. *Suppose that the assumptions* (A_2) *and* (A_3) *hold. Further if there exists a real number* $r > 0$ *such that condition* (3.3) *holds with* $\|q\|_{BM} = |x_0|$*, then the FDE* (3.6) *has a solution* u *on* J *with* $\|u\| \le r$.

Proof. Notice that FDE (3.6) is equivalent to

$$x(t) = x_0 + \int_0^t f(s, x(\theta(s)))ds + \int_0^t gf(s, x(\eta(s)))ds \tag{3.7}$$

for $t \in J$. Applying Theorem 3.1 directly to the FIE (3.7) yields that it has a solution u in $\overline{B}_r(0) \subset BM(J, R)$. From the nature of the equation (3.7), it follows that u is continuous on J. As a result $u \in AC(J, R)$ with $\|u\| = \sup_{t \in J} |u(t)| \le r$. □

4. Remarks and Conclusion

Applications of fixed point theory to nonlinear differential and integral equations is an art and it depends upon the clever selection of the fixed point theorem suitable for the given data or conditions. Notice that the local solution of the FIE(3.1) could also be obtained via a nonlinear alternative of Leray-Schauder type recently proved in Dhage and Regan [4]. But in that case, we need the equation to satisfy certain boundary conditions. In the present situation this is not the case. At the present, we do not know which approach out of above two is better for dealing with the nonlinear equations. We conclude this paper by a remark that Corollary 2.2 could also be used to discuss the existence of the local solution to the FIE of the type,

$$x(t) = q(t) + \int_0^{\mu(t)} k(t,s)x(\theta(s))ds + \int_0^{\sigma(t)} v(t,s)g(s,x(\eta(s)))ds. \tag{4.1}$$

For a another approach to the FDEs (3.1) and (4.1) the readers are referred to Dhage [2] and Dhage and Ntouyas [3].

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