MATHEMATICA, 68 (91), N^{o} 1, 2026, pp. 101–115

PRIMAL m-SPACE

ARIJIT SAHA, SIDHARTHA RAJGURU, GITARTHA RAJGURU, YUVRAJ SINGH, SATABDI SARMA, SANTANU ACHARJEE, and DEBASISH BHATTACHARJEE

Abstract. The primal is a recently introduced structure in general topology. It is the dual structure of a grill and was introduced by Acharjee et al. In this paper, we introduce the primal m-space by combining the concepts of a primal and an m-space. We study several fundamental properties of this new generalized topological structure.

MSC 2020. 54A05, 54A20, 54E99.

Key words. Primal, m-space, grill, topology.

1. INTRODUCTION

Topology provides powerful tools for tackling a wide range of mathematical problems. Owing to its numerous generalized applications in both the natural and social sciences, several associated structures, such as ideals [10], filters [11], grills [9], etc., have been introduced. An ideal is the dual structure of a filter. Applications of filters are widely available in the literature [13, 14]. Moreover, ideals and their generalized fuzzy counterparts have found numerous applications in general topology [5], summability theory [18], fuzzy summability theory [19], and many other areas. Similar to ideals and filters, another classical structure in topology is the grill. The concept of a grill was introduced by Choquet [9] in 1947. Like ideals and filters, grills also have several applications in general topology [4], fuzzy topology [8], and many other fields.

Recently, the structure called primal was introduced by Acharjee et al. [1]. A primal is the dual structure of a grill. It has attracted significant attention from researchers and has been used to construct several new operators [1, 2]. These operators have, in turn, been employed to develop various topologies. Al-Omari et al. [6] used the primal structure to study primal-proximity spaces. Furthermore, Şaşmaz and Özkoç [7] introduced new operators derived from a primal and established an associated topology on primal topological spaces. Al-Omari et al. [3] constructed a primal local closure operator and investigated some of its fundamental properties.

Corresponding author: Santanu Acharjee.

DOI: 10.24193/mathcluj.2026.1.09

In [5], Al-Omari and Noiri introduced the concept of an m-space, and later, Modak et al. [16] introduced several new structures on m-spaces. Furthermore, in [15], Modak introduced the grill m-space by combining the grill structure with an m-space. Motivated by these developments, in this paper, we combine the primal structure with the m-space and introduce a new structure called the 'primal m-space'. Further, we define two operators on primal m-spaces and, using these operators, construct two topologies. We also characterize these topologies with the aid of various topological properties.

2. PRELIMINARIES

In this section, we review some preliminary definitions and theorems that will be used throughout the subsequent sections.

Definition 2.1 ([5]). A subfamily $\mathcal{M}$ of the power set 2^X of a non-empty set X is called an m-structure on X if $\mathcal{M}$ satisfies the following conditions:

(1) $\mathcal{M}$ contains ϕ and X,
(2) $\mathcal{M}$ is closed under the finite intersection.

The pair $(X, \mathcal{M})$ is called a m-space.

Definition 2.2 ([5]). A set $A \in 2^X$ is called an m-open set if $A \in \mathcal{M}$. $B \in 2^X$ is called an m-closed set if $X \setminus B \in \mathcal{M}$. We define the m-interior and m-closure of A as $m\mathrm{Int}(A) = \cup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m\mathrm{Cl}(A) = \cap\{F : A \subseteq F, X \setminus F \in \mathcal{M}\}$, respectively.

Theorem 2.3 ([16]). *Let $(X, \mathcal{M})$ be an m-space. Then, $x \in m\mathrm{Cl}(A)$ if and only if for every m-open set U_x containing x, we have $U_x \cap A \neq \phi$.*

Theorem 2.4 ([16]). *Let $(X, \mathcal{M})$ be an m-space and $A \subseteq X$. Then, $m\mathrm{Int}(A) = X \setminus m\mathrm{Cl}(X \setminus A)$.*

Theorem 2.5 ([15]). *Let $(X, \mathcal{M})$ be an m-space. Then, for $G \in \mathcal{M}$, $G \cap m\mathrm{Cl}(A) \subseteq m\mathrm{Cl}(G \cap A)$*

Definition 2.6 ([9]). A subcollection $\mathcal{G}$ of 2^X is called a grill on X if $\mathcal{G}$ satisfies the following conditions:

(1) $\phi \notin \mathcal{G}$;
(2) if $A \in \mathcal{G}$ and $A \subseteq B$, then $B \in \mathcal{G}$;
(3) if $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

A primal, which is the dual structure of a grill, is defined as follows:

Definition 2.7 ([1]). Let X be a non-empty set. A collection $\mathcal{P} \subseteq 2^X$ is called a primal on X if it satisfies the following conditions:

(1) $X \notin \mathcal{P}$;
(2) if $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$;
(3) if $A \cap B \in \mathcal{P}$, then $A \in \mathcal{P}$ *or* $B \in \mathcal{P}$.

Thus, we obtain the following definition from the above discussion.

Definition 2.8. An m-space $(X, \mathcal{M})$ with a primal $\mathcal{P}$ on X is called a primal m-space and is denoted as $(X, \mathcal{M}, \mathcal{P})$.

Definition 2.9. [12] A subset $A \subseteq X$ in a topological space $(X, \mathcal{T})$ is called semi-open if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$. The set of all semi-open sets in a topological space $(X, \mathcal{T})$ is denoted as $SO(X, \mathcal{T})$.

Definition 2.10. [17] A subset $A \subseteq X$ in a topological space $(X, \mathcal{T})$ is called α-set if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. The set of all α-sets in a topological space $(X, \mathcal{T})$ is denoted as $\mathcal{T}^{\alpha}$.

Definition 2.11. [17] A topological space $(X, \mathcal{T})$ is said to be resolvable if for a subset D of X, both D and $X \setminus D$ are dense in $(X, \mathcal{T})$, otherwise it is said to be irresolvable.

Throughout this paper, we denote the set of all m-closed sets by $\mathcal{M}^{\mathcal{CL}}$.

3. $\varphi_{\mathcal{P}}$-OPERATOR

In this section, we introduce the $\varphi_{\mathcal{P}}$-operator and study several of its fundamental properties.

Definition 3.1. Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space and $A \subseteq X$. A mapping $\varphi_{\mathcal{P}} : 2^X \to 2^X$ is defined by: $\varphi_{\mathcal{P}}(A) = \{x \in X : A^c \cup U^c \in \mathcal{P} \text{ for all } U \in \mathcal{M}(x)\}$, where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$. The mapping $\varphi_{\mathcal{P}}$ is called the operator associated with the primal $\mathcal{P}$ and the m-structure $\mathcal{M}$ on X.

Now, using the above definition, we discuss some properties of the $\varphi_{\mathcal{P}}$-operator.

Property 3.1. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, $\varphi_{\mathcal{P}}(\phi) = \phi$.*

Proof. Since $X \notin P$, we get $\varphi_{\mathcal{P}}(\phi) = \{x \in X : \phi^C \cup U^c \in \mathcal{P} \text{ for all } U \in \mathcal{M}(x)\} = \{x \in X : X \in \mathcal{P} \text{ for all } U \in \mathcal{M}(x)\} = \phi$. □

Corollary 3.2. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, for $P^C \notin \mathcal{P}$, $\varphi_{\mathcal{P}}(P) = \phi$.*

Proof. If $P^c \notin \mathcal{P}$, then $P^c \cup U^c \notin \mathcal{P}$ for all $U \in \mathcal{M}(x)$. Therefore, $\varphi_{\mathcal{P}}(P) = \phi$. □

Property 3.2. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, for $A \subseteq X$, $\varphi_{\mathcal{P}}(A) \subseteq m\mathrm{cl}(A)$.*

Proof. Let $x \in \varphi_{\mathcal{P}}(A)$. Therefore, we have $A^c \cup U^c \in \mathcal{P}$ for all $U \in \mathcal{M}(x)$, i.e., $(A \cap U)^c \in \mathcal{P}$ for all $U \in \mathcal{M}(x)$. Thus, by Definition 2.7, $(A \cap U)^c \neq X$ for all $U \in \mathcal{M}(x)$. Therefore, $(A \cap U) \neq \phi$. Hence, by Theorem 2.3, $x \in m\mathrm{Cl}(A)$. Thus, $\varphi_{\mathcal{P}}(A) \subseteq m\mathrm{Cl}(A)$. □

Property 3.3. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, for $A \subseteq X$, $m\mathrm{cl}(\varphi_{\mathcal{P}}(A)) \subseteq \varphi_{\mathcal{P}}(A)$.*

Proof. Let $x \in m\mathrm{cl}(\varphi_{\mathcal{P}}(A))$. Therefore, for any $U \in \mathcal{M}(x)$, $U \cap \varphi_{\mathcal{P}}(A) \neq \phi$. Let $y \in U \cap \varphi_{\mathcal{P}}(A)$. It implies $y \in U$ and $y \in \varphi_{\mathcal{P}}(A)$. So, $A^c \cup U^c \in \mathcal{P}$. Since, U is arbitrary in $\mathcal{M}(x)$, we have $x \in \varphi_{\mathcal{P}}(A)$. □

