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FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS

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Abstract. We present fixed point results for generalized contractions on spaces with two metrics. The focus is on continuation results for such type of mappings.
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1. Introduction

This paper presents fixed point theorems for some classes of generalized contraction on metric spaces. The results are in connection with similar theorems established by Granas [7], [8], Frigon [6], Granas and Frigon [5], Precup [11], [12], Agarwal and O'Regan [1], O'Regan [10], O'Regan and Precup [9], and Avramescu [3]. Such type of results apply to semilinear equations and inclusions. Section 2 present new local and global fixed point results for contractions of the Riech-Rus type

$$d(Fx, Fy) \leq ad(x, Fx) + bd(y, Fy) + cd(x, y),$$

where a, b, c are non-negative numbers with $a + b + c < 1$ (see Rus[13]).

Section 3 is devote to similar results for a contraction of the type

$$d(Fx, Fy) \leq q \max\{d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)\}$$

where $q \in [0, \frac{1}{2})$ (see Ćirić [4]).

Throughout this article (X, d') will be a complete metric space and d another metric on X. If $x_0 \in X$ and $r > 0$ denote by $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ and by $\overline{B(x_0, r)}^{d'}$ the $d'-$closure of $B(x_0, r)$.

2. Fixed Point Results for Reich-Rus Generalized Contractions

Theorem 1. *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $F : \overline{B(x_0, r)}^{d'} \longrightarrow X$. Suppose for any $x, y \in \overline{B(x_0, r)}^{d'}$ we have*

$$d(Fx, Fy) \leq ad(x, Fx) + bd(y, Fy) + cd(x, y),$$

where a,b,c are non-negative numbers with $a + b + c < 1$.

In addition assume the following three properties hold:

$$d(x_0, Fx_0) < (1 - \frac{a+c}{1-b})r, \tag{1}$$

(2)

if $d \ngeq d'$ then F is uniformly continuous from $(B(x_0, r), d)$ into (X, d'),

and

(3) *if $d \neq d'$ then F is continuous from $(\overline{B(x_0, r)}^{d'}, d')$ into (X, d').*

Then F has a fixed point, that is there exists $x \in \overline{B(x_0, r)}^{d'}$ with $Fx = x$.

Proof. Let $x_1 = Fx_0$. From (1), since $a + b + c < 1$, we have

$$d(x_1, x_0) < (1 - \frac{a+c}{1-b})r \leq r$$

so $x_1 \in B(x_0, r)$.

Next let $x_2 = Fx_1$ and note that

$$\begin{aligned} d(x_1, x_2) &= d(Fx_0, Fx_1) \\ &\leq ad(x_0, Fx_0) + bd(x_1, Fx_1) + cd(x_0, x_1) \\ &= ad(x_0, x_1) + bd(x_1, x_2) + cd(x_0, x_1). \end{aligned}$$

Hence

$$(1 - b)d(x_1, x_2) \leq (a + c)d(x_0, x_1).$$

It follows that

$$d(x_1, x_2) \leq \frac{a+c}{1-b} d(x_0, x_1) \leq \frac{a+c}{1-b}(1 - \frac{a+c}{1-b})r.$$

Then

$$\begin{aligned} d(x_0,x_2) &\le d(x_0,x_1)+d(x_1,x_2) \\ &< (1-\frac{a+c}{1-b})r+\frac{a+c}{1-b}(1-\frac{a+c}{1-b})r \\ &= (1-\frac{a+c}{1-b})r(1+\frac{a+c}{1-b}) \\ &\le (1-\frac{a+c}{1-b})r[1+\frac{a+c}{1-b}+(\frac{a+c}{1-b})^2+(\frac{a+c}{1-b})^3+...] \\ &= (1-\frac{a+c}{1-b})r\frac{1}{1-\frac{a+c}{1-b}}=r. \end{aligned}$$

So we have $d(x_0,x_2)<r$, that is $x_2\in B(x_0,r)$. Proceeding inductively we obtain

$$\begin{aligned} d(x_{n+1},x_n) &\le \frac{a+c}{1-b}d(x_n,x_{n-1}) \\ &\le ... \le (\frac{a+c}{1-b})^n d(x_0,x_1) < (\frac{a+c}{1-b})^n(1-\frac{a+c}{1-b})r \end{aligned}$$

where $x_n=Fx_{n-1}$,$n=3,4,...$. Since $\frac{a+c}{1-b}\in[0,1)$ it follows that $(\frac{a+c}{1-b})^n\in[0,1)$ and thus

$$d(x_{n+1},x_n)\le(1-\frac{a+c}{1-b})r.$$

The last inequality implies $x_{n+1}\in B(x_0,r)$ and, the sequence (x_n) is a Cauchy sequence with respect to d. We claim that

(4) (x_n) is a Cauchy sequence with respect to d'.

