MATHEMATICA, 68 (91), N° 1, 2026, pp. 127–145

# STABILITY ANALYSIS OF KDV EQUATION WITH TIME-VARYING DELAY AND INPUT SATURATION

AHMAT MAHAMAT TABOYE and TOUFIK ENNOUARI

**Abstract.** This paper deals with the stability analysis of the Korteweg–de Vries equation with time-varying delay on internal feedback in the presence of a saturated source term. The well-posedness is proven under specific assumptions regarding the time-varying delay. Moreover, the exponential stability is shown using an appropriate Lyapunov functional.

**MSC 2020.** 93B05, 35R02, 93C20.

**Key words.** KdV equation, stability, saturated source term, time-varying delay.

## 1. INTRODUCTION

The Korteweg-de Vries equation, formulated as: $u_t + u_x + u_{xxx} + uu_x = 0$ originated to model long waves in shallow water channels. It serves as a pivotal model for understanding nonlinear wave dynamics and soliton behaviors. Various techniques have been employed to investigate the well-posedness of the KdV equation, as detailed in literature such as [12, 2, 24]. Several works examine this equation, notably with a focus on control theory aspects see for instance [5, 25, 23]. Specific research on boundary and distributed control for the KdV equation is particularly discussed in works such as [13, 28, 6], and [23, 21].

In this paper, we focus on the analysis of a class of Korteweg-de Vries equations with time-varying delay on internal feedback in the presence of a source term. The system under consideration is formulated as follows:

$$(1)\qquad \begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + a(x)u(x,t) + b(x)u(x,t-\theta(t)) \\ = f(x,t), \quad t>0, x \in (0,L); \\ u(0,t) = u(L,t) = u_x(L,t) = 0, \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(x,t) = z_0(x,t), \quad x \in (0,L), 0 < t < \theta(0). \end{cases}$$

Here, the variable $u$ signifies the system's state, while $f$ represents the source term. $L > 0$ denotes the length of the spatial domain, the delay denoted by $\theta$ is a function of time $t$. Additionally, $a = a(x)$ and $b = b(x)$ correspond to

Corresponding author: Ahmat Taboye.

DOI: 10.24193/mathcluj.2026.1.11

real functions that satisfy specific assumptions, which will be detailed subsequently in this paper.
The presence of delay in some areas raises significant practical problems. The stability problems of partial differential equations with delay have recently attracted very active research. Studies have shown that even a small delay in feedback can make a system unstable, as discussed in several references [8, 7]. However, the existence of a delay can also improve system performance, as observed in studies such as [1]. It is clear that delay affects the stability of systems, both in theory and in practical applications. When delay is time-dependent, the analysis of system stability becomes significantly more complex. Several studies have looked at the stability of partial differential equations involving time-varying delay, with notable references such as [16, 9, 20, 10].

In recent years, researchers have shown an increased interest in addressing stability issues related to delay in the Korteweg-de Vries equation(see e.g. [4, 27, 19, 18]). Baudouin et al. in [4] investigated the following KdV equation with boundary time-delay feedback:

$$(2)\quad \begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) = 0, & t > 0, x \in (0, L); \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-h), & t > 0; \\ u(x,0) = u_0(x), & x \in (0, L); \\ u_x(0,t) = z_0(t), & t \in [-h, 0]. \end{cases}$$

The authors in [4] presented two distinct approaches proving the exponential stability of system (2). The first approach uses a Lyapunov functional with an estimation of the decay rate, assuming that the length $L$ of the spatial domain satisfies $L < \pi\sqrt{3}$. Secondly, using an observability inequality approach without providing an estimation of the decay rate, and for any length such that $L \notin \mathbb{N} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, k, l \in \mathbb{N}^\star \right\}$, the authors also establish the exponential stability of the system (2).

In the study presented in [29], the asymptotic stability of the following nonlinear Korteweg-de Vries equation with time-delay internal feedback was investigated:

$$(3)\qquad \begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) \\ +a(x)u(x,t) + b(x)u(x,t-h) = 0, \quad t > 0, x \in (0, L); \\ u(0,t) = u(L,t) = u_x(L,t) = 0, \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in (0, L); \\ u_x(x,t) = z_0(x,t), \quad x \in (0, L), t \in [-h, 0], \end{cases}$$

The authors established the semiglobal stability of (3) for any lengths under the following assumption: $\exists c_0 > 0, \quad b(x) + c_0 \leq a(x) \quad$ a.e. in $\omega$, where $\omega$ is an open subset of $(0, L)$. This result was proved by using an observability inequality on the nonlinear system (3). Additionally, when $\operatorname{supp} b \nsubseteq \operatorname{supp} a$, the local exponential stability result is demonstrated.

In [18], the authors study the stabilization of the Korteweg-de Vries equation with time-varying delay on internal and boundary feedback. With specific assumptions concerning time-varying delay, they established well-posedness and analyzed the stability results, using an appropriate Lyapunov functional.

In this paper, the source term $f$ of system (1) is given by

$$f(x,t) = -\operatorname{sat}(\tilde{a}u(x,t)) \tag{4}$$

where $\tilde{a} = \tilde{a}(x) \in L^\infty(0,L)$ is a nonnegative function satisfying

$$\begin{cases} \tilde{a} = \tilde{a}(x) \geq \tilde{a}_0 > 0 \quad \text{on} \quad \omega \subseteq [0,L], \\ \omega \quad \text{is a nonempty open subset of} \quad [0,L], \end{cases} \tag{5}$$

and the function $\operatorname{sat}(\cdot)$ is the saturation function defined as follows

$$\operatorname{sat}(s) = \begin{cases} s, & \text{if } \|s\|_{L^2} \leq 1; \\ \frac{s}{\|s\|_{L^2}}, & \text{if } \|s\|_{L^2} \geq 1. \end{cases} \tag{6}$$

In control systems, the presence of a saturated source term has attracted considerable attention in recent decades because physical constraints or practical limitations can impose restrictions on the amplitudes of input signals, leading to unfavorable outcomes. Consequently, several studies have investigated the stability of the Korteweg-de Vries (KdV) equation with saturation, see for instance [14, 17, 3]. In particular, the study by [15] examines the global stabilization of the following nonlinear KdV equation:

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + u(x,t)u_x(x,t) & \\ +\operatorname{sat}(a(x)u(t,x)) = 0, & t \geq 0, ; x \in (0,L); \\ u(t,0) = u(t,L) = u_x(t,L) = 0, & t \geq 0; \\ u(0,x) = u_0(x). & \end{cases} \tag{7}$$

To investigate this problem, the authors worked with two different types of saturation, the saturation function given by (6) and the following saturation function

$$\operatorname{sat}(t) = \begin{cases} -u_0, & \text{if } t \leq -u_0 \\ t, & \text{if } -u_0 \leq t \leq u_0 \\ u_0, & \text{if } t \geq u_0 \end{cases} \tag{8}$$

where $u_0$ represents positive constants. Thanks to the Banach fixed-point theorem, well-posedness is proven. To establish asymptotic stability, the authors have considered two cases. When the control acts on the entire saturated domain, the saturation under consideration is the function given by (6). In this case, they used a sector condition and Lyapunov theory for infinite-dimensional systems. The two saturation functions given by (6) and (8) were used when the control acts only on a portion of the saturated domain. For this second case, they prove the asymptotic stability of the closed-loop system using an argument by contradiction.

