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ON CERTAIN REAL-VALUED FUNCTIONS WHICH GENERATE FREE GROUPS

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Abstract. In this paper, we provide a sufficient condition for a real-valued function to generate a free group. In fact, a one-one onto increasing function of $\mathbb{R}$ generates a free group. This is not true for one-one onto decreasing functions of $\mathbb{R}$. But decreasing functions can also generate free groups. We also prove that any pair of lines fails to generate a free group. In the last part, we discuss the fixed points of the non-empty reduced words in the maps $x \mapsto x+1$ and $x \mapsto x^p$, where p is an odd prime. Finally, we pose some problems regarding the fixed points of the words in these functions.

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Key words. Free groups, real-valued functions, fixed points.

1. INTRODUCTION

In 1947, I. N. Sanov [5] proved that $I+2e_{12}$ and $I+2e_{21}$, where e_{ij} is a matrix unit, generate a free group. Further, in 1957, K. Goldberg and M. Newman [8] proved that the rational integral unimodular matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ which are not of finite period generate a free group if a_{12} is dominant in A and b_{21} is dominant in B. Again, in 1958, B. Cheng et al. [3] proved the result for the matrices $I+2e_{12}$ and $I+\lambda e_{21}$, where λ is a complex number satisfying

$$|\lambda| \geq 1,\ |\lambda - 1| \geq 1,\ |\lambda + 1| \geq 1.$$

In 1958, S. Świerczkowski [12] produced a free group generated by a pair of rotations (matrices of order 3) of a regular tetrahedron. Later in 2000, A. S. Oliinyk and V. I. Sushchanskii [2] constructed a free subgroup of rank two in the group of infinite unitriangular matrices over a field of rank 2. Recently, Waldemar Hołubowski [14], in 2003, and, A. I. Generalov and A. S. Mironov [1], in 2019, studied free subgroups of the group of infinite unitriangular matrices. Samuel White [11] (1988) proved that the maps $x \mapsto x+1$ and $x \mapsto x^p$

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generate a free group under the composition of functions, where p is an odd prime, and C. D. Bennett [4] (1995) constructed a free group generated by n real-valued functions.

To prove that the letters $a_1, \ldots, a_k$ generate a free group, we show that two reduced words cannot represent the same word unless they are equal. It is the same as showing that no non-empty reduced word represents the identity element. For many rational numbers a, satisfying $1 \leq |a| < 2$, the group generated by the matrices $A = I_2 + ae_{12}$ and $B = I_2 + ae_{12}$ is found to be not free. It is still not known whether there exists such a rational number a such that the group generated by A and B is free. To prove that the letters $a_1, \ldots, a_k$ fail to generate a free group, we show that two different reduced words represent the same word. It is the same as showing that a non-empty reduced word represents the identity element. M. Newman, in 1947, conjectured that for any root ζ of unity, the matrix group G_ζ generated by $I + \zeta e_{12}$ and $I + \zeta e_{21}$ is non-free. Characterisation of ζ for which the group G_ζ is non-free was established extensively by J. L. Brenner et al. [6] in 1975. The above conjecture was proven by R. J. Evans [9] in 1979. Again, in 2022, S. Kim and T. Koberda [13] produced a computational characterisation which proves that the groups generated by $I + qe_{12}$ and $I + e_{21}$, $q \in \mathbb{C}$, is non-free for $q = \dfrac{s}{r}$, $|s| \leq 27$ except $s = 24$.

In this paper, we establish a sufficient condition for which a real-valued function can generate a free group. Again, we also prove that the group generated by any pair of lines is non-free. Finally, we discuss the fixed points of the words generated by the maps $x \mapsto x + 1$ and $x \mapsto x^p$ and pose some problems whose positive answer proves that the group generated by these functions is free.

2. FREE GROUPS GENERATED BY FUNCTIONS

In this section, we prove that being one-one onto increasing is a sufficient condition for a function f defined on $\mathbb{R}$ to generate a free group under the composition of functions. However, this is not a necessary condition; there are other functions which generate free groups. We begin with the following lemmas.