If $d\ge d'$ this is trivial. Next suppose $d\not\ge d'$. Let $\varepsilon>0$ be given. Now (2) guarantees that there exists $\delta>0$ such that

(5) $d'(Fx,Fy)<\varepsilon$ whenever $x,y\in B(x_0,r)$ and $d(x,y)<\delta$.

From above the sequence (x_n) is a Cauchy sequence with respect to d, so we know that there exists N with

(6) $d(x_n,x_m)<\delta$ for all $n,m\ge N$.

Now (5) and (6) imply

$$d'(x_{n+1},x_{m+1})=d'(Fx_n,Fy_m)<\varepsilon \text{ whenever } n,m\ge N,$$

which proves (4). Now since (X, d') is complete there exists $x \in \overline{B(x_0, r)}^{d'}$ with $d'(x_n, x) \to 0$ as $n \to \infty$. We claim now that

$$x = Fx. \tag{7}$$

First consider the case when $d \neq d'$. Notice

$$d'(x, Fx) \leq d'(x, x_n) + d'(x_n, Fx) = d'(x, x_n) + d'(Fx_{n-1}, Fx).$$

Let $n \to \infty$ and using (3) we obtain

$$d'(x, Fx) \leq d'(x, x) + d'(Fx, Fx)$$

so $d'(x, Fx) = 0$, and thus (7) is true in this case. Next suppose $d = d'$ ((2) and (3) do not hold). Then

$$\begin{aligned} d(x, Fx) &\leq d(x, x_n) + d(x_n, Fx) = d(x, x_n) + d(Fx_{n-1}, Fx) \\ &\leq d(x, x_n) + ad(x_{n-1}, Fx_{n-1}) + bd(x, Fx) + cd(x_{n-1}, x). \end{aligned}$$

Hence

$$(1 - b)d(x, Fx) \leq d(x, x_n) + cd(x_{n-1}, x) + ad(x_{n-1}, x_n).$$

In the last inequality letting $n \to \infty$ we obtain

$$(1 - b)d(x, Fx) \leq 0.$$

So $d(x, Fx) = 0$, and (7) holds. Thus, the proof of the theorem is complete. □

Next we present an homotopy result for this type of generalized contractions.

Theorem 2. *Let (X, d') be a complete metric space and d another metric on X. Let $Q \subset X$ be $d'-$closed and let $U \subset X$ be $d-$open and $U \subset Q$. Suppose $H : Q \times [0, 1] \longrightarrow X$ satisfies the following five properties:*

(i) $x \neq H(x, \lambda)$ for $x \in Q \backslash U$ and $\lambda \in [0, 1]$;

(ii) for any $\lambda \in [0, 1]$ and $x, y \in Q$ we have

$$d(H(x, \lambda), H(y, \lambda)) \leq ad(x, H(x, \lambda) + bd(y, H(y, \lambda)) + cd(x, y)$$

with a, b, c non-negative numbers and $a + b + c < 1$;

(iii) $H(x, \lambda)$ is continuous in λ with respect to d, uniformly for $x \in Q$;

(iv) if $d \ngeq d'$ assume H is uniformly continuous from $U \times [0, 1]$ endowed with the metric d on U into (X, d');

(v) if $d \neq d'$ assume H is continuous from $Q \times [0, 1]$ endowed with the metric d' on Q into (X, d').

In addition assume H_0 has a fixed point. Then for each $\lambda \in [0,1]$ we have that H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(.) = H(.,\lambda)$).