To the best of our knowledge, no prior work has addressed to the linear Korteweg-de Vries equations with time-varying delay on internal feedback in the presence of a saturated source term. The objective of our work is to investigate the question of the local stability of system (1) with the saturated source term (4). Specifically, we analyze the well-posedness and stability result of this equation. We demonstrate exponential stability using an appropriate Lyapunov functional.

The paper is organized as follows. In section 2, the problem is stated. In section 3, we prove that our problem is well-posed. The exponential stability is investigated in section 4. Finally, some conclusion is given in section 5.

## 2. PROBLEM STATMENT

In this paper, we focus on the following semilinear system

$$
\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + a(x)u(x,t) + b(x)u(x,t-\theta(t)) \\ = -\mathrm{sat}(\tilde{a}(x)u), \quad t > 0, x \in (0,L); \\ u(0,t) = u(L,t) = u_x(L,t) = 0, \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(x,t) = z_0(x,t), \quad x \in (0,L), 0 < t < \theta(0), \end{cases} \tag{9}
$$

where $u$ stands for the state, $L > 0$ is the length of the spatial domain and $\tilde{a} = \tilde{a}(x) \in L^\infty(0,L)$, satisfying

$$
\begin{cases} \tilde{a} = \tilde{a}(x) \geq \tilde{a}_0 > 0 \quad \text{on} \quad \omega \subseteq [0,L], \\ \omega \quad \text{is a nonempty open subset of} \quad (0,L). \end{cases} \tag{10}
$$

Suppose that the delay $\theta(\cdot)$ satisfies the following conditions

$$
\theta(\cdot) \in W^{2,\infty}[0,T] \text{ for all } T > 0, \tag{11}
$$

$$
0 < \theta_0 \leq \theta(t) \leq K \text{ for all } t \geq 0, \tag{12}
$$

and

$$
\dot{\theta}(t) \leq d \leq 1 \text{ for all } t \geq 0, \tag{13}
$$

with $d \geq 0$. Moreover, we assume that the real functions $a = a(x)$ and $b = b(x)$ are nonnegative functions in $L^\infty(0,L)$. We also suppose that the functions $b$ satisfies $\operatorname{supp} b = \omega$ and

$$
b(x) \geq b_0 \geq 0, \text{ in } \omega \subset (0,L), \tag{14}
$$

where $\omega$ is a nonempty open subset of $(0,L)$. Furthermore, we suppose along the whole paper that the coefficients $a$ and $b$ satisfy the following assumption

$$
\exists c_0 > 0, \quad \frac{2-d}{2-2d} b(x) + c_0 \leq a(x) \quad \text{a.e in } \omega. \tag{15}
$$

From (14) and (15), we have

$$a(x) \geq b(x) + c_0 \geq 0, \text{ in } \omega \subset (0, L). \tag{16}$$

Now, we introduce a new variable $z$ that takes into account the delay $\theta(\cdot)$. Let $z(x, \mu, t) = u_{|\omega}(x, t - \theta(t)\mu)$ for any $x \in \omega$, $\mu \in (0, 1)$ and $t > 0$. Hence, $z$ satisfies the following system

$$\begin{cases} \theta(t)z_t(x, \mu, t) + (1 - \dot{\theta}(t)\mu)z_\mu(x, \mu, t) = 0, & x \in \omega, t > 0, \mu \in (0, 1); \\ z(x, 0, t) = u_{|\omega}(x, t), & x \in \omega, t > 0; \\ z(x, \mu, 0) = z_{0|\omega}(x, -\theta(0)\mu), & x \in \omega, \mu \in (0, 1). \end{cases} \tag{17}$$

Let the following Hilbert space $H = L^2(0, L) \times L^2(\omega \times (0, 1))$ be equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle = \int_0^L uu_1 \mathrm{d}x + \int_\omega \int_0^1 zz_1 \mathrm{d}\mu \mathrm{d}x,$$

for all $(u, z), (u_1, z_1) \in H$ and its norm is given by

$$\left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_H^2 = \int_0^L u^2 \mathrm{d}x + \int_\omega \int_0^1 z^2 \mathrm{d}\mu \mathrm{d}x.$$

Let $Y = \begin{pmatrix} u \\ z \end{pmatrix}$. Then the system (9) and (17) can be rewritten as the following fist-order system

$$\begin{cases} Y_t = A(t)Y(t) + \begin{pmatrix} -\mathrm{sat}(\tilde{a}(x)u) \\ 0 \end{pmatrix}, \quad t > 0, \\ Y(0) = \begin{pmatrix} u_0, & z_{0|\omega}(x, -\theta(0)\mu) \end{pmatrix}^T, \end{cases} \tag{18}$$

where the time-dependent operator $A(t)$ is defined as follows:

$$D(A(t)) = \{(u, z) \in H^3((0, L) \times L^2(\omega, H^1(0, L)), u(0) = u(L) = u_x(L) = 0, \\ z(x, 0) = u_{|\omega}(x), \}$$

(19)

$$A(t) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} -u_x - u_{xxx} - au - b\tilde{z}(\cdot, 1) \\ \frac{\dot{\theta}(t)\mu - 1}{\theta(t)} z_\mu \end{pmatrix} \text{ for all } \begin{pmatrix} u \\ z \end{pmatrix} \in D(A(t)),$$

where $\tilde{z}(\cdot, 1) \in L^2(0, L)$ is the extension of $z(\cdot, 1)$ by zero outside $\omega$. From [18], the domain of operator $A(t)$ is independent of $t$, i.e. $D(A(t)) = D(A(0))$. We equipped the Hilbert space $H$ with the following time-dependent inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle = \int_0^L uu_1 \mathrm{d}x + \theta(t) \int_\omega \int_0^1 \xi(x) zz_1 \mathrm{d}\mu \mathrm{d}x, \tag{20}$$

where $\xi$ is a nonnegative function in $L^\infty(0, L)$ such that $\operatorname{supp} \xi = \operatorname{supp} b = \omega$, and

$$\frac{1}{1 - d} b(x) + c_0 \leq \xi(x) \leq 2a(x) - b(x) - c_0 \quad \text{a.e. in } \omega. \tag{21}$$

Thanks to (15) the choice of $\xi$ is feasible.

