Laboratory 9: Mathematical Models for Interacting Populations

1. Consider the basic 2-species Lotka–Volterra competition model with each species x_1 and x_2 having logistic growth in the absence of the other:

$$\begin{cases} x_1'(t) &= r_1 x_1(t)\left(1-\frac{x_1(t)}{K_1}\right) - b_{12}x_1(t)\,x_2(t) \\ x_2'(t) &= r_2 x_2(t)\left(1-\frac{x_2(t)}{K_2}\right) - b_{21}x_1(t)\,x_2(t) \end{cases}$$

   where r_1, K_1, r_2, K_2, b_{12} and b_{21} are all positive constants and, the r 's are the linear birth rates and the K's are the carrying capacities. The b_{12} and b_{21} measure the competitive effect of x_2 on x_1 and x_1 on x_2 respectively: they are generally not equal.

   (a) Find the equilibrium points;

   (b) Study the linearized stability;

   (c) Draw the phase portrait and find numerical solution for different initial conditions in the case of $r_1 = r_2 = 0.5$, $K_1 = 1$, $K_2 = 2$, $b_{12} = \frac{1}{12}$, $b_{21} = \frac{1}{6}$.

2. Let's consider the following predator–prey model with child care of J. M. A. Danby. Suppose that the prey $x(t)$ is divided into two classes, $x_1(t)$ and $x_2(t)$, of young and adults. Suppose that the young are protected from predators $y(t)$. Assume that the young increase in proportion to the number of adults and decrease due to death or to moving into the adult class. Then

$$x_1'(t) = a \cdot x_2(t) - b \cdot x_1(t) - c \cdot x_1(t)$$

   The number of adults is increased by the young growing up and decreased by natural death and predation, so that we model

$$x_2'(t) = b \cdot x_1(t) - d \cdot x_2(t) - e \cdot x_2(t)\,y(t)$$

   Finally, for the predators, we take

$$y'(t) = -f \cdot y(t) + g \cdot x_2(t)\,y(t)\,.$$

   (a) Find the equilibrium points;

   (b) Study the linearized stability;

   (c) Draw the phase portrait and find numerical solution for different initial conditions in the case of $a = 2$, $b = c = d = \frac{1}{2}$ and $e = f = g = 1$.