Proof. Let

$$A := \{\lambda \in [0,1];\ \text{there exists } x \in U \text{ such that } H(x,\lambda) = x\}.$$

Since H_0 has a fixed point and (i) holds we have $0 \in A$, and so the set A is nonempty. We will show A is open and closed in $[0,1]$ and so by the connectedness of $[0,1]$ we have $A = [0,1]$ (see [2]) and the proof is finished.

First we show that A is closed in $[0,1]$.

Let (λ_k) be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition of A for each k, there exists $x_k \in U$ such that $x_k = H(x_k, \lambda_k)$. Now we have

$$\begin{aligned} d(x_k,x_j) &= d(H(x_k,\lambda_k),H(x_j,\lambda_j)) \\ &\leq d(H(x_k,\lambda_k),H(x_k,\lambda)) + d(H(x_k,\lambda),H(x_j,\lambda)) \\ &\quad + d(H(x_j,\lambda),H(x_j,\lambda_j)) \\ &\leq d(H(x_k,\lambda_k),H(x_k,\lambda)) \\ &\quad + ad(x_k,H(x_k,\lambda)) + bd(x_j,H(x_j,\lambda)) + cd(x_k,x_j) \\ &\quad + d(H(x_j,\lambda),H(x_j,\lambda_j)). \end{aligned}$$

Hence

$$\begin{aligned} (1-c)d(x_k,x_j) &\leq d(H(x_k,\lambda_k),H(x_k,\lambda)) \\ &\quad + d(H(x_j,\lambda),H(x_j,\lambda_j)) \\ &\quad + ad(H(x_k,\lambda_k),H(x_k,\lambda)) + bd(H(x_j,\lambda),H(x_j,\lambda_j)) \\ &= (1+a)d(H(x_k,\lambda_k),H(x_k,\lambda)) \\ &\quad + (1+b)d(H(x_j,\lambda),H(x_j,\lambda_j)) \end{aligned}$$

and (iii) guarantees that (x_k) is a Cauchy sequence with respect to d. We claim that

$$(8) \qquad (x_k) \text{ is a Cauchy sequence with respect to } d'.$$

If $d \geq d'$ this is trivial. If $d \not\geq d'$ then

$$d'(x_k,x_j) = d'(H(x_k,\lambda_k),H(x_j,\lambda_j))$$

and (iv) guarantees that (8) holds (note as well that (x_k) is a Cauchy sequence with respect to d and (λ_k) is Cauchy sequence in $[0,1]$). Now since (X,d') is complete there exists an $x \in Q$ such that $d'(x_k, x) \to 0$ as $k \to \infty$. Claim now that

$$x = H(x,\lambda). \tag{9}$$

We consider first the case $d \neq d'$. Then

$$\begin{aligned} d'(x, H(x,\lambda)) &\leq d'(x,x_k) + d'(x_k, H(x,\lambda)) \\ &= d'(x,x_k) + d'(H(x_k,\lambda_k), H(x,\lambda)) \end{aligned}$$

together with (v), letting $k \to \infty$, we have $d'(x, H(x,\lambda)) = 0$, so (9) holds.

We consider now the case $d = d'$. Then

$$\begin{aligned} d(x, H(x,\lambda)) &\leq d(x,x_k) + d(H(x_k,\lambda_k), H(x,\lambda)) \\ &\leq d(x,x_k) + d(H(x_k,\lambda_k), H(x,\lambda_k)) \\ &\quad + d(H(x,\lambda_k), H(x,\lambda)) \\ &\leq d(x,x_k) + ad(x_k, H(x_k,\lambda_k)) \\ &\quad + bd(x, H(x,\lambda_k)) + cd(x,x_k) \\ &\quad + d(H(x,\lambda_k), H(x,\lambda)) \\ &= (1+c)d(x,x_k) + a.0 \\ &\quad + bd(x, H(x,\lambda_k)) + d(H(x,\lambda_k), H(x,\lambda)) \\ &\leq (1+c)d(x,x_k) + bd(x, H(x,\lambda)) \\ &\quad + bd(H(x,\lambda), H(x,\lambda_k)) \\ &\quad + d(H(x,\lambda_k), H(x,\lambda)) \end{aligned}$$

Now we have

$$(1-b)d(x, H(x,\lambda)) \leq (1+c)d(x,x_k) + (1+b)d(H(x,\lambda_k), H(x,\lambda))$$

Letting $k \to \infty$ and using (iii) we obtain

$$(1-b)d(x, H(x,\lambda)) \leq 0.$$

So we have $d(x, H(x,\lambda)) = 0$, that is (9) holds. Now from (9) and (i) we have $x \in U$. Consequently $\lambda \in A$ and so A is closed in [0,1].