REMARK 2.1. Using (12) and (14), the norm $\|\cdot\|_t$ and the usual norm $\|\cdot\|_H$ is clearly equivalent. Indeed, $\forall t>0, \forall (u,z) \in H$, we have

$$(1+\theta_0 b_0)\|(u,z)\|_H^2 \leq \|(u,z)\|_t^2 \leq (1+2K\|a\|_{L^\infty(0,L)})\|(u,z)\|_H^2. \tag{22}$$

We recall the definition of mild solution. Let us consider the abstract system in a Hilbert space

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t) + f(t), \quad t>0, \\ u(0) = u_0, \end{cases} \tag{23}$$

where $\mathcal{A}$ is an infinitesimal generator of linear $C_0-$semigroup $(T(t))_{t\geq 0}$ defined on its domain $D(A) \subseteq H$, where $H$ is a Hilbert space and $f \in L^1_{loc}([0,T],H)$.

DEFINITION 2.2 ([22, Definition 2.3]). Let $\mathcal{A}$ be the infinitesimal generator of a $C_0-$semigroup $(T(t))_{t\geq 0}$. Let $u_0 \in H$ and $f \in L^1(0,T,H)$. Then the function $u \in \mathcal{C}([0,T],H)$ given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\mathrm{d}s \quad 0 \leq t \leq T, \tag{24}$$

is the unique mild solution of the initial value problem (23) on $[0,T]$.

It should be noted that the saturation function is Lipschitzian in $L^2(0,L)$, as established by the following lemma.

LEMMA 2.3 ([26, Theorem 5.1]). *For all $(u,v) \in L^2(0,L)$, we have*

$$\|\mathrm{sat}(u) - \mathrm{sat}(v)\|_{L^2(0,L)} \leq 3\|u-v\|_{L^2(0,L)}.$$

## 3. WELL-POSEDNESS

Before stating the main result of this section, we recall the result of well-posedness of the following KdV equation with time-varying delay on the internal feedback without source term

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + a(x)u(x,t) + b(x)u(x,t-\theta(t)) \\ = 0, \quad t>0, x \in (0,L); \\ u(0,t) = u(L,t) = u_x(L,t) = 0, \quad t>0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(x,t) = z_0(x,t), \quad x \in (0,L), t \in [-h,0] \end{cases} \tag{25}$$

has been treated by Parada et al. [18]. As before, let $Y = \begin{pmatrix} u \\ z \end{pmatrix}$. We then rewrite (25) and (17) as the first-order system

$$\begin{cases} Y_t = A(t)Y(t), & t>0, \\ Y(0) = \begin{pmatrix} u_0, & z_{0|\omega}(x,-\theta(0)\mu) \end{pmatrix}^T \end{cases} \tag{26}$$

where the time-dependent operator $A(t)$ is given by (19). The proof of well-posedness result of (26) is based on the following Theorem.

THEOREM 3.1. *Assume that*

(1) $D(A(0))$, *is a dense subset of* $H$.
(2) $D(A(t)) = D(A(0)) \quad \forall t \geq 0$.
(3) *For all* $t \in [0,T]$ $A(t)$ *generates a strongly continuous semigroup on* $H$ *and the family* $A = \{A(t) : t \in [0,T]\}$ *is stable with stability constant* $C$ *and* $m$ *independent of* $t$, *i.e. the semigroup* $(T_t(s))_{s\geq 0}$ *generated by* $A(t)$ *satisfies* $\|T_t(s)Y\|_H \leq C\mathrm{e}^{ms}\|Y\|_H$, *for all* $Y \in H$, *and* $s \geq 0$.
(4) $\partial_t A(t)$ *belong to* $L^\infty_*([0,T], B(D(A(0))))$, *the space of equivalent of essentially classes bounded strongly measure functions from* $[0,T]$ *into the set* $B(D(A(0)), H)$ *of bounded operator from* $D(A(0)$ *to* $H$.

*Then, the system* (25) *has a unique solution*

$$Y \in \mathcal{C}([0,T], D(A(0))) \cap \mathcal{C}^1([0,T], H).$$

More precisely, the authors of [18] prove in their paper that if (11), (12), (13) and (14)-(15) holds, then the operator $A(t)$ satisfies all the assumptions of Theorem 3.1 and the system (26) has a unique mild solution $Y \in \mathcal{C}([0,+\infty[, H)$. Moreover if $Y_0 \in D(A(0))$, then $Y \in \mathcal{C}([0,+\infty[, D(A(0))) \cap \mathcal{C}^1([0,+\infty[, H)$.

The following theorem gives the well-posedness result of (18).

THEOREM 3.2. *We assume that* (11), (12) *and* (13) *holds and the function* $\tilde{a} = \tilde{a}(x) \in L^\infty(0,L)$ *satisfies* (5). *We also suppose that the real functions* $a$ *and* $b$ *belonging to* $L^\infty(0,L)$ *satisfy* (14)-(15), $u \in L^2(0,T,H^1(0,L))$, *and that* $Y_0 = (u_0, z_0) \in H$. *Then there exists a unique solution* $Y = (u,z) \in \mathcal{C}([0,T], H)$ *to* (18).

*Proof.* Firstly, we aim to prove that $\begin{pmatrix} -\mathrm{sat}(\tilde{a}(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$. Let $G(u) = \begin{pmatrix} -\mathrm{sat}(\tilde{a}(x)u) \\ 0 \end{pmatrix}$. Let $u_1, u_2 \in L^2(0,T,H^1(0,L))$, by using the Hölder inequality, ([15, Proposition 3.4]) and ([26, Theorem 5.1]), we get

$$\begin{aligned}
\|G(u_1) - G(u_2)\|_{L^1(0,T,H)} &= \int_0^T G(u_1) - G(u_2)\|_H \mathrm{d}t \\
&= \int_0^T \|\mathrm{sat}(\tilde{a}u_1) - \mathrm{sat}(\tilde{a}u_2)\|_{L^2(0,L))} + \|0\|_{L^2[0,1]}\mathrm{d}t \\
&= \int_0^T \|\mathrm{sat}(\tilde{a}u_1) - \mathrm{sat}(\tilde{a}u_2)\|_{L^2(0,L))}\mathrm{d}t \\
&\leq 3\int_0^T \|\tilde{a}u_1 - \tilde{a}u_2\|_{L^2(0,L))}\mathrm{d}t \\
&\leq 3\|\tilde{a}\|_{L^\infty(0,L)} \int_0^T \|u_1 - u_2\|_{L^2(0,L))}\mathrm{d}t \\
&\leq 3\|\tilde{a}\|_{L^\infty(0,L)}\sqrt{T}\sqrt{L}\|u_1 - u_2\|_{L^2(0,T,H^1(0,L))} < +\infty.
\end{aligned}$$

Therefore, $\begin{pmatrix} -\text{sat}(\tilde{a}(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$. Moreover, the operator $A(t)$ satisfies all the assumptions of Theorem 3.1. Thus, since $\begin{pmatrix} -\text{sat}(\tilde{a}(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$, then if $Y_0 \in H$, the system (18) has a unique solution $Y = (u,z) \in \mathcal{C}([0,T],H)$, according to [11, Theorem 2].