LEMMA 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a one-one onto increasing function such that $f(x) \neq x$ for some $x \in \mathbb{R}$. Then $f^n(x) \neq x$ for every $n \in \mathbb{N}$.*

Proof. Since $f(x) \neq x$ for some $x \in \mathbb{R}$, without loss of generality, we may assume that $f(x) < x$. Again, f is increasing. So, we have

$$f(f(x)) < f(x) < x, \text{ i.e., } f^2(x) < x.$$

By induction, we have $f^n(x) < x$. Thus, $f^n(x) \neq x$ for every $n \in \mathbb{N}$. □

LEMMA 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a one-one onto increasing function such that $f(x) \neq x$ for some $x \in \mathbb{R}$. Then $f^{-n}(x) \neq x$ for every $n \in \mathbb{N}$.*

Proof. Since f is one-one onto and increasing, f^{-1} is also increasing. Again, $f(x) \neq x$. So, $f^{-1}(x) \neq x$. Now, by Lemma 2.1, we see that $(f^{-1})^n(x) \neq x$, i.e., $f^{-n}(x) \neq x$ for every $n \in \mathbb{N}$. □

Now, we will use the above lemmas to prove the following theorem.

THEOREM 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a one-one onto increasing function such that $f(x) \neq x$ for some $x \in \mathbb{R}$. Then f generates a free group under the composition of functions.*

Proof. Using Lemmas 2.1 and 2.2, we have

$$f^n(x) \neq x \text{ for every } n \in \mathbb{Z} - \{0\}.$$

Thus, f^n is not the identity function for every $n \in \mathbb{Z}-\{0\}$. Hence, the function f generates a free group under the composition of functions. □

From the above theorem, we see that any non-identity, one-one onto increasing function of $\mathbb{R}$ generates a free group under the composition of functions. However, this is not the case for one-one onto decreasing functions of $\mathbb{R}$. We discuss an example.

EXAMPLE 2.4. Any one-one onto decreasing function of $\mathbb{R}$ which is symmetric about the identity function is the inverse of itself. For example, the function $x \mapsto -x$ is the inverse of itself.

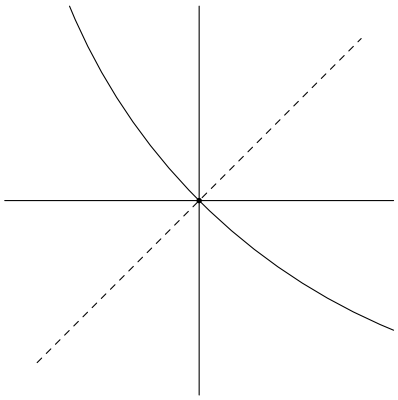

Fig. 2.1 – A function f symmetric about the line $y = x$.

We see that if f is such a function, then the non-empty reduced word f^2 represents the identity function. So, such functions do not generate free groups.

It can be seen from the following Theorem 2.5 that decreasing functions can also generate free groups under the composition of functions.

THEOREM 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero function defined by*

$$f(x) = kx + m, \text{ where } k, m \in \mathbb{R}, k \neq 0.$$

Then f generates a free group under the composition of functions only when $k \neq -1$ and $(k, m) \neq (1, 0)$.

Proof. Two cases arise depending on $k > 0$ and $k < 0$. Consider the first case when $k > 0$. In this case, the function f is one-one onto increasing. Also, $(k, m) \neq (1, 0)$. So, f is not the identity function, i.e., $f(x) \neq x$ for some $x \in \mathbb{R}$. Thus, by Theorem 2.3, f generates a free group under the composition of functions. Now, if $k < 0$, we have

$$f^{-1}(x) = \frac{1}{k}x - \frac{m}{k},$$

$$f^{-2}(x) = \frac{1}{k^2}x - \frac{m}{k}\left(\frac{1}{k} + 1\right),$$

$$\vdots$$

$$f^{-n}(x) = \frac{1}{k^n}x - \frac{m}{k}\left(\frac{1}{k^{n-1}} + \cdots + \frac{1}{k} + 1\right).$$

Since $k \neq -1$, we have $\dfrac{1}{k^n} \neq 1$ for all $n \in \mathbb{N}$. So, f^{-n} is not the identity function for all $n \in \mathbb{N}$. Again,

$$f^n(x) = k^n x + m(k^{n-1} + \cdots + 1) \text{ for all } n \in \mathbb{N}.$$

Since $k \neq -1$ and $k < 0$, we have $k^n \neq 1$ for all $n \in \mathbb{N}$. So, f^n is not the identity function for all $n \in \mathbb{N}$. Thus, f generates a free group under the composition of functions. □