We prove now A is open in $[0,1]$.

Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. From U $d-$open there exists a $d-$ball $B(x_0, \delta) = \{x \in X; d(x, x_0) < \delta\}$, $\delta > 0$, and $B(x_0, \delta) \subset U$. From (iii) we have that H is uniformly continuous on $B(x_0, \delta)$.

Let $\varepsilon = (1 - \frac{a+c}{1-b})\delta > 0$ and using the uniform continuity of H we have: there exists $\eta = \eta(\delta) > 0$ such that for each $\lambda \in [0,1]$ $\mid \lambda - \lambda_0 \mid \leq \eta$ with $d(H(x,\lambda), H(x,\lambda_0)) < \varepsilon$ for any $x \in B(x_0, \delta)$. So this property holds for $x = x_0$, and then we have

$$\begin{aligned} d(x_0, H(x_0,\lambda)) &= d(H(x_0,\lambda_0), H(x_0,\lambda)) < (1 - \frac{a+c}{1-b})\delta \\ \text{for } \lambda &\in [0,1] \text{ and } \mid \lambda - \lambda_0 \mid \leq \eta. \end{aligned}$$

Using now (ii), (iv) and (v) together with the theorem (2.1) (in this case $r = \delta$ and $F = H_\lambda$) we get: there exists $x_\lambda \in \overline{B(x_0,\delta)}^{d'} \subset Q$ with $x_\lambda = H_\lambda(x_\lambda)$ for $\lambda \in [0,1]$ and $\mid \lambda - \lambda_0 \mid \leq \eta$. But $x_\lambda \in U$ ((i) guarantees that) and so A contains all $\lambda \in [0,1]$ with $\mid \lambda - \lambda_0 \mid \leq \eta$. Consequently A is open in $[0,1]$. □

3. Fixed Point Result for Ćirić Generalized Contractions

Theorem 3. *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $F : \overline{B(x_0,r)}^{d'} \longrightarrow X$. Assume that there exists $q \in [0, \frac{1}{2})$ such that for any $x, y \in \overline{B(x_0,r)}^{d'}$ we have*

$$d(Fx, Fy) \leq q \max\{d(x,y), d(x,Fx), d(y,Fy), d(x,Fy), d(y,Fx)\} \tag{10}$$

In addition assume:

$$d(x_0, Fx_0) < (1 - \frac{q}{1-q})r \tag{11}$$

(12)
if $d \ngeq d'$ assume F is uniformly continuous from $(B(x_0,r), d)$ to (X, d')

(13) *if $d \neq d'$ assume F continuous from $(\overline{B(x_0,r)}^{d'}, d')$ to (X, d')*

Then F has a fixed point i.e., there exists $x \in \overline{B(x_0,r)}^{d'}$ with $x = Fx$.

Proof. Let $x_1 = Fx_0$. From inequality (11) we have

$$d(x_1, x_0) = d\,(x_0, Fx_0) < (1 - \frac{q}{1-q})r \leq r$$

so $x_1 \in B(x_0, r)$. Next let $x_2 = Fx_1$ and note that

$$\begin{aligned} d(x_1, x_2) &= d(Fx_0, Fx_1) \\ &\leq q \max\{d(x_0, x_1), d(x_0, Fx_0), d(x_1, Fx_1), d(x_0, Fx_1), d(x_1, Fx_0)\} \\ &= q \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), d(x_1, x_1)\} \\ &= q \max\{d(x_0, x_1), d(x_0, x_2), d(x_1, x_2)\} \\ &\leq q \max\{d(x_0, x_1), d(x_0, x_1) + d(x_1, x_2), d(x_1, x_2)\} \\ &= q[d(x_0, x_1) + d(x_1, x_2)]. \end{aligned}$$

Then

$$d(x_1, x_2) \leq \frac{q}{1-q} d(x_0, x_1).$$

It follows that

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq (1 + \frac{q}{1-q}) d(x_0, x_1) \\ &< [1 + \frac{q}{1-q} + (\frac{q}{1-q})^2 + ...](1 - \frac{q}{1-q})r = r. \end{aligned}$$

It follows that $x_2 \in B(x_0, r)$.