Furthermore, $\text{sat}(\tilde{a}(x)u) \in L^1(0,T,L^2(0,L))$, hence if $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$, then from [4, Proposition 2] the solution is a regular solution. $\square$

## 4. EXPONENTIAL STABILITY

The aim of this section is to study the local stability of (9). Before that, let us consider the energy of the system (9)

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)\mathrm{d}x + \frac{\theta(t)}{2}\int_\omega \int_0^1 \xi(x)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x, \tag{27}$$

where $\xi$ is given by (21). In the following proposition, we prove that the energy (27) is not increasing.

Proposition 4.1. *We assume that* (11), (12) *and* (13) *holds and the function* $\tilde{a} = \tilde{a}(x) \in L^\infty(0,L)$ *satisfies* (5). *We also suppose that the real functions* $a$ *and* $b$ *belonging to* $L^\infty(0,L)$ *satisfy* (14)-(15), $u \in L^2(0,T,H^1(0,L))$, *and that* $Y_0 = (u_0,z_0) \in D(A(0))$. *Then for any regular solution of* (9), *the energy* (27) *is non-increasing and satisfies the following inequality*

$$\begin{aligned} \dot{E}(t) \leq & -\frac{1}{2}u_x^2(0,t) - \int_{(0,L)\setminus\omega} a(x)u^2(x,t)\mathrm{d}x \\ & + \frac{1}{2}\int_\omega (-2a(x)+b(x)+\xi(x))u^2(x,t)\mathrm{d}x \\ & + \frac{1}{2}\int_\omega (\xi(x)(d-1)+b(x))u^2(x,t-\theta(t))\mathrm{d}x \\ & - \int_0^l \text{sat}(\tilde{a}u)u\mathrm{d}x \leq 0. \end{aligned} \tag{28}$$

*Proof.* Let us consider a regular solution of (9). Noting that

$$u_t(x,t-\theta(t)\mu) = \left(\frac{\dot{\theta}(t)\mu - 1}{\theta(t)}\right)u_\mu(x,t-\theta(t)\mu) \text{ for all } x \in \omega,$$

then by differentiating $E(\cdot)$, we get

$$
\begin{aligned}
\dot{E}(t) = & \int_0^L u u_t \mathrm{d}x + \frac{\dot{\theta}(t)}{2} \int_\omega \int_0^1 \xi(x) u^2(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& + \theta(t) \int_\omega \int_0^1 \xi(x) u_t(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& = - \int_0^L (u u_x + u u_{xxx} + a(x) u^2) \mathrm{d}x \\
& - \int_0^L b(x) u(x,t) u(x, t-\theta(t)) \mathrm{d}x \\
& - \int_0^L \mathrm{sat}(\tilde{a} u) u \mathrm{d}x + \frac{\dot{\theta}(t)}{2} \int_\omega \int_0^1 \xi(x) u^2(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& + \int_\omega \int_0^1 \xi(x) (\dot{\theta}(t)\mu - 1) u_\mu(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x.
\end{aligned} \tag{29}
$$

After some integration by parts, we have

$$
- \int_0^L u u_x \mathrm{d}x = 0; \quad - \int_0^L u u_{xxx} \mathrm{d}x = -\frac{1}{2} u_x^2(0,t). \tag{30}
$$

Moreover,

$$
\begin{aligned}
& \int_\omega \int_0^1 \xi(x) (\dot{\theta}(t)\mu - 1) u_\mu(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& = \int_\omega \int_0^1 \xi(x) \dot{\theta}(t) \mu u_\mu(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& - \int_\omega \int_0^1 \xi(x) u_\mu(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x
\end{aligned}
$$

Integrating by parts $\int_\omega \int_0^1 \xi(x) \dot{\theta}(t) \mu u_\mu(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x$, we obtain

$$
\begin{aligned}
& \dot{\theta}(t) \int_\omega \xi(x) \int_0^1 \mu u_\mu(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& = \dot{\theta}(t) \int_\omega \xi(x) \left[ \mu \frac{1}{2} u^2(x, t-\theta(t)\mu) \right]_0^1 \mathrm{d}x - \frac{\dot{\theta}(t)}{2} \int_\omega \xi(x) \int_0^1 u^2(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& = \frac{\dot{\theta}(t)}{2} \int_\omega \xi(x) u^2(x, t-\theta(t)) \mathrm{d}x - \frac{\dot{\theta}(t)}{2} \int_\omega \xi(x) \int_0^1 u^2(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x.
\end{aligned}
$$

Using (13), we get

$$
\begin{aligned}
& \dot{\theta}(t) \int_\omega \xi(x) \int_0^1 \mu u_\mu(x, t-\theta(t)\mu) u(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x \\
& \leq \frac{d}{2} \int_\omega \xi(x) u^2(x, t-\theta(t)) \mathrm{d}x - \frac{\dot{\theta}(t)}{2} \int_\omega \int_0^1 \xi(x) u^2(x, t-\theta(t)\mu) \mathrm{d}\mu \mathrm{d}x.
\end{aligned} \tag{31}
$$

Now integrating by parts $-\int_\omega \int_0^1 \xi(x) u_\mu(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x$, we have

$$\begin{aligned}
&-\int_\omega \xi(x)\int_0^1 u_\mu(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x\\
&= -\frac{1}{2}\int_\omega \xi(x)\left[u^2(x,t-\theta(t)\mu)\right]_0^1\mathrm{d}x \qquad (32)\\
&= \frac{1}{2}\int_\omega \xi(x)u^2(x,t)\mathrm{d}x - \frac{1}{2}\int_\omega \xi(x)u^2(x,t-\theta(t))\mathrm{d}x.
\end{aligned}$$