Corollary 3.3. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, for $A \subseteq X$, $\varphi_{\mathcal{P}}(A)$ is an m-closed set.*

Property 3.4. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, for $A \subseteq X$, $\varphi_{\mathcal{P}}(\varphi_{\mathcal{P}}(A)) \subseteq \varphi_{\mathcal{P}}(A)$.*

Proof. From property 3.2, we have $\varphi_{\mathcal{P}}(\varphi_{\mathcal{P}}(A)) \subseteq m\mathrm{cl}(\varphi_{\mathcal{P}}(A))$. Again from property 3.3, $m\mathrm{cl}(\varphi_{\mathcal{P}}(A)) \subseteq \varphi_{\mathcal{P}}(A)$. So, $\varphi_{\mathcal{P}}(\varphi_{\mathcal{P}}(A)) \subseteq \varphi_{\mathcal{P}}(A)$. □

Property 3.5. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, for $A, B \subseteq X$ and $A \subseteq B$, $\varphi_{\mathcal{P}}(A) \subseteq \varphi_{\mathcal{P}}(B)$.*

Proof. Let $x \in \varphi_{\mathcal{P}}(A)$. Therefore, $A^c \cup U^c \in \mathcal{P}$ for all $U \in \mathcal{M}(x)$. Now, $A \subseteq B$ $B^c \subseteq A^c$. So, $B^c \cup U^c \subseteq A^c \cup U^c$ for all $U \in \mathcal{M}(x)$. Therefore, as $A^c \cup U^c \in \mathcal{P}$ for all $U \in \mathcal{M}(x)$, so $B^c \cup U^c \in \mathcal{P}$ for all $U \in \mathcal{M}(x)$. Therefore, $\varphi_{\mathcal{P}}(A) \subseteq \varphi_{\mathcal{P}}(B)$. □

Property 3.6. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ be two primals on m-space $(X, \mathcal{M})$ and $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\varphi_{\mathcal{P}_1}(A) \subseteq \varphi_{\mathcal{P}_2}(A)$.*

Proof. Let $x \in \varphi_{\mathcal{P}_1}(A)$. Therefore, $A^c \cup U^c \in \mathcal{P}_1$ for all $U \in \mathcal{M}(x)$. Since $\mathcal{P}_1 \subseteq \mathcal{P}_2$, we have $A^c \cup U^c \in \mathcal{P}_2$ for all $U \in \mathcal{M}(x)$. It implies $x \in \varphi_{\mathcal{P}_2}(A)$. Thus, $\varphi_{\mathcal{P}_1}(A) \subseteq \varphi_{\mathcal{P}_2}(A)$. □

Property 3.7. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, for $A, B \subseteq X$, $\varphi_{\mathcal{P}}(A \cup B) = \varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(B)$.*

Proof. $A, B \subseteq A \cup B$. Thus, we have $\varphi_{\mathcal{P}}(A) \subseteq \varphi_{\mathcal{P}}(A \cup B)$ and $\varphi_{\mathcal{P}}(B) \subseteq \varphi_{\mathcal{P}}(A \cup B)$. It implies $\varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(B) \subseteq \varphi_{\mathcal{P}}(A \cup B)$.

Now, let $x \notin \varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(B)$. It implies $x \notin \varphi_{\mathcal{P}}(A)$ and $x \notin \varphi_{\mathcal{P}}(B)$. So, $A^c \cup U_1^c \notin \mathcal{P}$ and $B^c \cup U_2^c \notin \mathcal{P}$ for some $U_1, U_2 \in \mathcal{M}(x)$. Therefore, $(A^c \cup U_1^c) \cap (B^c \cup U_2^c) \notin \mathcal{P}$. Therefore, $(A^c \cap B^c) \cup (U_1 \cap U_2)^c \notin \mathcal{P}$. Therefore, $(A \cup B)^c \cup (U_1 \cap U_2)^c \notin \mathcal{P}$. It implies $x \notin \varphi_{\mathcal{P}}(A \cup B)$. Therefore, $\varphi_{\mathcal{P}}(A \cup B) \subseteq \varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(B)$. Hence, we have $\varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(B) = \varphi_{\mathcal{P}}(A \cup B)$. □

Property 3.8. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. If $U \in \mathcal{M}$, then $U \cap \varphi_{\mathcal{P}}(A) = U \cap \varphi_{\mathcal{P}}(U \cap A)$, for any $A \subseteq X$.*

Proof. Since $U \cap A \subseteq A$, thus $\varphi_{\mathcal{P}}(U \cap A) \subseteq \varphi_{\mathcal{P}}(A)$. Therefore, $U \cap \varphi_{\mathcal{P}}(U \cap A) \subseteq U \cap \varphi_{\mathcal{P}}(A)$.

Again, let $x \in U \cap \varphi_{\mathcal{P}}(A)$. It implies $x \in U$ and $x \in \varphi_{\mathcal{P}}(A)$. So, $x \in U$ and $A^c \cup V^c \in \mathcal{P}$ for all $V \in \mathcal{M}(x)$. So, $A^c \cup (V \cap U)^c \in \mathcal{P}$ for all $V \in \mathcal{M}(x)$. It yields $(A \cap U)^c \cup V^c \in \mathcal{P}$ for all $V \in \mathcal{M}(x)$. Thus, $x \in \varphi_{\mathcal{P}}(A \cap U)$. So, $x \in U \cap \varphi_{\mathcal{P}}(A \cap U)$. Therefore, $U \cap \varphi_{\mathcal{P}}(A) \subseteq U \cap \varphi_{\mathcal{P}}(A \cap U)$. Hence, $U \cap \varphi_{\mathcal{P}}(A) = U \cap \varphi_{\mathcal{P}}(A \cap U)$. □

Property 3.9. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space and $A, B \subseteq X$. Then, $\varphi_{\mathcal{P}}(A) \setminus \varphi_{\mathcal{P}}(B) = \varphi_{\mathcal{P}}(A \setminus B) \setminus \varphi_{\mathcal{P}}(B)$.*

Proof. We know $\varphi_{\mathcal{P}}(A) = \varphi_{\mathcal{P}}((A\setminus B)\cup(A\cap B)) = \varphi_{\mathcal{P}}(A\setminus B)\cup\varphi_{\mathcal{P}}(A\cap B) \subseteq \varphi_{\mathcal{P}}(A \setminus B) \cup \varphi_{\mathcal{P}}(B)$. Therefore, $\varphi_{\mathcal{P}}(A) \setminus \varphi_{\mathcal{P}}(B) \subseteq \varphi_{\mathcal{P}}(A \setminus B) \setminus \varphi_{\mathcal{P}}(B)$. Again, $\varphi_{\mathcal{P}}(A\setminus B) \subseteq \varphi_{\mathcal{P}}(A)$. It implies $\varphi_{\mathcal{P}}(A\setminus B)\setminus\varphi_{\mathcal{P}}(B) \subseteq \varphi_{\mathcal{P}}(A)\setminus\varphi_{\mathcal{P}}(B)$. Therefore, $\varphi_{\mathcal{P}}(A) \setminus \varphi_{\mathcal{P}}(B) = \varphi_{\mathcal{P}}(A \setminus B) \setminus \varphi_{\mathcal{P}}(B)$. □

Property 3.10. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. If $A, B \subseteq X$, then $\varphi_{\mathcal{P}}(A \cap B) \subseteq \varphi_{\mathcal{P}}(A) \cap \varphi_{\mathcal{P}}(B)$.*

Proof. Since $A \cap B \subseteq A$, we have $\varphi_{\mathcal{P}}(A \cap B) \subseteq \varphi_{\mathcal{P}}(A)$. Again, $A \cap B \subseteq B$, therefore $\varphi_{\mathcal{P}}(A \cap B) \subseteq \varphi_{\mathcal{P}}(B)$. Therefore, we get $\varphi_{\mathcal{P}}(A \cap B) \subseteq \varphi_{\mathcal{P}}(A) \cap \varphi_{\mathcal{P}}(B)$. □

Remark 3.4. The converse of the above property is not always true. For example, let $X = \{a, b, c\}$ and $\mathcal{M} = \{\phi, X\}$. We consider a primal $\mathcal{P} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ on X. Now, if A={b} and B={c}, then we have $\varphi_{\mathcal{P}}(A) \cap \varphi_{\mathcal{P}}(B) = X \cap X = X \neq \phi = \varphi_{\mathcal{P}}(\phi) = \varphi_{\mathcal{P}}(A \cap B)$.