Now let $x_3 = Fx_2$.We have

$$\begin{aligned} d(x_2, x_3) &= d(Fx_1, Fx_2) \\ &\leq q \max\{d(x_1, x_2), d(x_1, Fx_1), d(x_2, Fx_2), d(x_1, Fx_2), d(x_2, Fx_1)\} \\ &= q \max\{d(x_1, x_2), d(x_1, x_3), d(x_2, x_3)\} \\ &\leq q \max\{d(x_1, x_2), d(x_1, x_2) + d(x_2, x_3), d(x_2, x_3)\} \\ &= q[d(x_1, x_2) + d(x_2, x_3)]. \end{aligned}$$

Hence

$$d(x_2, x_3) \leq \frac{q}{1-q} d(x_1, x_2) \leq (\frac{q}{1-q})^2 d(x_0, x_1).$$

Then

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_2) + d(x_2, x_3) \\ &\leq [1 + \frac{q}{1-q} + (\frac{q}{1-q})^2] d(x_0, x_1) \\ &< [1 + \frac{q}{1-q} + (\frac{q}{1-q})^2 + ...](1 - \frac{q}{1-q})r = r. \end{aligned}$$

Thus $x_3 \in B(x_0, r)$.

Inductively we obtain

$$(14) \qquad \begin{aligned} d(x_{n+1}, x_n) &\leq \frac{q}{1-q} d(x_n, x_{n-1}) \leq ... \leq (\frac{q}{1-q})^n d(x_0, x_1) \\ &< (\frac{q}{1-q})^n (1 - \frac{q}{1-q}) r \end{aligned}$$

for $x_n = Fx_{n-1}, n = 3, 4, ...$which implies that

$$d(x_0, x_{n+1}) \leq d(x_0, x_n) + d(x_n, x_{n+1}) < r.$$

Hence $x_{n+1} \in B(x_0, r)$.

Now,because $q \in [0, \frac{1}{2})$ from (14) we deduce that (x_n) is Cauchy sequence with respect to d. We will prove that

$$(15) \qquad (x_n) \text{ is a Cauchy sequence with respect to } d'.$$

If $d \geq d'$ this is trivial. Next assume $d \ngeq d'$.

Let $\varepsilon > 0$. From (12) we have: there exists $\delta > 0$ such that

$$(16) \qquad d'(Fx, Fy) < \varepsilon \text{ for any } x, y \in B(x_0, r) \text{ and } d(x, y) < \delta.$$

From the start we know that there exists a positive natural number N with

$$(17) \qquad d(x_n, x_m) < \delta \ \text{ for all } n, m \geq N.$$

Now (16)+(17) implies that

$$d'(x_{n+1}, x_{m+1}) = d'(Fx_n, Fx_m) < \varepsilon, \text{for all } n, m \geq N,$$

so (15) holds. Since (X, d') is complete we have that there exists $x \in \overline{B(x_0, r)}^{d'}$ with $d'(x_n, x) \to 0$,as $n \to \infty$.

Claim now that

$$(18) \qquad Fx = x.$$

If (18) holds then the proof is complete. First take the case $d \neq d'$. Then we have

$$d'(x, Fx) \leq d'(x, x_n) + d'(x_n, Fx) = d'(x, x_n) + d'(Fx_{n-1}, Fx).$$

Letting $n \to \infty$ and using (13) we obtain

$$d'(x, Fx) \leq 0 + 0 = 0 \text{ which implies } x = Fx.$$

So, in this case (18) holds.