Corollary 2.6. *Let T be a linear operator on $\mathbb{R}$. Then it generates a free group under the composition of functions if and only if $T(x) \neq \lambda x$, where $\lambda \neq 0, \pm 1$.*

Proof. To each linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$, there is an $m \times n$ matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}$. For a linear operator T on $\mathbb{R}$, there is a real number λ such that $T(x) = \lambda x$. Thus, by Theorem 2.5, T generates a free group under the composition of functions if and only if $\lambda \neq 0, \pm 1$. □

Theorem 2.7. *Let ρ_ϕ be a rotation of $\mathbb{R}^3$ (or $\mathbb{R}^2$) about an axis (or origin) by an angle ϕ. Then ρ_ϕ generates a free group under the composition of functions if and only if ϕ (in degrees) is irrational.*

Proof. We know that $\rho_\phi^{-1} = \rho_{-\phi}$ and $\rho_\alpha \rho_\beta = \rho_{\alpha+\beta}$ for all $\phi, \alpha, \beta \in \mathbb{R}$. So, we have $\rho_\phi^n = \rho_{n\phi}$ for all $n \in \mathbb{Z}$. Let ρ_ϕ generate a free group. Let, if possible, ϕ be rational. Then $\phi = \left(\dfrac{p}{q}\right)^\circ$, for some integers p, q, where $q > 0$. Now, $\rho_\phi^{360q} = \rho_{360q\phi} = \rho_{(360p)^\circ}$. Thus, ρ_ϕ^{360q} is the identity function, and so ρ_ϕ does not generate a free group. This is a contradiction. Hence, ϕ is irrational. Conversely, if ϕ is irrational, then $\phi \neq (360r)^\circ$ for all $r \in \mathbb{Q}$. So, $n\phi \neq (360k)^\circ$

for all $n, k \in \mathbb{Z}$. Hence, ρ_ϕ^n is not the identity rotation for all $n \in \mathbb{Z}$. Hence, ρ_ϕ generates a free group. □

This theorem can be extended to $\mathbb{R}^m$, where $m \geq 2$ is an integer.

COROLLARY 2.8. *Let ρ_ϕ be a rotation of $\mathbb{R}^m$ $(m \geq 2)$ about $\mathbb{R}^{m-2}$ by an angle ϕ. Then ρ_ϕ generates a free group under the composition of functions if and only if ϕ (in degrees) is irrational.*

The proof is same as for Theorem 2.7. Note that $\mathbb{R}^0$ represents the origin.

3. FUNCTIONS NOT GENERATING FREE GROUPS

In 1961, Rimhak Ree [10] produced certain pairs of matrices which do not generate free groups. In this section, we prove that any pair of lines fails to generate a free group under the composition of functions. Now, we discuss some examples first.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = ax$ and $g(x) = bx$, where a and b are any two nonzero real numbers. The inverse functions of f and g are defined by

$$f^{-1}(x) = a^{-1}x \text{ and } g^{-1}(x) = b^{-1}x.$$

For any integer n,

$$f^n(x) = a^n x \text{ and } g^n(x) = b^n x.$$

Let $S = \{f, g\}$. Now, a non-empty reduced word h in the set S is of the form

$$h = f^{l_d} g^{k_d} \cdots f^{l_1} g^{k_1},$$

where l_i's, k_j's are integers, and $l_i \neq 0$ for all $i = 1, 2, \ldots, d-1$, $k_j \neq 0$ for all $j = 2, 3, \ldots, d$. Now, after some tedious calculations, we have

$$h(x) = a^{l_d + \cdots + l_1} b^{k_d + \cdots + k_1} x.$$

If $l_d + \cdots + l_1 = 0$ and $k_d + \cdots + k_1 = 0$, then $h(x) = x$, i.e., h is the identity function. Now, we can produce such integers satisfying the above equation. It is easy to see that the word $h = f^{-1}g^{-1}fg$ is a non-empty reduced word in the set S and h is the identity function. This shows that f and g cannot generate a free group under the composition of functions.

