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ON AN ELEMENTARY OPERATOR WITH (n,k)-QUASI-$*$-PARANORMAL OPERATORS ENTRIES

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Abstract. We prove that every (n,k)-quasi-$*$-paranormal operator is polaroid. Let A, $B^* \in B(H)$ be (n,k)-quasi-$*$-paranormal operators and let $d_{A,B} \in B(B(H))$ denote either the generalized derivation $\delta_{A,B}(X) = AX - XB$ or the elementary operator $\Delta_{A,B}(X) = AXB - X$. We show that the null space of $(d_{A,B} - \lambda)$, $N(d_{A,B} - \lambda) \subseteq N(d_{A^*,B^*} - \overline{\lambda})$, for all non null complex λ. Let $H\left(\sigma(d_{f(A),g(B)})\right)$ denote the space of analytic functions on a neighborhood of $\sigma(d_{f(A),g(B)})$, and let $H_c(\sigma(T))$ be the set of $f \in H(\sigma(T))$ which are nonconstant on every connected component of $\sigma(T)$ the spectrum of the operator T. We prove that $h\left(d_{f(A),g(B)}\right)$ satisfies generalized Weyl's theorem and $h\left(d^*_{f(A),g(B)}\right)$ satisfies generalized a-Weyl's theorem for every $h \in H\left(\sigma(d_{f(A),g(B)})\right)$, where $f \in H_c(\sigma(A))$ and $g \in H_c(\sigma(B))$. Other related results are also given.

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INTRODUCTION

Let H be a separable infinite dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H into itself. An operator $T \in B(H)$ is said to be $*$-paranormal if $\|T^*x\|^2 \leq \left\|T^2x\right\|$ for any unit vector x in H. The operator T is called n-$*$-paranormal for some positive integer n if

$$\|T^*x\| \leq \left\|T^{1+n}x\right\|^{1/n+1} \|x\|^{n/1+n} \text{ for all } x \in H.$$

Further, T is said to be k-quasi-$*$-paranormal operator if it satisfies the following inequality [16]:

$$\left\|T^*T^kx\right\|^2 \leq \left\|T^{k+2}x\right\| \left\|T^kx\right\| \text{ for all } x \in H.$$

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In order to extend the classes of n-$*$-paranormal and k-quasi-$*$-paranormal operators, Zeng and Zhong [29] introduced a very interesting class of operators which is called (n,k)-quasi-$*$-paranormal operators.

An operator T is said to be (n,k)-quasi-$*$-paranormal if it satisfies the following inequality:

$$\left\|T^*(T^k x)\right\| \leq \left\|T^{1+n}(T^k x)\right\|^{1/n+1} \left\|T^k x\right\|^{n/1+n} \text{ for all } x \in H.$$

This definition is equivalent by [29, Lemma 2.2] to

$$T^{*k}T^{*(1+n)}T^{1+n}T^k - (1+n)\lambda^n T^{*k}TT^*T^k + n\lambda^{1+n}T^{*k}T^k \geq 0 \text{ for all } \lambda > 0.$$

For $T \in B(H)$, let $\sigma(T)$, $\sigma_a(T)$ and $\sigma_p(T)$ denote the spectrum, the approximate point spectrum and point spectrum of T respectively. We shall write $N(T)$ and $R(T)$ for the kernel and range of T. Let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim(H/R(T))$. If the range $R(T)$ of T is closed and either $\alpha(T) < \infty$ or $\beta(T) < \infty$, then T is called a semi-Fredholm operator, and the index of T is defined by $i(T) = \alpha(T) - \beta(T)$. If both $\alpha(T) < \infty$ and $\beta(T) < \infty$, then T is called a Fredholm operator. An operator $T \in B(H)$, is called a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum of T is defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : (T-\lambda) \text{ is not a Weyl operator}\}.$$

We consider the sets

$$\Phi_+(H) = \{T \in B(H) : R(T) \text{ is closed and } \alpha(T) < \infty\},$$
$$\Phi_+^-(H) = \{T \in B(H) : R(T) \text{ is closed and } i(T) \leq 0\}.$$

We define

$$\sigma_{\Phi_+^-}(T) = \left\{\lambda \in \mathbb{C} : T-\lambda \notin \Phi_+^-(H)\right\},$$
$$\pi_{00}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda) < \infty\},$$
$$\pi_{00}^a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\}.$$

An operator T is said to satisfy Weyl's theorem if $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$ and T satisfies a-Weyl's theorem if $\sigma_a(T)\backslash\sigma_{\Phi_+^-}(T) = \pi_{00}^a(T)$.