Property 3.11. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space and $A, B \subseteq X$. If A is m-open in X, then $A \cap \varphi_{\mathcal{P}}(B) \subseteq \varphi_{\mathcal{P}}(A \cap (B))$.*

Proof. Let $A \in \mathcal{M}$ and $x \in A \cap \varphi_{\mathcal{P}}(B)$. It $x \in A$ and $x \in \varphi_{\mathcal{P}}(B)$. Then, for all $U \in \mathcal{M}(x)$, $B^c \cup U^c \in \mathcal{P}$. Since $A \in \mathcal{M}$, we get $(A \cap B)^c \cup U^c = B^c \cup (A^c \cup U^c) = B^c \cup (A \cap U)^c \in \mathcal{P}$. So, $x \in \varphi_{\mathcal{P}}(A \cap B)$. Thus, $A \cap \varphi_{\mathcal{P}}(B) \subseteq \varphi_{\mathcal{P}}(A \cap B)$. □

Property 3.12. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal topological space. If $A^c \in \mathcal{M}$, then $\varphi_{\mathcal{P}}(A) \subseteq A$.*

Proof. Given $A^c \in \mathcal{M}$ and let $x \in \varphi_{\mathcal{P}}(A)$. Suppose $x \notin A$. Then, $A^c \in \mathcal{M}(x)$. Since $x \in \varphi_{\mathcal{P}}(A)$, so for all $U \in \mathcal{M}(x), U^c \cup A^c \in \mathcal{P}$. Thus, $(A^c)^c \cup A^c \in \mathcal{P}$, i.e., $X \in \mathcal{P}$. This contradicts the fact that $X \notin \mathcal{P}$. So, $\varphi_{\mathcal{P}}(A) \subseteq A$. □

Corollary 3.5. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space and $A, B \subseteq X$ with $B^c \notin \mathcal{P}$. Then, $\varphi_{\mathcal{P}}(A \cup B) = \varphi_{\mathcal{P}}(A) = \varphi_{\mathcal{P}}(A \setminus B)$.*

Proof. Since $B^c \notin \mathcal{P}$, it gives $\varphi_{\mathcal{P}}(B) = \phi$. Now, $\varphi_{\mathcal{P}}(A \cup B) = \varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(B) = \varphi_{\mathcal{P}}(A) \cup \phi = \varphi_{\mathcal{P}}(A)$. Again, $\varphi_{\mathcal{P}}(A \setminus B) \subseteq \varphi_{\mathcal{P}}(A)$. Also, we have $\varphi_{\mathcal{P}}(A) \setminus \varphi_{\mathcal{P}}(B) \subseteq \varphi_{\mathcal{P}}(A \setminus B)$. It implies $\varphi_{\mathcal{P}}(A) \subseteq \varphi_{\mathcal{P}}(A \setminus B)$. Thus, $\varphi_{\mathcal{P}}(A) = \varphi_{\mathcal{P}}(A \setminus B)$. Therefore, $\varphi_{\mathcal{P}}(A \cup B) = \varphi_{\mathcal{P}}(A) = \varphi_{\mathcal{P}}(A \setminus B)$. □

Definition 3.6. Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. A mapping $CL : 2^X \to 2^X$ is defined as follows: $CL(A) = A \cup \varphi_{\mathcal{P}}(A)$ for all $A \in 2^X$.

Theorem 3.7. *The above map CL satisfies Kuratowski closure axioms.*

Proof. Due to Property 3.1, $CL(\phi) = \phi$, and obviously $A \subseteq CL(A)$. Now, $CL(A\cup B) = (A\cup B)\cup\varphi_{\mathcal{P}}(A\cup B) = (A\cup B)\cup\varphi_{\mathcal{P}}(A)\cup\varphi_{\mathcal{P}}(B) = CL(A)\cup CL(B)$. Again, for any $A \subseteq X$, we have $CL(CL(A)) = CL(A\cup\varphi_{\mathcal{P}}(A)) = (A\cup\varphi_{\mathcal{P}}(A))\cup \varphi_{\mathcal{P}}(A \cup \varphi_{\mathcal{P}}(A)) = A \cup \varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(A) = A \cup \varphi_{\mathcal{P}}(A) = CL(A)$. □

Since CL satisfies the Kuratowski closure axioms, it induces a unique topology on X. This topology is described in the following theorem.

THEOREM 3.8. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, $\tau_{\mathcal{MP}} = \{W \subseteq X : CL(X \setminus W) = X \setminus W\}$ is a topology on X, where $CL(A) = A \cup \varphi_{\mathcal{P}}(A)$. If $A \in \tau_{\mathcal{MP}}$, then A is said to be $\tau_{\mathcal{MP}}$-open set. $B \subseteq X$ is said to be $\tau_{\mathcal{MP}}$-closed set if $B^c \in \tau_{\mathcal{MP}}$.*

Henceforth, throughout this paper, $\mathrm{Int}_{\mathcal{MP}}$ and $\mathrm{Cl}_{\mathcal{MP}}$ denote the interior and closure operators, respectively, with respect to the $\tau_{\mathcal{MP}}$-topology on $(X, \mathcal{M}, \mathcal{P})$.

THEOREM 3.9. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. If $A \subseteq B$, then $CL(A) \subseteq CL(B)$.*

Proof. Since $A \subseteq B$, then from Property 3.5, we get $\varphi_{\mathcal{P}}(A) \subseteq \varphi_{\mathcal{P}}(B)$. So, $A \cup \varphi_{\mathcal{P}}(A) \subseteq B \cup \varphi_{\mathcal{P}}(B)$. It implies $CL(A) \subseteq CL(B)$. □

THEOREM 3.10. *(a) If $\mathcal{P}_1$ and $\mathcal{P}_2$ be two primals on X with $\mathcal{P}_1 \subseteq \mathcal{P}_2$. Then, $\tau_{\mathcal{MP}_2} \subseteq \tau_{\mathcal{MP}_1}$.*

(b) If $\mathcal{P}$ is a primal on X and $B^c \notin \mathcal{P}$, then B is $\tau_{\mathcal{MP}}$-closed set.

(c) For any subset A of an m-space $(X, \mathcal{M})$ and any primal P on X, $\varphi_{\mathcal{P}}(A)$ is $\tau_{\mathcal{MP}}$ -closed.

Proof. (a) Let $U \in \tau_{\mathcal{MP}_2}$. Then $CL_{\mathcal{MP}_2}(X \setminus U) = (X \setminus U)$. It $(X \setminus U) = (X \setminus U) \cup \varphi_{\mathcal{P}2}(X \setminus U)$. Thus, $\varphi_{\mathcal{P}2}(X \setminus U) \subseteq (X \setminus U)$. It $\varphi_{\mathcal{P}_1}(X \setminus U) \subseteq (X \setminus U)$. So, $(X \setminus U) = CL_{\mathcal{MP}_1}(X \setminus U)$. Hence, $U \in \tau_{\mathcal{MP}_1}$.

(b) It is obvious that, due to Corollary3.2, for $B^c \notin \mathcal{P}$, $\varphi_{\mathcal{P}}(B) = \phi$. Then, $CL_{\mathcal{MP}}(B) = B \cup \varphi_{\mathcal{P}}(B) = B$. Hence, B is $\tau_{\mathcal{MP}}$ - closed.

(c) We have $CL(\varphi_{\mathcal{P}}(A)) = \varphi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(\varphi_{\mathcal{P}}(A)) = \varphi_{\mathcal{P}}(A)$. Therefore, $\varphi_{\mathcal{P}}(A)$ is $\tau_{\mathcal{MP}}$-closed. □

We now determine a basis for the topology $\tau_{\mathcal{MP}}$ on X. The following theorem provides such a basis.

THEOREM 3.11. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, $\beta(\mathcal{M}, \mathcal{P}) = \{V \setminus A : V \in \mathcal{M}$ and $A^c \notin \mathcal{P}\}$ is the base for $\tau_{\mathcal{MP}}$.*

Proof. Let $U \in \tau_{\mathcal{MP}}$ and $x \in U$. Then, $(X \setminus U)$ is $\tau_{\mathcal{MP}}$-closed. So, $CL(X \setminus U)$ $= X \setminus U$. Hence, $\varphi_{\mathcal{P}}(X \setminus U) \subseteq (X \setminus U)$. Then, $x \notin \varphi_{\mathcal{P}}(X \setminus U)$ and so, there exists $V \in \mathcal{M}(x)$ such that $((X \setminus U) \cap V)^c \notin \mathcal{P}$. Let $A^c = ((X \setminus U) \cap V)^c$. Then, $x \notin A$ and $A^c \notin \mathcal{P}$. Thus,

$$x \in (V \setminus A) = V \setminus [(X \setminus U) \cap V] = V \setminus (V \setminus U) \subseteq U,$$

and $V \setminus A \in \beta(\mathcal{M}, \mathcal{P})$. It now suffices to observe that $\beta(\mathcal{M}, \mathcal{P})$ is closed under finite intersection. Let $V_1 \setminus A, V_2 \setminus A \in \beta(\mathcal{M}, \mathcal{P})$. So, $V_1, V_2 \in \mathcal{M}$ and $A^c, B^c \notin \mathcal{P})$. Then, $(V_1 \cap V_2) \in \mathcal{M}$ and $A^c \cap B^c \notin \mathcal{P}$. Now, $(V_1 \setminus A) \cap (V_2 \setminus B)$= $(V_1 \cap V_2) \setminus (A \cup B) \in \beta(\mathcal{M}, \mathcal{P})$. Hence, $\beta(\mathcal{M}, \mathcal{P})$ is a base for $\tau_{\mathcal{MP}}$. □

COROLLARY 3.12. *For any primal $\mathcal{P}$ on an m-space $(X, \mathcal{M})$, $\mathcal{M} \subseteq \beta(\mathcal{M}, \mathcal{P}) \subseteq \tau_{\mathcal{MP}}$.*

Proof. As $\beta(\mathcal{M}, \mathcal{P})$ serves as a base of $\tau_{\mathcal{MP}}$, we have $\beta(\mathcal{M}, \mathcal{P}) \subseteq \tau_{\mathcal{MP}}$. Again, for any $U \in \mathcal{M}$, $\phi^c = X \notin \mathcal{P}$. Therefore, by the definition of $\beta(\mathcal{M}, \mathcal{P})$, $U = U \setminus \phi \in \beta(\mathcal{M}, \mathcal{P})$ for any $U \in \mathcal{M}$. Therefore, $\mathcal{M} \subseteq \beta(\mathcal{M}, \mathcal{P})$. Thus, $\mathcal{M} \subseteq \beta(\mathcal{M}, \mathcal{P}) \subseteq \tau_{\mathcal{MP}}$. $\square$

4. $\psi_{\mathcal{P}}$- OPERATOR

In this section, we introduce $\psi_{\mathcal{P}}$-operator and study some of its properties.