Now assume $d = d'$. Then

$$\begin{aligned} d(x, Fx) &\leq d(x, x_n) + d(x_n, Fx) = d(x, x_n) + d(Fx_{n-1}, Fx) \\ &\leq d(x, x_n) + q \max\{d(x_{n-1}, x), d(x_{n-1}, Fx_{n-1}), \\ &\qquad d(x, Fx), d(x_{n-1}, Fx), d(x, Fx_{n-1})\} \\ &= d(x, x_n) + q \max\{d(x_{n-1}, x), d(x_{n-1}, x_n), \\ &\qquad d(x, Fx), d(x_{n-1}, Fx), d(x, x_n)\} \\ &\leq d(x, x_n) + q \max\{d(x_{n-1}, x), d(x_{n-1}, x_n) d(x, x_n), \\ &\qquad d(x, Fx), d(x_{n-1}, x) + d(x, Fx)\}. \end{aligned}$$

Hence

$$\begin{aligned} d(x, Fx) &\leq d(x, x_n) \\ &\quad + q \max\left\{d(x, x_n), d(x, Fx), d(x_{n-1}, x_n), d(x_{n-1}, x) + d(x, Fx)\right\}. \end{aligned}$$

Letting $n \to \infty$ we obtain

$$d(x, Fx) \leq q \max\{0, d(x.Fx), 0, 0 + d(x, Fx)\} = q d(x, Fx).$$

This implies

$$d(x, Fx) = 0.$$

So $x = Fx$ and (18) holds. □

The following global result can be easy obtained from the above theorem.

Theorem 4. *Let (X, d') be a complete metric space, d another metric on X, and $F : X \longrightarrow X$. Assume there exists $q \in [0, \frac{1}{2})$ such that $\forall x, y \in X$ we have*

$$d(Fx, Fy) \leq q \max\{d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)\}.$$

In addition assume that the following proprieties hold:

if $d \ngeq d'$ F is uniformly continuous from (X, d') to (X, d');

if $d \neq d'$ F continuous from (X, d') to (X, d').

Then F has a fixed point.

Proof. Let $x_0 \in X$ and take any $r > 0$ such that

$$d(x_0, Fx_0) < (1 - \frac{q}{1-q})r.$$

Then from the above theorem there exists $x \in \overline{B(x_0, r)}^{d'}$ with $x = Fx$. □

Next we present an homotopy result for this type of generalized contractions.

Theorem 5. *Let (X, d') be a complete metric space and let d another metric on X. Let $Q \subset X$ $d'-$closed and let $U \subset X$ $d-$open and $U \subset Q$. Suppose $H : Q \times [0, 1] \longrightarrow X$ with the following properties:*

(i) $x \neq H(x, \lambda)$ for $x \in Q \backslash U$ and $\lambda \in [0, 1]$;

(ii)there exists $q \in [0, \frac{1}{2})$ such that, for any $\lambda \in [0, 1]$ and $x, y \in Q$ we have

$$\begin{aligned} & d(H(x,\lambda), H(y,\lambda)) \\ \leq\ & q \max\{d(x,y), d(x, H(x,\lambda)), d(y, H(y,\lambda)), d(x, H(y,\lambda)), d(y, H(x,\lambda))\} \end{aligned}$$

(iii) $H(x, \lambda)$ is continuous in λ with respect to d, uniformly for $x \in Q$;

(iv) if $d \ngeq d'$ assume H is uniformly continuous from $U \times [0, 1]$ endowed with the metric d on U into (X, d'); and

(v) if $d \neq d'$ assume H is continuous from $Q \times [0, 1]$ endowed with the metric d' on Q into (X, d').

In addition assume H_0 has a fixed point. Then for each $\lambda \in [0, 1]$ we have that H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(.) = H(., \lambda)$).

Proof. Let

$$A = \{\lambda \in [0, 1];\ \text{exists } x \in U \text{ such that } H(x, \lambda) = x\}.$$

Since H_0 has a fixed point and (i) holds we have $0 \in A$, and so the set A is nonempty. We will show A is open and closed in $[0, 1]$ and so by the connectedness of $[0, 1]$ we have $A = [0, 1]$ (see [2])and the proof is finished.

First we show that A is closed in $[0, 1]$.