Again, let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = ax$ and $g(x) = x + b$, where a and b are real numbers and $a \neq 0$. The inverse functions of f and g are defined by

$$f^{-1}(x) = a^{-1}x \text{ and } g^{-1}(x) = x - b.$$

For any integer n,

$$f^n(x) = a^n x \text{ and } g^n(x) = x + nb.$$

Let $S = \{f, g\}$. Now, a non-empty reduced word h in the set S is of the form

$$h = f^{l_d} g^{k_d} \cdots f^{l_1} g^{k_1},$$

where l_i's, k_j's are integers, and $l_i \neq 0$ for all $i = 1, 2, \ldots, d-1$, $k_j \neq 0$ for all $j = 2, 3, \ldots, d$. Now, after some tedious calculations, the function h is explicitly given by

$$h(x) = a^{l_d+\cdots+l_1}x + a^{l_d+\cdots+l_1}k_1 b + \cdots + a^{l_d+l_{d-1}}k_{d-1}b + a^{l_d}k_d b.$$

For h to be the identity function, we must have

$$a^{l_d+\cdots+l_1} = 1$$

and

$$a^{l_d+\cdots+l_1}k_1 b + a^{l_d+\cdots+l_2}k_2 b + \cdots + a^{l_d+l_{d-1}}k_{d-1}b + a^{l_d}k_d b = 0.$$

Now, we can produce such integers satisfying the above equations. Let us consider the non-empty reduced word

$$h = g^{-1}f^{-1}g^{-1}fgf^{-1}gf.$$

Then, clearly, $h(x) = x$. Hence, the functions f and g cannot generate a free group under the composition of functions.

LEMMA 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = x + a$ and $g(x) = x + b$, where a and b are real numbers. Then f and g do not generate a free group under the composition of functions.*

The inverse functions of f and g are defined by

$$f^{-1}(x) = x - a \text{ and } g^{-1}(x) = x - b.$$

For any integer n,

$$f^n(x) = x + na \text{ and } g^n(x) = x + nb.$$

Let $S = \{f, g\}$. Now, a non-empty reduced word h in the set S is of the form

$$h = f^{l_d}g^{k_d}\cdots f^{l_1}g^{k_1},$$

where l_i's, k_j's are integers, and $l_i \neq 0$ for all $i = 1, 2, \ldots, d-1$, $k_j \neq 0$ for all $j = 2, 3, \ldots, d$. Now, we have

$$h(x) = x + (k_1 + \cdots + k_d)b + (l_1 + \cdots + l_d)a.$$

For h to be the identity function, we must have

$$(k_1 + \cdots + k_d)b + (l_1 + \cdots + l_d)a = 0.$$

Now, we can produce such integers satisfying the above equation.

Proof of Lemma 3.1. Let us consider the non-empty reduced word

$$h = g^{-1}f^{-1}gf.$$

Then, clearly, $h(x) = x$. Hence, the functions f and g cannot generate a free group under the composition of functions. □

THEOREM 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c and d are real numbers, and $a \neq 0, c \neq 0$. Then f and g do not generate a free group under the composition of functions.*

Proof. It is easy to see that $fgf^{-1}g^{-1}(x) = x + r$ and $g^{-1}fgf^{-1}(x) = x + s$ for some real numbers r and s. Then, by Lemma 3.1, we can conclude that the functions f and g cannot generate a free group under the composition of functions. □

In fact, we can always find a non-empty reduced word (in f and g) of length less than or equal to 14 which represents the identity function. For example, $fg^{-1}f^{-1}g^2fg^{-1}f^{-1}g^{-1}fg^2f^{-1}g^{-1}(x) = x$.

4. FUNCTIONS $x \mapsto x + 1$ AND $x \mapsto x^p$

In this section, we discuss the following theorem which states that the functions $x \mapsto x + 1$ and $x \mapsto x^p$, where p is an odd prime, generate a free group under the composition of functions. In 1988, Samuel White [11] used algebraic results to prove this theorem. In the last part of this section, we draw some results about the fixed points of the non-empty reduced words in these functions.

THEOREM 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the mappings defined by*

$$f(x) = x + 1 \text{ and } g(x) = x^p, \text{ where } p \text{ is an odd prime.}$$

Then f and g generate a free group under the composition of functions.

From this theorem, we have a corollary.