DEFINITION 4.1. Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. An operator $\psi_{\mathcal{P}} : 2^X \to 2^X$ is defined as : for every $A \in 2^X$, $\psi_{\mathcal{P}}(A) = \{x \in X :$ there exists a $U \in \mathcal{M}(x)$ such that $(U \setminus A)^c \notin \mathcal{P}\}$.

It is easy to find that $\psi_{\mathcal{P}}(A) = X \setminus \varphi_{\mathcal{P}}(X \setminus A)$.

THEOREM 4.2. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, the following properties hold:*

(1) *if $A \subseteq X$, then $\psi_{\mathcal{P}}(A)$ is m-open in $(X, \mathcal{M})$,*
(2) *if $A \subseteq B$, then $\psi_{\mathcal{P}}(A) \subseteq \psi_{\mathcal{P}}(B)$,*
(3) *if $A, B \in 2^X$, then $\psi_{\mathcal{P}}(A \cap B) = \psi_{\mathcal{P}}(A) \cap \psi_{\mathcal{P}}(B)$,*
(4) *if $U \in \tau_{\mathcal{MP}}$, then $U \subseteq \psi_{\mathcal{P}}(U)$,*
(5) *if $A \subseteq X$, then $\psi_{\mathcal{P}}(A) \subseteq \psi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$,*
(6) *let $A \subseteq X$, then $\psi_{\mathcal{P}}(A) = \psi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$ if and only if $\varphi_{\mathcal{P}}(X \setminus A) = \varphi_{\mathcal{P}}(\varphi_{\mathcal{P}}(X \setminus A))$,*
(7) *if $A^c \notin \mathcal{P}$, then $\psi_{\mathcal{P}}(A) = X \setminus \varphi_{\mathcal{P}}(X)$,*
(8) *if $A \subseteq X$, then $A \cap \psi_{\mathcal{P}}(A) = \mathrm{Int}_{\mathcal{MP}}(A)$,*
(9) *if $A \subseteq X$ and $P^c \notin \mathcal{P}$, then $\psi_{\mathcal{P}}(A \setminus P) = \psi_{\mathcal{P}}(A)$,*
(10) *if $A \subseteq X$ and $P^c \notin \mathcal{P}$, then $\psi_{\mathcal{P}}(A \cup P) = \psi_{\mathcal{P}}(A)$,*
(11) *if $A, B \subseteq X$ and $(A \setminus B)^c \cap (B \setminus A)^c \notin \mathcal{P}$, then $\psi_{\mathcal{P}}(A) = \psi_{\mathcal{P}}(B)$.*

Proof. (1) We have $\psi_{\mathcal{P}}(A) = X \setminus \varphi_{\mathcal{P}}(X \setminus A)$. By Corollary 3.3, we get that $\varphi_{\mathcal{P}}(X \setminus A)$ is m-closed. Therefore, $X \setminus \varphi_{\mathcal{P}}(X \setminus A)$ is m-open. Therefore, $\psi_{\mathcal{P}}(A)$ is m-open.

(2) Consider $A \subseteq B$. Then, from Property 3.5, we have $\varphi_{\mathcal{P}}(X \setminus B) \subseteq \varphi_{\mathcal{P}}(X \setminus A)$. Therefore, $\psi_{\mathcal{P}}(A) \subseteq \psi_{\mathcal{P}}(B)$.

(3) From (2), we have $\psi_{\mathcal{P}}(A \cap B) \subseteq \psi_{\mathcal{P}}(A)$ and $\psi_{\mathcal{P}}(A \cap B) \subseteq \psi_{\mathcal{P}}(B)$. Therefore, $\psi_{\mathcal{P}}(A \cap B) \subseteq \psi_{\mathcal{P}}(A) \cap \psi_{\mathcal{P}}(B)$. Now, let $x \in \psi_{\mathcal{P}}(A) \cap \psi_{\mathcal{P}}(B)$. Then, there exist $U, V \in \mathcal{M}(x)$ such that $(U \setminus A)^c \notin \mathcal{P}$ and $(V \setminus B)^c \notin \mathcal{P}$. Let $P = U \cap V \in \mathcal{M}(x)$. Then we have, $(P \setminus A)^c \notin \mathcal{P}$ and $(P \setminus B)^c \notin \mathcal{P}$. Therefore, we have

$$[P \setminus (A \cap B)]^c = [(P \setminus A) \cup (P \setminus B)]^c = (P \setminus A)^c \cap (P \setminus B)^c \notin \mathcal{P}.$$

Therefore, $x \in \psi_{\mathcal{P}}(A \cap B)$. Therefore, $\psi_{\mathcal{P}}(A) \cap \psi_{\mathcal{P}}(B) \subseteq \psi_{\mathcal{P}}(A \cap B)$. Hence, the prove is done.

(4) If $U \in \tau_{\mathcal{MP}}$, then $(X \setminus U)$ is $\tau_{\mathcal{MP}}$-closed. It implies $\varphi_{\mathcal{P}}(X \setminus U) \subseteq (X \setminus U)$. Hence $U \subseteq [X \setminus \varphi_{\mathcal{P}}(X \setminus U)] = \psi_{\mathcal{P}}(U)$.

(5) From (1), we have $\psi_{\mathcal{P}}(A)$ is m-open. Therefore, by Corollary 3.12, $\psi_{\mathcal{P}}(A)$ is $\tau_{\mathcal{MP}}$-open. Therefore, from (4), $\psi_{\mathcal{P}}(A) \subseteq \psi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$.

(6) The proof follows from the following two facts:
$\psi_{\mathcal{P}}(A) = X \setminus \varphi_{\mathcal{P}}(X \setminus A)$ and thus, $\psi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)) = [X \setminus \varphi_{\mathcal{P}}(X \setminus (X \setminus \varphi_{\mathcal{P}}(X \setminus A)))]$.

(7) By Corollary 3.5, we have $\varphi_{\mathcal{P}}(X \setminus A) = \varphi_{\mathcal{P}}(X)$ if $A^c \notin \mathcal{P}$. Then, $\psi_{\mathcal{P}}(A) = X \setminus \varphi_{\mathcal{P}}(X)$.

(8) If $x \in A \cap \psi_{\mathcal{P}}(A)$, then $x \in A$ and there exists a $U \in \mathcal{M}(x)$ such that $(U \setminus A)^c \notin \mathcal{P}$. Then, $[U \setminus (U \setminus A)]$ is a $\tau_{\mathcal{MP}}$ -open neighbourhood of x and $x \in \mathrm{Int}_{\mathcal{MP}}(A)$.

Conversely, suppose $x \in \mathrm{Int}_{\mathcal{MP}}(A)$. Then, there exists a basic $\tau_{\mathcal{MP}}$-open neighbourhood $V \setminus P$ of x where $V \in \mathcal{M}(x)$ and $P^c \notin \mathcal{P}$, such that $x \in V \setminus P \subseteq A$. It $V \setminus A \subseteq P$. Therefore, $P^c \subseteq (V \setminus A)^c$. Hence, $(V \setminus A)^c \notin \mathcal{P}$. Hence, $x \in \psi_{\mathcal{P}}(A)$. Therefore, $x \in A \cap \psi_{\mathcal{P}}(A)$.

(9) $\psi_{\mathcal{P}}(A \setminus P) = [X \setminus \varphi_{\mathcal{P}}(X \setminus (A \setminus P))] = [X \setminus \varphi_{\mathcal{P}}((X \setminus A) \cup P)] = [X \setminus \varphi_{\mathcal{P}}(X \setminus A)] = \psi_{\mathcal{P}}(A)$.

(10) $\psi_{\mathcal{P}}(A \cup P) = X \setminus \varphi_{\mathcal{P}}[X \setminus (A \cup P)] = X \setminus \varphi_{\mathcal{P}}[(X \setminus A) \setminus P] = X \setminus \varphi_{\mathcal{P}}(X \setminus A) = \psi_{\mathcal{P}}(A)$.