Let (λ_k) be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition of A for each k, there exists $x_k \in U$ such that $x_k = H(x_k, \lambda_k)$. Now we have

$$\begin{aligned}
d(x_k, x_j) &= d(H(x_k,\lambda_k), H(x_j,\lambda_j)) \\
&\leq d(H(x_k,\lambda_k), H(x_k,\lambda)) \\
&\quad + d(H(x_k,\lambda), H(x_j,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\
&\leq d(H(x_k,\lambda_k), H(x_k,\lambda)) \\
&\quad + q\max\{d(x_k,x_j), d(x_k, H(x_k,\lambda)), \\
&\quad d(x_j, H(x_j,\lambda)), d(x_k, H(x_j,\lambda)), d(x_j, H(x_k,\lambda))\} \\
&\quad + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\
&\leq d(H(x_k,\lambda_k), H(x_k,\lambda)) \\
&\quad + q\max\{d(x_k,x_j), d(H(x_k,\lambda_k), H(x_k,\lambda)), \\
&\quad d(H(x_j,\lambda_j), H(x_j,\lambda)),\ d(H(x_k,\lambda_k), H(x_j,\lambda)), d(H(x_j,\lambda_j), H(x_k,\lambda))\} \\
&\quad + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\
&\leq d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\
&\quad + q\max\{d(x_k,x_j), d(H(x_k,\lambda_k), H(x_k,\lambda)), d(H(x_j,\lambda_j), H(x_j,\lambda)), \\
&\qquad\quad d(H(x_k,\lambda_k), H(x_j,\lambda_j)) + d(H(x_j,\lambda_j), H(x_j,\lambda)), \\
&\qquad\quad d(H(x_j,\lambda_j), H(x_k,\lambda_k)) + d(H(x_k,\lambda_k), H(x_k,\lambda))\} \\
&\leq d(x_k, H(x_k,\lambda)) + d(x_j, H(x_j,\lambda)) \\
&\quad + q\max\{d(x_k,x_j) + d(x_k, H(x_k,\lambda)), d(x_k,x_j) + d(x_j, H(x_j,\lambda))\} \\
&\leq d(x_k, H(x_k,\lambda)) + d(x_j, H(x_j,\lambda)) \\
&\quad + q[2d(x_k,x_j) + d(x_k, H(x_k,\lambda)) + d(x_j, H(x_j,\lambda))]
\end{aligned}$$

$$\begin{aligned}
&= d(x_k, H(x_k,\lambda)) + d(x_j, H(x_j,\lambda)) \\
&\quad + q\max\{d(x_k,x_j), d(x_k, H(x_k,\lambda)), d(x_j, H(x_j,\lambda)) \\
&\quad d(x_k,x_j) + d(x_j, H(x_j,\lambda)), \\
&\quad d(x_k,x_j) + d(x_k, H(x_k,\lambda))\}
\end{aligned}$$

and

$$(1-2q)d(x_k,x_j) \leq (1+q)[d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j))].$$

Hence

$$d(x_k, x_j) \leq \frac{1+q}{1-2q}[d(H(x_k, \lambda_k), H(x_k, \lambda)) + d(H(x_j, \lambda), H(x_j, \lambda_j))]$$

and (iii) guarantees that (x_k) is a Cauchy sequence with respect to d. We claim that

$$(x_k) \text{ is a Cauchy sequence with respect to } d'. \tag{19}$$

If $d \geq d'$ this is trivial. If $d \not\geq d'$ then

$$d'(x_k, x_j) = d'(H(x_k, \lambda_k), H(x_j, \lambda_j))$$

and (iv) guarantees that (19) holds (note as well that (x_k) is a Cauchy sequence with respect to d and (λ_k) is Cauchy sequence in $[0, 1]$). Now since (X, d') is complete there exists an $x \in Q$ such that $d'(x_k, x) \to 0$ as $k \to \infty$. Claim now that

$$x = H(x, \lambda). \tag{20}$$

We consider first the case $d \neq d'$. Then

$$\begin{aligned} d'(x, H(x, \lambda)) &\leq d'(x, x_k) + d'(x_k, H(x, \lambda)) \\ &= d'(x, x_k) + d'(H(x_k, \lambda_k), H(x, \lambda)) \end{aligned}$$

together with (v), letting $k \to \infty$, we have $d'(x, H(x, \lambda)) = 0$, so (20) holds.