Hence, using (29), (30), (31), (32) and Cauchy Schwarz inequality, we get

$$\begin{aligned}
\dot{E}(t) =& -\frac{1}{2}u_x^2(0,t) - \int_0^L a(x)u^2\mathrm{d}x - \int_0^L b(x)u(x,t)u(x,t-\theta(t))\mathrm{d}x\\
&-\int_0^l \mathrm{sat}(\tilde{a}u)u\mathrm{d}x + \frac{1}{2}\int_\omega \xi(x)u^2(x,t)\mathrm{d}x - \frac{1}{2}\int_\omega \xi(x)u^2(x,t-\theta(t))\mathrm{d}x\\
&+\frac{d}{2}\int_\omega \xi(x)u^2(x,t-\theta(t))\mathrm{d}x\\
&\leq -\frac{1}{2}u_x^2(0,t) - \int_{(0,L)\ \ \omega} a(x)u^2\mathrm{d}x - \int_\omega a(x)u^2\mathrm{d}x + \frac{1}{2}\int_\omega b(x)u^2(x,t)\mathrm{d}x\\
&+\frac{1}{2}\int_\omega b(x)u^2(x,t-\theta(t))\mathrm{d}x - \int_0^l \mathrm{sat}(\tilde{a}u)u\mathrm{d}x + \frac{1}{2}\int_\omega \xi(x)u^2(x,t)\mathrm{d}x\\
&-\frac{1}{2}\int_\omega \xi(x)u^2(x,t-\theta(t))\mathrm{d}x + \frac{d}{2}\int_\omega \xi(x)u^2(x,t-\theta(t))\mathrm{d}x.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\dot{E}(t) \leq& -\frac{1}{2}u_x^2(0,t) - \int_{(0,L)\setminus\omega} a(x)u^2(x,t)\mathrm{d}x\\
&+\frac{1}{2}\int_\omega (-2a(x)+b(x)+\xi(x))u^2(x,t)\mathrm{d}x\\
&+\frac{1}{2}\int_\omega (\xi(x)(d-1)+b(x))u^2(x,t-\theta(t))\mathrm{d}x - \int_0^l \mathrm{sat}(\tilde{a}u)u\mathrm{d}x \leq 0.
\end{aligned}$$

Indeed, using (21), we have $-2a(x)+b(x)+\xi(x) \leq -c_0 < 0$, and $\frac{1}{1-d}b(x) \leq \xi(x)$, thus $\xi(x)(d-1)+b(x) < 0$. Moreover, $\int_0^L \mathrm{sat}(\tilde{a}u)u\mathrm{d}x \geq 0$, in fact, if $\|\tilde{a}u\|_{L^2} \leq 1$, then $\mathrm{sat}(\tilde{a}u)u = \tilde{a}u^2 \geq 0$. If $\|\tilde{a}u\|_{L^2} \geq 1$, then

$$\mathrm{sat}(\tilde{a}u)u = \frac{\tilde{a}u}{\|\tilde{a}u\|_{L^2}}u = \frac{\tilde{a}u^2}{\|\tilde{a}u\|_{L^2}} \geq 0,$$

where $\tilde{a} = \tilde{a}(x)$ is a nonnegative function. which concludes the proof. □

Let us take the following Lyapunov function

$$V(t) = E(t) + \lambda V_1(t) + \gamma V_3(t), \qquad (33)$$

where $\lambda$ and $\gamma$ are fixed positive constants taken small enough, $E(\cdot)$ is the energy given by (27), $V_1(\cdot)$ and $V_2(\cdot)$ are defined as follows

$$V_1(t) = \int_0^L x u^2(x,t)\mathrm{d}x, \tag{34}$$

$$V_2(t) = \theta(t)\int_\omega \int_0^1 (1-\mu)u^2(x, t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x. \tag{35}$$

For the stability analysis, the following lemmas will be needed.

Lemma 4.2. *Suppose that $a$ and $b$ are nonnegative functions belonging to $L^\infty(0,L)$ satisfying (14) and (15), and $Y_0 = (u_0, z_0(\cdot, -\theta(0)\cdot) \in D(A(0))$. Moreover we assume that $\tilde{a} \in L^\infty(0,L)$ satisfies (5) and $u \in L^2(0,T,H^1(0,L))$. Then for any regular solution of (9), the following equation is satisfied:*

$$\begin{aligned}\frac{d}{\mathrm{d}t}V_1(t) \leq &\int_0^L u^2(x,t)\mathrm{d}x - 3\int_0^L u_x^2(x,t)\mathrm{d}x - 2\int_0^L xa(x)u^2(x,t)\mathrm{d}x \\ &+L\int_\omega b(x)u^2(x,t)\mathrm{d}x + L\int_\omega b(x)u^2(x,t-\theta(t))\mathrm{d}x \\ &-2\int_0^L x sat(\tilde{a}(x)u(x,t))u(x,t)\mathrm{d}x.\end{aligned} \tag{36}$$

*Proof.* Let us consider a regular solution of (9). Then by differentiating $V_1(\cdot)$, we get

$$\begin{aligned}\frac{d}{\mathrm{d}t}V_1(t) &= 2\int_0^L xu(x,t)u_t(x,t)\mathrm{d}x \\ &= -2\int_0^L xu(x,t)u_x(x,t)\mathrm{d}x - 2\int_0^L xu(x,t)u_{xxx}(x,t)\mathrm{d}x \\ &\quad - 2\int_0^L xa(x)u^2(x,t)\mathrm{d}x - 2\int_0^L xb(x)u(x,t)u(x,t-\theta(t))\mathrm{d}x \\ &\quad - 2\int_0^L x\mathrm{sat}(\tilde{a}u(x,t))u(x,t)\mathrm{d}x.\end{aligned}$$

After integrations by parts, we get $-2\int_0^L xu(x,t)u_x(x,t)\mathrm{d}x = \int_0^L u^2(x,t)\mathrm{d}x$ and $-2\int_0^L xu(x,t)u_{xxx}(x,t)\mathrm{d}x = -3\int_0^L u_x^2(x,t)\mathrm{d}x$. Consequently, using the last equations and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\frac{d}{\mathrm{d}t}V_1(t) &= \int_0^L u^2(x,t)\mathrm{d}x - 3\int_0^L u_x^2(x,t)\mathrm{d}x - 2\int_0^L xa(x)u^2(x,t)\mathrm{d}x \\ &\quad - 2\int_0^L xb(x)u(x,t)u(x,t-\theta(t))\mathrm{d}x - 2\int_0^L x\mathrm{sat}(\tilde{a}u(x,t))u(x,t)\mathrm{d}x \\ &\leq \int_0^L u^2(x,t)\mathrm{d}x - 3\int_0^L u_x^2(x,t)\mathrm{d}x - 2\int_0^L xa(x)u^2(x,t)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\int_0^L xb(x)u^2(x,t)\mathrm{d}x+\int_0^L xb(x)u^2(x,t-\theta(t))\mathrm{d}x\\
&-2\int_0^L x\mathrm{sat}(\tilde{a}u(x,t))u(x,t)\mathrm{d}x\\
&\leq \int_0^L u^2(x,t)\mathrm{d}x-3\int_0^L u_x^2(x,t)\mathrm{d}x-2\int_0^L xa(x)u^2(x,t)\mathrm{d}x\\
&+L\int_0^L b(x)u^2(x,t)\mathrm{d}x+L\int_0^L b(x)u^2(x,t-\theta(t))\mathrm{d}x\\
&-2\int_0^L x\mathrm{sat}(\tilde{a}u(x,t))u(x,t)\mathrm{d}x.
\end{aligned}
$$