(11) Assume $(A \setminus B)^c \cap (B \setminus A)^c \notin \mathcal{P}$. Let $(A \setminus B)^c = P_1^c$ *and* $(B \setminus A)^c = P_2^c$. Then, $P_1^c, P_2^c \notin \mathcal{P}$. Also, observe that $B = (A \setminus P_1) \cup P_2$. Thus, $\psi_{\mathcal{P}}(A) = \psi_{\mathcal{P}}(A \setminus P_1) = \psi_{\mathcal{P}}[(A \setminus P_1) \cup P_2] = \psi_{\mathcal{P}}(B)$. □

Corollary 4.3. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, $U \subseteq \psi_{\mathcal{P}}(U)$ for every $U \in \mathcal{M}$*

Proof. $U \in \mathcal{M}$ implies $U \in \tau_{\mathcal{MP}}$. Then, from (4) of Theorem 4.2, we have $U \subseteq \psi_{\mathcal{P}}(U)$. □

5. SOME PROPERTIES RELATED TO THE ABOVE TWO OPERATORS

In this section, we prove some properties related to $\varphi_{\mathcal{P}}$ and $\psi_{\mathcal{P}}$ operators.

Theorem 5.1. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$ if and only if $\varphi_{\mathcal{P}}(X) = X$.*

Proof. Suppose that $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. It is obvious that $\varphi_{\mathcal{P}}(X) \subseteq X$. For the reverse inclusion, let $x \in X$ but $x \notin \varphi_{\mathcal{P}}(X)$. Then, there exists $U \in \mathcal{M}(x)$ such that $U^c \cup X^c \notin \mathcal{P}$, i.e., $U^c \notin \mathcal{P}$. But, this contradicts the fact that $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. Therefore, $x \in X$ implies $x \in \varphi_{\mathcal{P}}(X)$. Hence, $X \subseteq \varphi_{\mathcal{P}}(X)$. Therefore, $\varphi_{\mathcal{P}}(X) = X$.

Conversely, suppose that $\varphi_{\mathcal{P}}(X) = X$. Let $V \in \mathcal{M}^{\mathcal{CL}} \setminus \{X\}$. Therefore, $V^c \in \mathcal{M} \setminus \{\phi\}$. Therefore, $\varphi_{\mathcal{P}}(X) = X$ implies $(V^c)^c \cup X^c \in \mathcal{P}$. Therefore, $V \in \mathcal{P}$. Hence, $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. □

Theorem 5.2. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space and $A \in \mathcal{M}$. Then, $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$ if and only if $\varphi_{\mathcal{P}}(A) = m\mathrm{Cl}(A)$.*

Proof. Suppose that $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. From Property 3.2 , we have $\varphi_{\mathcal{P}}(A) \subseteq m\mathrm{Cl}(A)$. For the reverse inclusion, let $x \in m\mathrm{Cl}(A)$. Then, for every $U_x \in \mathcal{M}(x)$, $U_x \cap A \neq \phi$. Therefore, $U_x \cap A \in \mathcal{M} \setminus \{\phi\}$, which implies $U_x^c \cup A^c \in \mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. So, $x \in \varphi_{\mathcal{P}}(A)$ and hence, $m\mathrm{Cl}(A) \subseteq \varphi_{\mathcal{P}}(A)$. Therefore, $\varphi_{\mathcal{P}}(A) = m\mathrm{Cl}(A)$.

For the converse part, consider $\varphi_{\mathcal{P}}(A) = m\mathrm{Cl}(A)$. Therefore, $\varphi_{\mathcal{P}}(X) = m\mathrm{Cl}(X) = X$. Therefore, from Theorem 5.1, we have $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. □

THEOREM 5.3. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. Then, the following properties are equivalent:*

(1) $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$,
(2) *for every* $P \in \mathcal{M},\ P \subseteq \varphi_{\mathcal{P}}(P)$,
(3) $X = \varphi_{\mathcal{P}}(X)$,
(4) *if* $A \in \mathcal{M}$, *then* $\varphi_{\mathcal{P}}(A) = m\mathrm{Cl}(A)$,
(5) $\varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)) = m\mathrm{Cl}(\psi_{\mathcal{P}}(A))$.

Proof. (1) $\Longrightarrow$ (2) Suppose $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. Take $P \in \mathcal{M}$. Then, either $P = \phi$ or $P \in \mathcal{M} \setminus \{\phi\}$. If $P = \phi$, then $P^c = X$. Therefore, $P^c \cup U^c = X \notin \mathcal{P}$ for all $U \in \mathcal{M}$. Therefore, we have $\phi = \varphi_{\mathcal{P}}(P) = \varphi_{\mathcal{P}}(\phi)$.
For $P \in \mathcal{M} \setminus \{\phi\}$, consider $x \in P$ and any $U_x \in \mathcal{M}(x)$. Then, $P \cap U_x \in \mathcal{M}(x)$. Therefore, $P^c \cup U_x^c = (P \cap U_x)^c \in \mathcal{M}^{\mathcal{CL}} \setminus \{X\}$. Therefore, $P^c \cup U_x^c \in \mathcal{P}$. Thus, we have $x \in \varphi_{\mathcal{P}}(P)$. Therefore, $P \subseteq \varphi_{\mathcal{P}}(P)$.

(2) $\Longrightarrow$ (3) Now, suppose property (2) holds. We have $X \in \mathcal{M}$. Therefore, from (2), we have $X \subseteq \varphi_{\mathcal{P}}(X)$. Also, $\varphi_{\mathcal{P}}(X) \subseteq X$. Therefore, $X = \varphi_{\mathcal{P}}(X)$.

(3) $\Longrightarrow$ (1) Again, from Theorem 5.1, we can say (3) $\Longrightarrow$ (1).

(1) $\Longleftrightarrow$ (4) It is the result from Theorem 5.2.

(4) $\Longrightarrow$ (5) Now, suppose property (4) holds. We know $\psi_{\mathcal{P}}(A) \in \mathcal{M}$. Therefore, we get $\varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)) = m\mathrm{Cl}(\psi_{\mathcal{P}}(A))$.

(5) $\Longrightarrow$ (3) Now, suppose property (5) holds. We have $\psi_{\mathcal{P}}(X) = X$ and $m\mathrm{Cl}(X) = X$. Therefore, $\varphi_{\mathcal{P}}(X) = X$. Hence, the theorem is proved. □

THEOREM 5.4. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space and $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. Then, $\psi_{\mathcal{P}}(A) \setminus A = \phi$, for m-closed subset A.*

Proof. By Theorem 5.2, we have $\psi_{\mathcal{P}}(A) \setminus A = [X \setminus \varphi_{\mathcal{P}}(X \setminus A)] \setminus A = [X \setminus m\mathrm{Cl}(X \setminus A)] \setminus A = m\mathrm{Int}(A) \setminus A = \phi$. □

6. $\psi_{\mathcal{P}}$-C SET AND PROPERTIES OF $\psi_{\mathcal{P}}(X, \mathcal{M})$

In this section, we define the $\psi_{\mathcal{P}}$-C sets and prove some related results.

DEFINITION 6.1. Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space. A subset A of X is called a $\psi_{\mathcal{P}}$-C set if $A \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A))$. The collection of all $\psi_{\mathcal{P}}$-C sets in $(X, \mathcal{M}, \mathcal{P})$ is denoted as $\psi_{\mathcal{P}}(X, \mathcal{M})$.

REMARK 6.2. It is obvious that $\mathcal{M} \subseteq \tau_{\mathcal{MP}} \subseteq \psi_{\mathcal{P}}(X, \mathcal{M})$, but the reverse inclusion does not hold in general. For example, let us consider $X = \{a, b, c, d\}$,

$\mathcal{M} = \{\phi, X, \{a\}, \{b,c\}, \{a,b,c\}\}$ and

$$\mathcal{P} = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{b,c,d\}, \{a,c,d\}, \{b,d\}, \{b,c\}, \{a,c\}, \{c,d\}\}.$$

Consider $A = \{a,d\}$, then $\psi_{\mathcal{P}}(A) = \{a\}$. Thus, $m\mathrm{Cl}(\psi_{\mathcal{P}}(A)) = \{a,d\}$. So, $\{a,d\} \in \psi_{\mathcal{P}}(X,\mathcal{M})$, but $\{a,d\} \notin \tau_{\mathcal{MP}}$.

THEOREM 6.3. *Let $\{A_\alpha\colon \alpha \in \Delta, \Delta$ is an index set$\}$ be a collection of non-empty $\psi_{\mathcal{P}}$-C sets in a primal m-space $(X,\mathcal{M},\mathcal{P})$. Then, $\cup_{\alpha\in\Delta} A_\alpha \in \psi_{\mathcal{P}}(X,\mathcal{M})$.*

Proof. For each $\alpha \in \Delta$, $A_\alpha \subseteq mCl(\psi_{\mathcal{P}}(A_\alpha)) \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(\cup_{\alpha\in\Delta} A_\alpha))$. This implies $\cup_{\alpha\in\Delta} A_\alpha \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(\cup_{\alpha\in\Delta} A_\alpha))$. Thus, $\cup_{\alpha\in\Delta} A_\alpha \in \psi_{\mathcal{P}}(X,\mathcal{M}))$. □

REMARK 6.4. The intersection of two $\psi_{\mathcal{P}}$-C sets may not be a $\psi_{\mathcal{P}}$-C set in general. In Remark 6.2, consider $A = \{a,d\}$ and $B = \{b,c,d\}$.Then, $\varphi_{\mathcal{P}}(X \setminus A) = \{b,c,d\}$ and $m\mathrm{Cl}(\psi_{\mathcal{P}}(A)) = \{a,d\}$. Hence, $A \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Again, $\varphi_{\mathcal{P}}(X \setminus B) = \{a,d\}$ and $m\mathrm{Cl}(\psi_{\mathcal{P}}(B)) = \{b,c,d\}$. Hence, $B \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Since, $A \cap B = \{d\}$, $\varphi_{\mathcal{P}}(X \setminus (A\cap B)) = X$ and $m\mathrm{Cl}(\psi_{\mathcal{P}}(A \cap B) = \phi$. Thus, $A \cap B \notin \psi_{\mathcal{P}}(X,\mathcal{M})$.