We consider now the case $d = d'$. Then

$$\begin{aligned}
d(x, H(x,\lambda)) &\leq d(x, x_k) + d(x_k, H(x,\lambda)) \\
&\leq d(x, x_k) + d(H(x_k,\lambda_k), H(x,\lambda_k)) + d(H(x,\lambda_k), H(x,\lambda)) \\
&\leq d(x, x_k) \\
&\quad + q \max\{d(x, x_k), d(x_k, H(x_k,\lambda_k)), \\
&\qquad d(x, H(x,\lambda_k)), d(x, H(x_k,\lambda_k)), d(x_k, H(x,\lambda_k))\} \\
&\quad + d(H(x,\lambda_k), H(x,\lambda)) \\
&\leq d(x, x_k) + q \max\{d(x, x_k), d(x_k, x_k), \\
&\qquad\qquad d(x, H(x,\lambda_k)), d(x_k, H(x,\lambda_k))\} \\
&\quad + d(H(x,\lambda_k), H(x,\lambda)) \\
&= d(x, x_k) \\
&\quad + q \max\{d(x, x_k), d(x, H(x,\lambda_k)), d(x_k, H(x,\lambda_k))\} \\
&\quad + d(H(x,\lambda_k), H(x,\lambda)) \\
&\leq d(x, x_k) \\
&\quad + q \max\{d(x, x_k), d(x, H(x,\lambda_k)), d(x, x_k) + d(x, H(x,\lambda_k))\} \\
&\quad + d(H(x,\lambda_k), H(x,\lambda)) \\
&\leq d(x, x_k) \\
&\quad + q[d(x, x_k) + d(x, H(x,\lambda_k))] \\
&\quad + d(H(x,\lambda_k), H(x,\lambda)) \\
&\leq d(x, x_k) \\
&\quad + q[d(x, x_k) + d(x, H(x,\lambda)) + d(H(x,\lambda), H(x,\lambda_k))] \\
&\quad + d(H(x,\lambda_k), H(x,\lambda))
\end{aligned}$$

and we have

$$\begin{aligned}
d(x, H(x,\lambda)) &\leq d(x, x_k) \\
&\quad + q[d(x, x_k) + d(x, H(x,\lambda)) + d(H(x,\lambda), H(x,\lambda_k))] \\
&\quad + d(H(x,\lambda_k), H(x,\lambda))
\end{aligned}$$

Letting $k \to \infty$ we have

$$d(x, H(x,\lambda)) \leq 0 + q[0 + d(x, H(x,\lambda)) + 0] + 0$$

$$d(x, H(x,\lambda)) \leq qd(x, H(x,\lambda))$$

so $d(x, H(x,\lambda)) = 0$ and (20) holds.We have now $H(x,\lambda) = x$ for $x \in Q$ and with (i) we have $H(x,\lambda) = x$ for $x \in U$. Consequently $\lambda \in A$ and so A is closed in [0,1].

We prove now A is open in $[0,1]$.

Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. From U $d-$open there exists a $d-$ball $B(x_0,\delta) = \{x \in X; d(x,x_0) < \delta\}$, $\delta > 0$, and $B(x_0,\delta) \subset U$. From (iii) we have that H is uniformly continuous on $B(x_0,\delta)$.

Let $\varepsilon = (1 - \frac{q}{1-q})\delta > 0$ and using the uniform continuity of H we have: there exists $\eta = \eta(\delta) > 0$ such that for each $\lambda \in [0,1]$ $\mid \lambda - \lambda_0 \mid \leq \eta$ with $d(H(x,\lambda), H(x,\lambda_0)) < \varepsilon$ for any $x \in B(x_0,\delta)$. So this property holds for $x = x_0$, and then we have

$$\begin{aligned} d(x_0, H(x_0,\lambda)) &= d(H(x_0,\lambda_0), H(x_0,\lambda)) < (1 - \frac{q}{1-q})\delta \\ \text{for } \lambda &\in [0,1] \text{ and } \mid \lambda - \lambda_0 \mid \leq \eta. \end{aligned}$$

Using now (ii), (iv) and (v) together with the theorem (3) (in this case $r = \delta$ and $F = H_\lambda$) we get: there exists $x_\lambda \in \overline{B(x_0,\delta)}^{d'} \subset Q$ with $x_\lambda = H_\lambda(x_\lambda)$ for $\lambda \in [0,1]$ and $\mid \lambda - \lambda_0 \mid \leq \eta$. But $x_\lambda \in U$ ((i) guarantees that) and so A contains all $\lambda \in [0,1]$ with $\mid \lambda - \lambda_0 \mid \leq \eta$. Consequently A is open in $[0,1]$. □

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