□

LEMMA 4.3. *Let* $Y_0=(u_0,z_0(\cdot,-h\cdot)\in D(A_1)$ *and* $u\in L^2(0,T,H^1(0,L))$. *Then for any regular solution of (9), the following equation is satisfied:*

$$
\frac{d}{\mathrm{d}t}V_2(t)=\int_\omega u^2(x,t)\mathrm{d}x-\int_\omega\int_0^1(1-\dot{\theta}(t)\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x. \tag{37}
$$

*Proof.* Let us consider a regular solution of (9). Then differentiating $V_3(\cdot)$ we get

$$
\begin{aligned}
\dot{V}_2(t)&=\dot{\theta}(t)\int_\omega\int_0^1(1-\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x\\
&+2\theta(t)\int_\omega\int_0^1(1-\mu)u_t(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x\\
&=\dot{\theta}(t)\int_\omega\int_0^1(1-\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x\\
&+2\int_\omega\int_0^1\theta(t)u_t(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x\\
&-2\int_\omega\int_0^1\mu\theta(t)u_t(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.
\end{aligned}
\tag{38}
$$

Note that $\theta(t)u_t(x,t-\theta(t)\mu)=(\dot{\theta}(t)\mu-1)u_\mu(x,t-\theta(t)\mu)$. We integrate by parts $2\int_\omega\int_0^1\theta(t)u_t(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x$, and we obtain

$$
\begin{aligned}
&2\int_\omega\int_0^1\theta(t)u_t(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x\\
&=2\int_\omega\int_0^1(\dot{\theta}(t)\mu-1)u_\mu(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&= 2\int_\omega \left[(\dot\theta(t)\mu - 1)\frac{u^2}{2}(x,t-\theta(t)\mu)\right]_0^1 \mathrm{d}x \\
&- 2\dot\theta(t)\int_\omega\int_0^1 \frac{u^2}{2}(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&= \int_\omega u^2(x,t)\mathrm{d}x + \int_\omega (\dot\theta(t)-1)u^2(x,t-\theta(t))\mathrm{d}x \\
&- \int_\omega\int_0^1 \dot\theta(t)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.
\end{aligned}
$$

Integrate by parts $-2\int_\omega\int_0^1 \mu\theta(t)u_t(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x$ to get:

$$
\begin{aligned}
&- 2\int_\omega\int_0^1 \mu\theta(t)u_t(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&= 2\int_\omega\int_0^1 (\mu-\dot\theta(t)\mu^2)u_\mu(x,t-\theta(t)\mu)u(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&= 2\int_\omega \left[(\mu-\dot\theta(t)\mu^2)\frac{u^2}{2}(x,t-\theta(t)\mu)\right]_0^1 \mathrm{d}x \\
&- 2\int_\omega\int_0^1 (1-2\dot\theta(t)\mu)\frac{u^2}{2}(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&= \int_\omega (1-\dot\theta(t))u^2(x,t-\theta(t))\mathrm{d}x \\
&- \int_\omega\int_0^1 (1-2\dot\theta(t)\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.
\end{aligned}
\tag{40}
$$

Therefore, using (38), (39) and (40), we have

$$
\begin{aligned}
\dot V_2(t) &= \dot\theta(t)\int_\omega\int_0^1 (1-\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x + \int_\omega u^2(x,t)\mathrm{d}x \\
&+ \int_\omega (\dot\theta(t)-1)u^2(x,t-\theta(t))\mathrm{d}x - \int_\omega\int_0^1 \dot\theta(t)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&+ \int_\omega (1-\dot\theta(t))u^2(x,t-\theta(t))\mathrm{d}x \\
&- \int_\omega\int_0^1 (1-2\dot\theta(t)\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&= \int_\omega u^2(x,t)\mathrm{d}x - \int_\omega\int_0^1 (1-\dot\theta(t)\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x,
\end{aligned}
$$

which concludes the proof. □

Now, we are able to state and prove the main result of this section.

THEOREM 4.4. *Assume that $a$ and $b$ are nonnegative functions belonging to $L^\infty(0,L)$ satisfying (14) and (15), and $L < \pi\sqrt{3}$. Moreover, suppose that $a\tilde a =$*

*$\tilde{a}(x) \in L^\infty(0,L)$ satisfies (5) and $u \in L^2(0,T,H^1(0,L))$. Then there exists $r > 0$, sufficiently small, such that for every $(u_0, z_0(\cdot, -\theta(0)\cdot) \in \mathcal{H}$ satisfying $\|(u_0, z_0(\cdot, -\theta(0)\cdot)\|_{\mathcal{H}} \leq r$, the energy of system (9) decays exponentially. More precisely, there exists $\delta > 0$ and $M > 0$, such that*

$$E(t) \leq M\mathrm{e}^{-\delta t}E(0), \quad \forall t > 0, \tag{41}$$

*where for $\lambda$ and $\gamma$ sufficiently small, the two positive constants $\delta$ and $M$ satisfy the following inequality*

$$\delta \leq \min\left\{\frac{(9\pi^2 - 3L^2 - 2L^{\frac{3}{2}})\lambda}{3L^2(1+2L\lambda)}, \frac{\gamma}{K(2\gamma + \|\xi\|_{L^\infty(0,L)})}\right\} \tag{42}$$

*and $M \leq \left(1 + \max\left\{L\lambda, \dfrac{\gamma}{b_0}\right\}\right)$. Moreover, $\lambda$ and $\gamma$ satisfy the following inequalities:*

$$\lambda \leq \inf_{x\in\omega}\left\{\frac{2a(x) - b(x) - \xi(x)}{Lb(x)}, \frac{\xi(x) - b(x)}{Lb(x)}\right\} \tag{43}$$

*and*

$$\gamma \leq \inf_{x\in\omega}\left\{\frac{2a(x) - b(x) - \xi(x) - 2\lambda Lb(x)}{2}\right\}. \tag{44}$$

Remark 4.5. The Lyapunov function $V(\cdot)$ given by (33) and the energy $E(\cdot)$ defined by (27) are equivalent. Indeed,

$$E(t) \leq V(t) \leq M_1E(t), \quad \forall t > 0, \tag{45}$$

where

$$M_1 = \left(1 + \max\left\{L\lambda, \frac{\gamma}{b_0}\right\}\right).$$

Thanks to inequality (45), in order to prove the local exponential stability of system (9), it is sufficient to show that for all $\delta > 0$, $\frac{d}{\mathrm{d}t}V(t) + 2\delta V(t) \leq 0$.