THEOREM 6.5. *Let $(X,\mathcal{M},\mathcal{P})$ be a primal m-space and $A \in \psi_{\mathcal{P}}(X,\mathcal{M})$. If $U \in \mathcal{M}$, then $U \cap A \in \psi_{\mathcal{P}}(X,\mathcal{M})$.*

Proof. Let $U \in \mathcal{M}$ and $A \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Therefore, $A \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A)$. Thus,

$$\begin{aligned} U \cap A &\subseteq U \cap m\mathrm{Cl}(\psi_{\mathcal{P}}(A)) \subseteq m\mathrm{Cl}(U \cap \psi_{\mathcal{P}}(A)) \\ &\subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(U) \cap \psi_{\mathcal{P}}(A)) = m\mathrm{Cl}(\psi_{\mathcal{P}}(U \cap A)). \end{aligned}$$

So, $U \cap A \in \psi_{\mathcal{P}}(X,\mathcal{M})$. □

7. $\tau^{\psi}_{\mathcal{MP}}$-TOPOLOGY AND PROPERTIES OF $\tau^{\psi}_{\mathcal{MP}}$

In this section, we introduce the $\tau^{\psi}_{\mathcal{MP}}$-topology and investigate several of its fundamental properties.

DEFINITION 7.1. Let $(X,\mathcal{M},\mathcal{P})$ be a primal m-space. Then, a subset A of X is called a $\psi_{\mathcal{P}}$-set if $A \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A)))$. The collection of all $\psi_{\mathcal{P}}$ sets in $(X,\mathcal{M},\mathcal{P})$ is denoted by $\tau^{\psi}_{\mathcal{MP}}$.

It can be checked that $\mathcal{M} \subseteq \tau^{\psi}_{\mathcal{MP}} \subseteq \psi_{\mathcal{P}}(X,\mathcal{M})$.

THEOREM 7.2. *Let $(X,\mathcal{M},\mathcal{P})$ be a primal m-space. Then, $\tau^{\psi}_{\mathcal{MP}} = \{A \subseteq X : A \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A)))\}$ forms a topology on X, where $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$.*

Proof. (1) We have

$$\psi_{\mathcal{P}}(\phi) = X \setminus \varphi_{\mathcal{P}}(X \setminus \phi) = X \setminus \varphi_{\mathcal{P}}(X),$$

and Theorem 5.3 gives $\varphi_{\mathcal{P}}(X) = X$. Therefore, we have $\psi_{\mathcal{P}}(\phi) = X \setminus X = \phi$. Therefore, $\psi_{\mathcal{P}}(\phi) = \phi$. So, $\phi \in \tau^{\psi}_{\mathcal{MP}}$. Also, we have $\psi_{\mathcal{P}}(X) = X \setminus \varphi_{\mathcal{P}}(X \setminus X) =$

$X \setminus \varphi_{\mathcal{P}}(\phi)$. Again, by Property 3.1, we have $\varphi_{\mathcal{P}}(\phi) = \phi$. Hence, $\psi_{\mathcal{P}}(X) = X$. Therefore, $X \in \tau^{\psi}_{\mathcal{MP}}$.

(2) Let $A_i \in \tau^{\psi}_{\mathcal{MP}}$ for all $i \in \Delta$, where Δ is an index set. Now, we show that $\cup_{i\in\Delta} A_i \in \tau^{\psi}_{\mathcal{MP}}$. Since, $A_i \subseteq \cup_{i\in\Delta} A_i$, we have $\psi_{\mathcal{P}}(A_i) \subseteq \psi_{\mathcal{P}}(\cup_i A_i)$. Then

$$m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_i))) \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(\cup_i A_i))).$$

So, we have:

$$A_i \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_i))) \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(\cup_i A_i)))$$

for all $i \in \Delta$. Therefore, $\cup_{i\in\Delta} A_i \in \tau^{\psi}_{\mathcal{MP}}$.

(3) Let $A_1, A_2 \in \tau^{\psi}_{\mathcal{MP}}$. We show that $A_1 \cap A_2 \in \tau^{\psi}_{\mathcal{MP}}$. If $A_1 \cap A_2 = \phi$, then we are done. Otherwise, let $x \in A_1 \cap A_2$. Now,

$$A_1 \subseteq m\mathrm{Int}(mCl(\psi_{\mathcal{P}}(A_1))),$$

$$A_2 \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_2))),$$

$$x \in m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1))) \cap m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_2))).$$

So, we have:

$$x \in m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1) \cap m\mathrm{Cl}(\psi_{\mathcal{P}}(A_2)))).$$

Therefore , there is an m-open set V_x containing x with the property that $V_x \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1) \cap m\mathrm{Cl}(\psi_{\mathcal{P}}(A_2)))$. Then $V_x \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1))$ and $V_x \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A_2))$. Let $y \in V_x$. Then, $\exists$ an m-open set G_y containing y such that $G_y \subseteq V_x$. Therefore, $G_y \cap \psi_{\mathcal{P}}(A_1) \neq \phi$. Therefore, there exists a $z \in G_y \cap \psi_{\mathcal{P}}(A_1)$. So, there exists a $U \in \mathcal{M}(z)$ such that $(U \setminus A_1)^c \notin \mathcal{P}$. We may assume that this U is chosen such that $U \subseteq G_y$. Again, $U \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A_2))$. So, there exists a non-empty m-open set $U' \subseteq U$ such that $(U' \setminus A_2)^c \notin \mathcal{P}$. Now,

$$U' \setminus (A_1 \cap A_2) = (U' \setminus A_1) \cup (U' \setminus A_2) \subseteq (U \setminus A_1) \cup (U' \setminus A_2).$$

Therefore,

$$(U' \setminus (A_1 \cap A_2))^c \supseteq ((U \setminus A_1) \cup (U' \setminus A_2))^c = (U \setminus A_1)^c \cap (U' \setminus A_2)^c.$$

Now, as $(U \setminus A_1)^c$ and $(U \setminus A_2)^c$ are not in $\mathcal{P}$, therefore, $(U' \setminus (A_1 \cap A_2))^c \notin \mathcal{P}$. Hence, we have $U' \subseteq \psi_{\mathcal{P}}(A_1 \cap A_2)$. As $U' \subseteq G_y$, we have $G_y \cap \psi_{\mathcal{P}}(A_1 \cap A_2) \neq \phi$. Therefore, $y \in m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1 \cap A_2))$. Since, y is any point of V_x, it follows that $V_x \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1 \cap A_2))$. Therefore, $x \in m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1 \cap A_2)))$. Thus, $(A_1 \cap A_2) \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A_1 \cap A_2)))$. Hence, $A_1 \cap A_2 \in \tau^{\psi}_{\mathcal{MP}}$.

Thus, we conclude that $\tau^{\psi}_{\mathcal{MP}}$ is a topology. □

PROPOSITION 7.3. *Let $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. Then, $\psi_{\mathcal{P}}(A) \neq \phi$ if and only if A contains a set with non-empty $\tau_{\mathcal{MP}}$ - interior.*

Proof. Let $\psi_{\mathcal{P}}(A) \neq \phi$. Then, there exists a non-empty set $U \in \mathcal{M}$ such that $(U \setminus A)^c = P^c$, where $P^c \notin \mathcal{P}$. Now, $U \setminus P \subseteq A$. Thus, we have $U \setminus P \in \tau_{\mathcal{MP}}$. Therefore, A contains a set with non-empty $\tau_{\mathcal{MP}}$ - interior.