COROLLARY 4.2. *The functions $x \mapsto x + 2$, $x \mapsto x^p + 1$ and $x \mapsto x^{p^{-1}} + 1$ generate a free group of rank 3 under the composition of functions.*

The above corollary follows from the fact that if F_2 is the free group of rank 2 generated by $\{a, b\}$, then the set $\{a^2, ab, ab^{-1}\}$ generates a free subgroup of rank 3 [7]. Now, we consider the free group generated by the functions f and g defined in the Theorem 4.1. The inverse functions of f and g are given by

$$f^{-1}(x) = x - 1 \text{ and } g^{-1}(x) = x^{p^{-1}}.$$

For any integer n, the functions f^n and g^n are given by

$$f^n(x) = x + n \text{ and } g^n(x) = x^{p^n}.$$

Let $S = \{f, g\}$. Now, a non-empty reduced word h in S can be put in the form

$$h = f^{l_d}g^{k_d} \cdots f^{l_1}g^{k_1},$$

where l_i's, k_j's are integers, and $l_i \neq 0$ for all $i = 1, 2, \ldots, d-1$, $k_j \neq 0$ for all $j = 2, 3, \ldots, d$. Now, the function h is explicitly given by

$$h(x) = \begin{cases} \left(\cdots\left((x+l_1)^{p^{k_2}} + l_2\right)^{p^{k_3}} + \cdots + l_{d-1}\right)^{p^{k_d}} & \text{if } k_1 = 0,\ l_d = 0; \\ \left(\cdots\left((x+l_1)^{p^{k_2}} + l_2\right)^{p^{k_3}} + \cdots + l_{d-1}\right)^{p^{k_d}} + l_d & \text{if } k_1 = 0,\ l_d \neq 0; \\ \left(\cdots\left((x^{p^{k_1}}+l_1)^{p^{k_2}} + l_2\right)^{p^{k_3}} + \cdots + l_{d-1}\right)^{p^{k_d}} & \text{if } k_1 \neq 0,\ l_d = 0; \\ \left(\cdots\left((x^{p^{k_1}}+l_1)^{p^{k_2}} + l_2\right)^{p^{k_3}} + \cdots + l_{d-1}\right)^{p^{k_d}} + l_d & \text{if } k_1 \neq 0,\ l_d \neq 0. \end{cases}$$

Since f and g generate a free group, the non-empty reduced word h is different from the identity function. As a result, there is a point which is not a fixed point of h. We know that the set of fixed points of a function is a closed set. So, the set of non-fixed points of the non-empty reduced word h is a union of open intervals.

Definition 4.3. For a nonnegative integer n, the *closed ball* B_n *of radius* n is defined to be the set of all reduced words of length less than or equal to n. The *boundary* ∂B_n of the closed ball of radius n is defined to be the set of all reduced words of length n.

In the remaining of this section, we consider the free group generated by f and g with the condition that $p = 3$. In this free group, the closed ball B_n has $1 + 2(3^n - 1)$ elements while the boundary ∂B_n has $4 \times 3^{n-1}$ elements. Now, we discuss some properties of the non-empty reduced word h. First, we prove the following theorem, and then discuss the fixed points of the word h using Python programming and GeoGebra.

Theorem 4.4. *Let* $\mu : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ *be a map defined by*

$$\mu(n) = \max_{w \in \partial B_n} \{\textit{number of fixed points of a reduced word } w \textit{ of length } n\}.$$

Then for each positive integer n*, the value* $\mu(n)$ *is attained at an even number of reduced words of length* n*.*

Proof. If a non-empty reduced word h has m fixed points, then h^{-1} also has m fixed points. It may be noted that the graph of $y = h^{-1}(x)$ is the reflection of the graph of $y = h(x)$ about the line $y = x$. The theorem follows from the fact that h and h^{-1} are distinct elements of ∂B_n. □

Now, we discuss the function μ for some positive integers. The graph of μ for the first few natural numbers is given in Fig. 4.2.

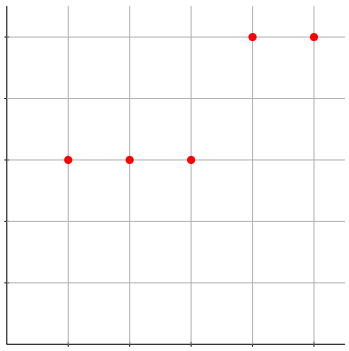

Fig. 4.2 – The map μ for the first five elements.