*Proof.* Let $(u_0, z_0(\cdot, -\theta(0)\cdot) \in D(A(0))$ and consider a regular solution. Then by using (28), (36) and (37) and the fact that $\int_0^L \mathrm{sat}(\tilde{a}u)u\mathrm{d}x \geq 0$,

$2\lambda\int_0^L xa(x)u^2\mathrm{d}x \geq 0$, and $2\lambda\int_0^L x\mathrm{sat}(\tilde{a}(x)u(x,t))u(x,t)\mathrm{d}x \geq 0$, we obtain:

$$
\begin{aligned}
\frac{d}{\mathrm{d}t}V(t) = &-\frac{1}{2}u_x^2(0,t) - \int_{(0,L)\setminus\omega} a(x)u^2(x,t)\mathrm{d}x \\
&+ \frac{1}{2}\int_\omega (-2a(x)+b(x)+\xi(x))u^2(x,t)\mathrm{d}x \\
&+ \frac{1}{2}\int_\omega (\xi(x)(d-1)+b(x))u^2(x,t-\theta(t))\mathrm{d}x \\
&- \int_0^L \mathrm{sat}(\tilde{a}(x)u(x,t))u(x,t)\mathrm{d}x \\
&+ \lambda\int_0^L u^2(x,t)\mathrm{d}x - 3\lambda\int_0^L u_x^2(x,t)\mathrm{d}x \\
&- 2\lambda\int_0^L xa(x)u^2(x,t)\mathrm{d}x + \lambda L\int_\omega b(x)u^2(x,t)\mathrm{d}x \\
&+ \lambda L\int_\omega b(x)u^2(x,t-\theta(t))\mathrm{d}x \\
&- 2\lambda\int_0^L x\mathrm{sat}(\tilde{a}(x)u(x,t))u(x,t)\mathrm{d}x \\
&+ \gamma\int_\omega u^2(x,t)\mathrm{d}x - \gamma\int_\omega\int_0^1 (1-\dot{\theta}(t)\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
\leq &\frac{1}{2}\int_\omega (-2a(x)+b(x)+\xi(x)+2L\lambda b(x)+2\gamma)u^2(x,t)\mathrm{d}x \\
&+ \frac{1}{2}\int_\omega (\xi(x)(d-1)+b(x)+2L\lambda b(x))u^2(x,t-\theta(t))\mathrm{d}x \\
&+ \lambda\int_0^L u^2(x,t)\mathrm{d}x \\
&- 3\lambda\int_0^L u_x^2(x,t)\mathrm{d}x - \gamma\int_\omega\int_0^1 (1-\dot{\theta}(t)\mu)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.
\end{aligned}
\tag{46}
$$

Now, by calculating $2\delta V(t)$, we have

$$
\begin{aligned}
2\delta V(t) =&2\delta E(t) + 2\delta\lambda V_1(t) + 2\delta\gamma V_2(t) \\
=&\delta\int_0^L u^2(x,t)\mathrm{d}x + \theta(t)\delta\int_\omega\int_0^1 \xi(x)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&+2\delta\lambda\int_0^L xu^2(x,t)\mathrm{d}x + 2\delta\gamma\theta(t)\int_\omega\int_0^1 (x)u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x \\
&-2\delta\gamma\theta(t)\int_\omega\int_0^1 \mu u^2(x,t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x
\end{aligned}
\tag{47}
$$

$$\leq (\delta + 2\delta\lambda L) \int_0^L u^2(x,t)\mathrm{d}x$$
$$+ \int_\omega \int_0^1 (K\delta\xi(x) + 2\delta\gamma K) u^2(x, t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.$$

Therefore, from (46) and (47) we deduce that

$$\begin{aligned}\frac{d}{\mathrm{d}t}V(t) + 2\delta V(t) \leq & \frac{1}{2}\int_\omega (-2a(x) + b(x) + \xi(x) + 2L\lambda b(x) + 2\gamma)u^2(x,t)\mathrm{d}x \\ & + \frac{1}{2}\int_\omega (\xi(x)(d-1) + b(x) + 2L\lambda b(x))u^2(x, t-\theta(t))\mathrm{d}x \\ & + (\lambda + \delta + 2\delta\lambda L)\int_0^L u^2(x,t)\mathrm{d}x - 3\lambda \int_0^L u_x^2(x,t)\mathrm{d}x \\ & + \int_\omega \int_0^1 (K\delta\xi(x) + 2\delta\gamma K - \gamma(1-d))u^2(x, t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.\end{aligned}$$

Using the Poincaré inequality, we get

$$\begin{aligned}\frac{d}{\mathrm{d}t}V(t) + 2\delta V(t) \leq & \frac{1}{2}\int_\omega (-2a(x) + b(x) + \xi(x) + 2L\lambda b(x) + 2\gamma)u^2(x,t)\mathrm{d}x \\ & + \frac{1}{2}\int_\omega (\xi(x)(d-1) + b(x) + 2L\lambda b(x))u^2(x, t-\theta(t))\mathrm{d}x \\ & + \left(\frac{L^2}{\pi^2}(\lambda + \delta + 2\delta\lambda L) - 3\lambda\right)\int_0^L u_x^2(x,t)\mathrm{d}x \\ & + \int_\omega \int_0^1 (K\delta\xi(x) + 2\delta\gamma K - \gamma(1-d))u^2(x, t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.\end{aligned}$$

By using (21), we can take $\lambda$ and $\gamma$ small enough as in (43) and (44), to get

$$-2a(x) + b(x) + \xi(x) + 2L\lambda b(x) + 2\gamma \leq 0,$$

and $\xi(x)(d-1) + b(x) + 2L\lambda b(x) \leq 0$, in $\omega$. Consequently, we have

$$\begin{aligned}\frac{d}{\mathrm{d}t}V(t) + 2\delta V(t) \leq & \left(\frac{L^2}{\pi^2}(\lambda + \delta + 2\delta\lambda L) - 3\lambda\right)\int_0^L u_x^2(x,t)\mathrm{d}x \\ & + \int_\omega \int_0^1 (K\delta\xi(x) + 2\delta\gamma K - \gamma(1-d))u^2(x, t-\theta(t)\mu)\mathrm{d}\mu\mathrm{d}x.\end{aligned}$$

Then, we can take $r$ small enough to have $r < \dfrac{3(3\pi^2 - L^2)}{2L^{\frac{3}{2}}\pi^2}$, which is possible since $L < \pi\sqrt{3}$. Hence, we can choose $\delta > 0$, as in (42), in order to have

$$\left(\frac{L^2}{\pi^2}(\lambda + \delta + 2\delta\lambda L) - 3\lambda\right) \leq 0,$$

and $(K\delta\xi(x) + 2\delta\gamma K - \gamma(1-d)) \leq 0$. Thus, we have

$$\frac{d}{\mathrm{d}t}V(t) + 2\delta V(t) \leq 0. \tag{48}$$

By integrating (48) over $[0,t]$, we obtain $V(t) \leq V(0)\mathrm{e}^{-2\delta t}$. Using the equivalence between the Lyapunov function $V(\cdot)$ and the energy $E(\cdot)$ (see Remark 4.5), we get $E(t) \leq M_1 E(0)\mathrm{e}^{-2\delta t}$, where $M_1 = 1 + \max\left\{L\lambda, \frac{\gamma}{b_0}\right\}$. Since $D(A(0))$ is dense in $H$, then we can extend the result to any initial condition $(u_0, z_0(\cdot, -\theta(0)\cdot)) \in H$. □