Conversely, suppose A contains a set with non-empty $\tau_{\mathcal{MP}}$- interior. So, there exists a non-empty set $U \in \mathcal{M}$ and $P^c \notin \mathcal{P}$ such that $(U \setminus P) \subseteq A$. So, $U \setminus A \subseteq P$. Therefore, $P^c \subseteq (U \setminus A)^c$. Since, $P^c \notin \mathcal{P}$, we have $(U \setminus A)^c \notin \mathcal{P}$. Thus, we have $\psi_{\mathcal{P}}(A) \neq \phi$. □

THEOREM 7.4. *If $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$ in $(X, \mathcal{M}, \mathcal{P})$, then $\mathcal{D}(X, \tau_{\mathcal{MP}}) = \mathcal{D}(X, \tau^{\psi}_{\mathcal{MP}})$, where $\mathcal{D}(X, \mathcal{T})$ denotes the collection of all dense subsets in a topological space $(X, \mathcal{T})$.*

Proof. Since $\tau_{\mathcal{MP}} \subseteq \tau^{\psi}_{\mathcal{MP}}$, we have $\mathcal{D}(X, \tau^{\psi}_{\mathcal{MP}}) \subseteq \mathcal{D}(X, \tau_{\mathcal{MP}})$. Next, let $D \in \mathcal{D}(X, \tau_{\mathcal{MP}})$. We have to show that $D \in \mathcal{D}(X, \tau^{\psi}_{\mathcal{MP}})$. Let $\phi \neq A \in \tau^{\psi}_{\mathcal{MP}}$. So, we have $\psi_{\mathcal{P}}(A) \neq \phi$. Thus, A has a set with non-empty $\tau_{\mathcal{MP}}$-interior. It yields, $\mathrm{Int}_{\mathcal{MP}}(A) \neq \phi$. Now, $Int_{\mathcal{MP}}(A) \cap D \subseteq A \cap D$ and $\mathrm{Int}_{\mathcal{MP}}(A) \cap D \neq \phi$, since $D \in (X, \tau_{\mathcal{MP}})$. Thus, $A \cap D \neq \phi$. So, $D \in \mathcal{D}(X, \tau^{\psi}_{\mathcal{MP}})$. Therefore, $\mathcal{D}(X, \tau_{\mathcal{MP}}) \subseteq \mathcal{D}(X, \tau^{\psi}_{\mathcal{MP}})$. Thus, we have $\mathcal{D}(X, \tau_{\mathcal{MP}}) = \mathcal{D}(X, \tau^{\psi}_{\mathcal{MP}})$. □

THEOREM 7.5. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space, where $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. Then, $(X, \tau^{\psi}_{\mathcal{MP}})$ is resolvable if and only if $(X, \tau_{\mathcal{MP}})$ is resolvable.*

Proof. Due to Theorem 7.4, we have $\mathcal{D}(X, \tau_{\mathcal{MP}}) = \mathcal{D}(X, \tau^{\psi}_{\mathcal{MP}})$. Thus, $(X, \tau^{\psi}_{\mathcal{MP}})$ is resolvable if and only if $(X, \tau_{\mathcal{MP}})$ is resolvable. □

Now, we provide a representation of the α-topology associated with the $\tau_{\mathcal{MP}}$-topology with the help of the $\psi_{\mathcal{P}}$-operator in the following theorem:

THEOREM 7.6. *Let $x \in X$. Then, $\{x\} \in \psi_{\mathcal{P}}(X, \mathcal{M})$ if and only if $\{x\}$ is $\tau_{\mathcal{MP}}$-open.*

Proof. Let $\{x\} \in \psi_{\mathcal{P}}(X, \mathcal{M})$. Then, $\psi_{\mathcal{P}}(\{x\}) \neq \phi$. Therefore, {x} contains a set with non-empty $\tau_{\mathcal{MP}}$-interior. Therefore, $\{x\}$ is open in $(X, \tau_{\mathcal{MP}})$.

Conversely, suppose {x} is open in $(X, \tau_{\mathcal{MP}})$. It implies $\{x\} \subseteq \psi_{\mathcal{P}}(\{x\})$. Therefore, $\{x\} \subseteq mCl(\psi_{\mathcal{P}}(\{x\}))$, i.e., $\{x\} \in \psi_{\mathcal{P}}(X, \mathcal{M})$. □

THEOREM 7.7. *Let $x \in X$. Then, $\{x\} \in \psi_{\mathcal{P}}(X, \mathcal{M})$ if and only if $\{x\} \in \tau^{\psi}_{\mathcal{MP}}$.*

Proof. Let $\{x\} \in \psi_{\mathcal{P}}(X, \mathcal{M})$. Therefore, $\{x\}$ is $\tau_{\mathcal{MP}}$-open. So, $\{x\} \subseteq \psi_{\mathcal{P}}(x)$. It $\{x\} \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(\{x\})))$, since $\psi_{\mathcal{P}}(\{x\})$ is an m-open set. Thus, $\{x\} \in \tau^{\psi}_{\mathcal{MP}}$.

Conversely, suppose that $\{x\} \in \tau^{\psi}_{\mathcal{MP}}$. Then, $\{x\} \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(\{x\})))$. It implies {x} $\subseteq m\mathrm{Cl}(\varphi_{\mathcal{P}}(\{x\}))$. Hence, $\{x\} \in \varphi_{\mathcal{P}}(X, \mathcal{M})$. □

COROLLARY 7.8. *$\tau^{\psi}_{\mathcal{MP}}$ is the collection such that A belongs to $\tau^{\psi}_{\mathcal{MP}}$ and B belongs to $\psi_{\mathcal{P}}(X, \mathcal{M})$ implies $(A \cap B) \in \psi_{\mathcal{P}}(X, \mathcal{M})$, where $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$.*

Proof. Let $A \in \tau^{\psi}_{\mathcal{MP}}$ and $B \in \psi_{\mathcal{P}}(X,\mathcal{M})$. We show that $A \cap B \in \psi_{\mathcal{P}}(X,\mathcal{M})$. If $(A \cap B) = \phi$, then we are done. Let $A \cap B \neq \phi$. Thus, we assume $x \in A \cap B$. This implies $x \in A \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A))$. Then, for every m-open set U_x such that $U_x \cap \psi_{\mathcal{P}}(A) \neq \phi$. Again $x \in B \subseteq mCl(\psi_{\mathcal{P}}(B))$.Then, for every m-open set V_x such that $V_x \cap \psi_{\mathcal{P}}(B) \neq \phi$. So, $W_x = U_x \cap V_x$ is an m-open set. So, $W_x \cap \psi_{\mathcal{P}}(A) \neq \phi$ and $W_x \cap \psi_{\mathcal{P}}(B) \neq \phi$. Again, $W_x \cap \psi_{\mathcal{P}}(A) \subseteq W_x$ and $W_x \cap \psi_{\mathcal{P}}(B) \subseteq W_x$. Therefore, $W_x \cap \psi_{\mathcal{P}}(A) \cap \psi_{\mathcal{P}}(B) \neq \phi$. It $x \in m\mathrm{Cl}(\psi_{\mathcal{P}}(A) \cap \psi_{\mathcal{P}}(B))$. Hence, it implies $x \in m\mathrm{Cl}(\psi_{\mathcal{P}}(A \cap B))$. Thus, $A \cap B \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Next, we consider a subset A of X such that $A \cap B \in \psi_{\mathcal{P}}(X,\mathcal{M})$ for each $B \in \psi_{\mathcal{P}}(X,\mathcal{M})$. We have to show that $A \in \tau^{\psi}_{\mathcal{MP}}$, i.e., $A \subseteq m\mathrm{Int}(mCl(\psi_{\mathcal{P}}(A)))$. It implies $A \subseteq m\mathrm{Int}(\varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)))$. Suppose that $x \in A$ but $x \notin m\mathrm{Int}(\varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)))$. So, $x \in A \cap (X \setminus m\mathrm{Int}(\varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)))$. It implies $x \in A \cap m\mathrm{Cl}(X \setminus \varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)))$. This implies $x \in A \cap m\mathrm{Cl}(C)$, where $C = X \setminus \varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$. C is a non-empty m-open set in $(X,\mathcal{M})$. Since, $\varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$ is an m-closed set. Since, $x \in m\mathrm{Cl}(C)$ then for all m-open set V_x containing x, we have $V_x \cap C \neq \phi$. Since $C \subseteq \psi_{\mathcal{P}}(C)$, so $V_x \cap \psi_{\mathcal{P}}(C) \neq \phi$. This implies $x \in m\mathrm{Cl}(\psi_{\mathcal{P}}(C)) \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(\{x\} \cup C))$, and $C \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(C)) \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(\{x\} \cup C))$. Hence, $\{x\} \cup C \subseteq m\mathrm{cl}(\psi_{\mathcal{P}}(\{x\} \cup C))$. Therefore, $\{x\} \cup C \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Now, by hypothesis, $A \cap (\{x\} \cup C) \in \psi_{\mathcal{P}}(X,\mathcal{M})$. We show that $A \cap (\{x\} \cup C) = \{x\}$.