From the graph of μ, it can be easily seen that $\mu(1) = \mu(2) = \mu(3) = 3$ and $\mu(4) = \mu(5) = 5$. When $n = 1$, the value $\mu(1)$ is attained at 2 reduced words, namely g and g^{-1}. When $n = 2$, the value $\mu(2)$ is attained at 2 reduced words, namely g^2 and g^{-2}. When $n = 3$, the value $\mu(3)$ is attained at 6 reduced words. When $n = 4$, the value $\mu(4)$ is attained at 8 reduced words. For $n = 5$, the value $\mu(5)$ is attained at 16 reduced words. In the following example, these 16 reduced words in pairs $\{h(x), h^{-1}(x)\}$ are explicitly discussed. We also draw the graphs of a pair of such words.

Example 4.5. Consider the case when the word length is 5. There are 324 reduced words of length 5. Among these words, the value $\mu(5)$ is attained at 16 reduced words. Each word has five distinct fixed points. The words occur in pairs of the form $\{h(x), h^{-1}(x)\}$. These pairs are given in Table 1.

Table 1 – The reduced words of length 5 having $\mu(5) = 5$ fixed points.

	$h(x)$	$h^{-1}(x)$
1	$(x^3-1)^9+1$	$((x-1)^{\frac{1}{9}}+1)^{\frac{1}{3}}$
2	$(x^3+1)^9-1$	$((x+1)^{\frac{1}{9}}-1)^{\frac{1}{3}}$
3	$((x+1)^3-1)^9$	$(x^{\frac{1}{9}}+1)^{\frac{1}{3}}-1$
4	$((x^3+1)^3-1)^3$	$((x^{\frac{1}{3}}+1)^{\frac{1}{3}}-1)^{\frac{1}{3}}$
5	$(x^9+1)^3-1$	$((x+1)^{\frac{1}{3}}-1)^{\frac{1}{9}}$
6	$((x^3-1)^3+1)^3$	$((x^{\frac{1}{3}}-1)^{\frac{1}{3}}+1)^{\frac{1}{3}}$
7	$((x-1)^3+1)^9$	$(x^{\frac{1}{9}}-1)^{\frac{1}{3}}+1$
8	$((x-1)^9+1)^3$	$(x^{\frac{1}{3}}-1)^{\frac{1}{9}}+1$

We generated the graphs of the functions $h(x) = (x^3-1)^9+1$ and $h^{-1}(x) = ((x-1)^{\frac{1}{9}}+1)^{\frac{1}{3}}$ using GeoGebra. The representations of these graphs are shown in Fig. 4.3.

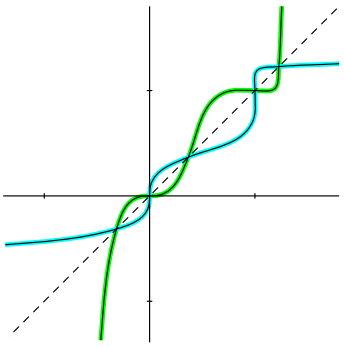

Fig. 4.3 – The graphs of $y = h(x)$ and $y = h^{-1}(x)$.

We see that each of the functions h and h^{-1} has five distinct fixed points. There are 972 reduced words of length 6. Out of these words, it can be seen that the functions $p(x) = (x^{\frac{1}{9}} + 1)^{\frac{1}{9}} - 1$ and $q(x) = (x^{\frac{1}{9}} - 1)^{\frac{1}{9}} + 1$ have five distinct fixed points. So, $\mu(1) \leq \cdots \leq \mu(6)$. We use Python programming and GeoGebra to generate the functions of different word lengths, draw their graphs and analyse their fixed points.

From the above discussion, we observe that the number of fixed points of each word of length up to 5 is finite and the function μ is increasing. Now, we pose the following three problems.

(1) Is the set of fixed points of the non-empty reduced word h finite?
(2) Does the set of fixed points of the non-empty reduced word h have measure zero?
(3) Is the map μ increasing?

The above problems may also be posed for any odd prime p. If the answer to any one of the first two problems is positive, then it gives a new proof of the fact that the maps $x \mapsto x + 1$ and $x \mapsto x^p$ generate a free group under the composition of functions. It may be noted that if the set of fixed points of each non-empty reduced word h is finite or of measure zero, then h is different from the identity map.

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