## 5. CONCLUSION

In this research paper, we have investigated the well-posedness and exponential stability of a Korteweg–de Vries equation with a time-varying delay on internal feedback, considering the presence of a saturated source term. This study demonstrates that, under certain assumptions, introducing a time-varying delay into the Korteweg-de Vries equation alongside a saturated source term results in a well-posed system. Moreover, by using an appropriate Lyapunov functional, we have shown that the studied equation is locally exponentially stable. Finally, let us address some issues that have arisen in this setting.

(1) In this paper, we have assumed that the function defining the time-varying delay, denoted as $\theta(.)$, is differentiable. A pertinent question to explore is whether the obtained results still hold true when the derivative of $\theta(.)$ does not exist.
(2) Can we investigate the exponential stability of the nonlinear KdV equation with a time-varying delay in the presence of a nonlinear source term as an intriguing topic for further research?

## REFERENCES

[1] C. Abdallah, P. Dorato, J. Benites-Read, and R. Byrne, *Delayed positive feedback can stabilize oscillatory systems.* In *American Control Conference*, IEEE, 1993, pp. 3106–3107.

[2] B. Alvarez-Samaniego and X. Carvajal, *On the local well-posedness for some systems of coupled KdV equations*, Nonlinear Analysis: Theory, Methods & Applications, **69** (2008), 692–715.

[3] H. Ayadi and M. Jlassi, *Global exponential stabilization of the linearized Korteweg-de Vries equation with a state delay*, IMA Journal of Mathematical Control and Information **40** (2023), p. 516–533.

[4] L. Baudouin, E. Crépeau, and J. Valein, *Two approaches for the stabilization of nonlinear KdV equation with boundary time-delay feedback*, IEEE Transactions on Automatic Control, **64** (2018), 1403–1414.

[5] E. Cerpa, *Control of a Korteweg-de Vries equation: a tutorial*, Math. Control Relat. Fields, **4** (2014), 45–99.

[6] E. Cerpa and J.-M. Coron, *Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition*, IEEE Transactions on Automatic Control, **58** (2013), 1688–1695.

[7] R. Datko, *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks*, SIAM Journal on Control and Optimization, **26** (1988), 697–713.

[8] R. Datko, J. Lagnese, and M. P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM Journal on Control and Optimization, **24** (1986), 152–156.

[9] E. Fridman, S. Nicaise, and J. Valein, *Stabilization of second order evolution equations with unbounded feedback with time-dependent delay*, SIAM Journal on Control and Optimization, **48** (2010), 5028–5052.

[10] Y. He, Q.-G. Wang, C. Lin, and M. Wu, *Delay-range-dependent stability for systems with time-varying delay*, Automatica, **43** (2007), 371–376.

[11] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations.* In *Spectral Theory and Differential Equations*, Proceedings of the Symposium held at Dundee, Scotland, 1–19 July, 1974, Springer, 2006, pp. 25–70.

[12] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*, Springer, 2014.

[13] S. Marx and E. Cerpa, *Output feedback control of the linear Korteweg-de Vries equation.* In *53rd IEEE Conference on Decision and Control*, pp. 2083–2087.

[14] S. Marx, E. Cerpa, C. Prieur, and V. Andrieu, *Stabilization of a linear Korteweg-de Vries equation with a saturated internal control.* In *European Control Conference (ECC)*, IEEE, 2015, pp. 867–872.

[15] S. Marx, E. Cerpa, C. Prieur, and V. Andrieu, *Global stabilization of a Korteweg–de Vries equation with saturating distributed control*, SIAM Journal on Control and Optimization, **55** (2017), 1452–1480.

[16] S. Nicaise, J. Valein, and E. Fridman, *Stability of the heat and of the wave equations with boundary time-varying delays*, Discrete and Continuous Dynamical Systems-Series S, **2** (2009), 559–581.

[17] H. Parada, *Stability analysis of a Korteweg-de Vries equation with saturated boundary feedback*, IFAC-PapersOnLine, **55** (2022), pp. 1–6.

[18] H. Parada, C. Timimoun, and J. Valein, *Stability results for the KdV equation with time-varying delay*, Systems & Control Letters, **177** (2023), 105547.

[19] H. Parada, E. Crépeau, and C. Prieur, *Delayed stabilization of the Korteweg–de Vries equation on a star-shaped network*, Mathematics of Control, Signals, and Systems, **34** (2022), 559–605.

[20] P. Park and J. W. Ko, *Stability and robust stability for systems with a time-varying delay*, Automatica, **43** (2007), 1855–1858.

[21] A. F. Pazoto, *Unique continuation and decay for the Korteweg-de Vries equation with localized damping*, ESAIM: Control, Optimisation and Calculus of Variations, **11** (2005), 473–486.

[22] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, **44**, Springer Science & Business Media, 2012.

[23] G. Perla Menzala, C. F. Vasconcellos, and E. Zuazua, *Stabilization of the Korteweg-de Vries equation with localized damping*, Quarterly of Applied Mathematics, **60** (2002), 111–129.

[24] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM: Control, Optimisation and Calculus of Variations, **2** (1997), 33–55.

[25] L. Rosier and B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation: recent progresses*, Journal of Systems Science and Complexity, **22** (2009), 647–682.

[26] M. Slemrod, *Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control*, Mathematics of Control, Signals and Systems, **2** (1989), 265–285.

[27] A. Taboye and T. Ennouari, *Stabilization of linear KdV equation with boundary time-delay feedback and internal saturation*, Gulf Journal of Mathematics, **17** (2024), 72–86.

[28] A. M. Taboye and T. Ennouari, *Stability result of the Kawahara equation with localized damping and time-varying delay*, Evolution Equations and Control Theory, **14** (2025), 36–56.

[29] J. Valein, *On the asymptotic stability of the Korteweg-de Vries equation with time-delayed internal feedback*, Mathematical Control and Related Fields, **12** (2022), 667–694.


Received June 25, 2024
Accepted December 29, 2024

*Chouaib Doukkali University*
*Department of Mathematics*
*El Jadida, Morocco*
*E-mail:* as.ahmat.taboye@gmail.com
https://orcid.org/0000-0001-9416-3203

*E-mail:* ennouari.t@ucd.ac.ma
https://orcid.org/0000-0002-1586-1925