We consider $y \in X$ and $x \neq y$ such that $y \in A \cap (\{x\} \cup C)$. So $y \in A$ and $y \in C$. Now, $A = A \cap X$ and $X \in \psi_{\mathcal{P}}(X,\mathcal{M})$. So, $A \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Hence, $A \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A))$. Since $y \in A$, thus $y \in m\mathrm{Cl}(\psi_{\mathcal{P}}(A))$. It means $y \in \varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$. But it is a contradiction to the fact that $y \in C$, i.e., $y \in X \setminus \varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$. This $y \notin \varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A))$. Thus, $A \cap (\{x\} \cup C) = \{x\}$. So, $\{x\} \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Hence, $\{x\} \in \tau^{\psi}_{\mathcal{MP}}$. Therefore,

$$\begin{aligned}\{x\} &\subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}\{x\}))\\ &= m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A \cap (\{x\} \cup C))))\\ &\subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A))).\end{aligned}$$

Hence, $x \in m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A)))$, which is a contradiction to the fact that $x \notin m\mathrm{Int}(\varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)))$. Therefore, $A \subseteq m\mathrm{Int}(m\mathrm{Cl}(\psi_{\mathcal{P}}(A)))$, i.e., $A \in \tau^{\psi}_{\mathcal{MP}}$. □

THEOREM 7.9. *Let $(X,\mathcal{M},\mathcal{P})$ be a primal m-space, where $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$. Then, $SO(X,\tau_{\mathcal{MP}}) = \psi_{\mathcal{P}}(X,\mathcal{M})$.*

Proof. Let $A \in SO(X,\tau_{\mathcal{MP}})$. Then, we have:

$$\begin{aligned}A &\subseteq \mathrm{Cl}_{\mathcal{MP}}(\mathrm{Int}_{\mathcal{MP}}(A)) = \mathrm{Cl}_{\mathcal{MP}}(A \cap \psi_{\mathcal{P}}(A)) \subseteq \mathrm{Cl}_{\mathcal{MP}}(\psi_{\mathcal{P}}(A))\\ &= (\psi_{\mathcal{P}}(A) \cup \varphi_{\mathcal{P}}(\psi_{\mathcal{P}}(A)) = \psi_{\mathcal{P}}(A) \cup m\mathrm{Cl}(\psi_{\mathcal{P}}(A)) = m\mathrm{Cl}(\psi_{\mathcal{P}}(A)).\end{aligned}$$

Hence, $A \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Therefore, we have $SO(X,\tau_{\mathcal{MP}}) \subseteq \psi_{\mathcal{P}}(X,\mathcal{M})$.

Conversely, consider $A \in \psi_{\mathcal{P}}(X,\mathcal{M})$. Take $x \in A$. Now, as $A \in \psi_{\mathcal{P}}(X,\mathcal{M})$, we have $A \subseteq m\mathrm{Cl}(\psi_{\mathcal{P}}(A))$. Therefore, for any $G \in \mathcal{M}(x)$, we have $G \cap \psi_{\mathcal{P}}(A) \neq \phi$. Let y $\in G \cap \psi_{\mathcal{P}}(A)$. Therefore, $y \in \psi_{\mathcal{P}}(A)$, which there exists $O_y \in \mathcal{M}$(y) such that $(O_y \setminus A)^c \notin \mathcal{P}$. As $y \in G$ and $y \in O_y$, we have $G \cap O_y \neq \phi$.

Therefore, as $((G \cap O_y) \setminus A)^c \supseteq (O_y \setminus A)^c$, we have $((G \cap O_y) \setminus A)^c \notin \mathcal{P}$. Let $G' = G \cap O_y$. Then, we have $G' \neq \phi, G' \in \mathcal{M}, (G' \setminus A)^c \notin \mathcal{P}$. Let $G' \setminus A = P$. Therefore, $(G' \setminus P) \subseteq A$. Now, as $\mathcal{M}^{\mathcal{CL}} \setminus \{X\} \subseteq \mathcal{P}$, we have $G' \setminus P \neq \phi$. Also $G' \setminus P \in \tau_{\mathcal{MP}}$. Therefore, we have a non-empty $\tau_{\mathcal{MP}}$-open set $G' \setminus \mathcal{P}$ contained in both A and G. Therefore, $x \in \mathrm{Cl}_{\mathcal{MP}}(\mathrm{Int}_{\mathcal{MP}}(A))$. Since, x is an arbitrary element of A, we have $A \subseteq \mathrm{Cl}_{\mathcal{MP}}(\mathrm{Int}_{\mathcal{MP}}(A))$. Therefore, we get $\psi_{\mathcal{P}}(X, \mathcal{M}) \subseteq SO(X, \tau_{\mathcal{MP}}))$. Hence, we have $SO(X, \tau_{\mathcal{MP}}) = \psi_{\mathcal{P}}(X, \mathcal{M})$. □

Now, we state the following results without providing their proofs.

THEOREM 7.10. *$\tau^{\psi}_{\mathcal{MP}}$ is exactly the collection such that A belongs to $\tau^{\psi}_{\mathcal{MP}}$ and $B \in SO(X, \tau_{\mathcal{MP}})$ implies $A \cap B \in SO(X, \tau_{\mathcal{MP}})$, where $\mathcal{M}^{\mathcal{CL}} \setminus X \subseteq \mathcal{P}$.*

THEOREM 7.11. [17] *Let (X, τ) be a topological space. Then, τ^{α} consists of exactly those sets A for which $A \cap B \in SO(X, \tau)$, for all $B \in SO(X, \tau)$.*

COROLLARY 7.12. *Let $(X, \mathcal{M}, \mathcal{P})$ be a primal m-space, where $\mathcal{M}^{\mathcal{CL}} \setminus X \subseteq \mathcal{P}$. Then, $\tau^{\psi}_{\mathcal{MP}} = (\tau_{\mathcal{MP}})^{\alpha}$.*

8. CONCLUSION

In this paper, we combined the primal structure with m-space and introduced a new structure called the primal m-space. We then defined two operators, $\varphi_{\mathcal{P}}$ and $\psi_{\mathcal{P}}$, on this structure. Using the $\varphi_{\mathcal{P}}$-operator, we constructed another operator, CL, which satisfies the Kuratowski closure axioms. Consequently, this operator generates a topology, namely the $\tau_{\mathcal{MP}}$-topology. Furthermore, using the $\psi_{\mathcal{P}}$-operator, we obtained a class of sets, which we call the $\psi_{\mathcal{P}}$-C sets. We then showed that the collection of these sets forms a topology, called the $\tau^{\psi}_{\mathcal{MP}}$-topology. Moreover, we established several important results concerning these structures. These newly introduced structures may find numerous applications in future research.

REFERENCES

[1] S. Acharjee, M. Özkoç, and F. Y. Issaka, *Primal topological spaces*, Bol. Soc. Paran. Mat., **43** (2025), 1–9.

[2] A. Al-Omari, S. Acharjee, and M. Özkoç, *A new operator of primal topological spaces*, Mathematica, **65(88)(2)** (2023), 175–183.

[3] A. Al-Omari and M. H. Alqahtani, *Primal structure with closure operators and their applications*, Mathematics, **11** (2023), Article 4946.

[4] A. Al-Omari and T. Noiri, *On Ψ-g-sets in grill topological spaces*, Filomat, **25** (2011), 187–196.

[5] A. Al-Omari and T. Noiri, *On ψ^*-operator in ideal m-spaces*, Bol. Soc. Paran. Mat., **30** (2012), 53–66.

[6] A. Al-Omari, M. Özkoç, and S. Acharjee, *Primal-proximity spaces*, Mathematica, **66(89)** (2024), 151–167.

[7] M. Özkoç, and P. Şaşmaz, *On a new operator based on a primal and its associated topology*, European Journal of Pure and Applied Mathematics, **17** (2024), 2800-2811.

[8] M. N. Mukherjee and S. Das, *Fuzzy grills and induced fuzzy topology*, Matematički vesnik, **62 (242)** (2010), 285–297.
[9] G. Choquet, *Sur les notions de filtre et de grill*, C. R. Acad. Sci. Paris, **224** (1947), 171–173.
[10] D. Janković and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, **97** (1990), 295–310.
[11] K. Kuratowski, *Topology*, Vol. I, Elsevier, 2014.
[12] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.
[13] S. Modak, *Grill-filter space*, J. Indian Math. Soc., **80** (2013), 313–320.
[14] S. Modak, *Topology on grill-filter space and continuity*, Bol. Soc. Paran. Mat., **31** (2013), 219–230.
[15] S. Modak, *Topology on grill m-space*, Jordan J. Math. Stat., **6** (2013), 183–195.
[16] S. Modak, B. Garai, and S. Mistry, *Remarks on ideal m-space*, Ann. Univ. Oradea Fasc. Mat., **19** (2012), 207–215.
[17] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961–970.
[18] T. Šalát, B. C. Tripathy, and M. Ziman, *On some properties of I-convergence*, Tatra Mt. Math. Publ., **28** (2004), 279–286.
[19] B. C. Tripathy and M. Sen, *On fuzzy I-convergent difference sequence spaces*, J. Intell. Fuzzy Syst., **25** (2013), 643–647.

Received September 1, 2024
Accepted June 19, 2025

Gauhati University
Department of Mathematics
Guwahati-781014, Assam, India
E-mail: asaha9063@gmail.com
https://orcid.org/0009-0002-6346-5350
E-mail: sidhartharajguru@gmail.com
https://orcid.org/0009-0007-5929-6132
E-mail: gitartharajguru2002@gmail.com
https://orcid.org/0009-0000-9567-2587
E-mail: yuvrajbkj083@gmail.com
https://orcid.org/0009-0007-0616-5446
E-mail: satabdi60020@gmail.com
https://orcid.org/0009-0000-0823-3675
E-mail: sacharjee326@gmail.com
https://orcid.org/0000-0003-4932-3305
E-mail: debabh2@gauhati.ac.in
https://orcid.org/0000-0003-0817-9562