An operator T is called B-Fredholm if there is an integer n for which $R(T^n)$ is closed and the operator $T_{[n]} : R(T^n) \longrightarrow R(T^n)$ defined by $T_{[n]}(x) = T(x)$ for every $x \in R(T^n)$ is a Fredholm operator (See [4, 5]). Similarly, an operator T is called a B-Weyl if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not a B-Weyl operator}\}.$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T)\backslash\sigma_{BW}(T) = E(T),$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues. We define $T \in SBF_+^-(H)$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $i(T_{[n]}) \le 0$. We define

$$\sigma_{SBF_+^-}(T) = \left\{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(H)\right\}.$$

We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \backslash \sigma_{SBF_+^-}(T) = E_a(T),$$

where $E_a(T)$ denote the set of all isolated points λ of the approximate point spectrum $\sigma_a(T)$ with $0 < \dim N(T - \lambda)$.

It is well known that if $T \in B(H)$, then generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem $\Rightarrow$ Weyl's theorem. Also, generalized a-Weyl's theorem $\Rightarrow$ generalized Weyl's theorem $\Rightarrow$ Weyl's theorem.

T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. Let $\mu \in \mathbb{C}$ be an isolated point of the spectrum of T. Then the Riesz idempotent E_μ of T with respect to μ is defined by

$$E_\mu = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} (z - T)^{-1} \mathrm{d}z,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. It is well known that the Riesz idempotent satisfies

$$E_\mu^2 = E_\mu,\ E_\mu T = T E_\mu,\ \sigma\left(T|_{R(E_\mu)}\right) = \{\mu\} \text{ and } N(T - \mu) \subseteq R(E_\mu).$$

An operator $T \in B(H)$ is said to be polaroid if $\operatorname{iso}\sigma(T) \subseteq \pi(T)$, where $\operatorname{iso}\sigma(T)$ denotes the isolated spectrum point of T and $\pi(T)$ denote the set of poles of the resolvent of T. Note that T is polaroid if and only if T^* is polaroid [2, Theorem 2.11].

An operator $T \in B(H)$ has Bishop's property (β) if for every open set $U \subset \mathbb{C}$ and every sequence of analytic functions $f_n : U \to H$, with the property that $(T - \lambda) f_n(\lambda) \to 0$ uniformly on every compact subset of U, it follows that $f_n \to 0$, again locally uniformly on U.

An operator $T \in B(H)$ is said to have the single valued extension property at λ_0 (abbreviated SVEP at λ_0), if for every open neighborhood U of λ_0, the only analytic function $f : U \to H$ which satisfies the equation $(T - \lambda) f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has this property at every $\lambda \in \mathbb{C}$.

Given operators $A, B \in B(H)$, let L_A and $R_B \in B(B(H))$ denote, respectively, the operators

$$L_A(X) = AX \ ,\ R_B(X) = XB$$

of left multiplication by A and right multiplication by B. The generalized derivation $\delta_{A,B} \in B(B(H))$, and the elementary operator $\Delta_{A,B} \in B(B(H))$ are then defined by

$$\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB,$$

$$\Delta_{A,B}(X) = (L_A R_B - I)(X) = AXB - X.$$

Let $d_{A,B}$ denote either of the operators $\delta_{A,B}$ and $\Delta_{A,B}$. The ranges and kernels of these operators have been studied intensively (see [6, 7, 8, 10, 11, 12, 13, 18, 21, 22, 25]).

An operator $T \in B(H)$ is said to be p-hyponormal, $0 < p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$, where $|T| = (T^*T)^{\frac{1}{2}}$. An invertible operator $T \in B(H)$ is log-hyponormal, if we have $\log |T^*|^2 \leq \log |T|^2$. For $s > 0$ and $t > 0$ a Hilbert space operator T belongs to class $w\mathcal{A}(s,t)$, if and only if

$$|T^*|^{2t} \leq (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}} \text{ and } |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}.$$

If A, B^* are p-hyponormal or log-hyponormal [11] or 2-isometric operators [25] and also if A, $B^* \in \omega\mathcal{A}(s,t)$ [22] such that $t + s = 1$, then for all complex λ we have $N(d_{A,B} - \lambda) \subseteq N(d_{A^*,B^*} - \bar{\lambda})$.

In this paper, we show that a (n,k)-quasi-$*$-paranormal operator is polaroid. Let A and B^* be (n,k)-quasi-$*$-paranormal operators, $H\left(\sigma(d_{f(A),g(B)})\right)$ denote the space of analytic functions on a neighborhood of $\sigma(d_{f(A),g(B)})$, and let $H_c(\sigma(T))$ be the set of $f \in H(\sigma(T))$ which are nonconstant on every connected component of $\sigma(T)$ the spectrum of the operator T. We prove that:

(1) $N(d_{A,B} - \lambda) \subseteq N(d_{A^*,B^*} - \overline{\lambda})$ for every non null complex number λ.
(2) $d_{A,B}$ satisfies the Fuglede-Putnam commutativity theorem $N(d_{A,B}) \subseteq N(d_{A^*,B^*})$ if $N(A) \subseteq N(A^*)$ and $N(B^*) \subseteq N(B)$.
(3) $f(d_{A,B})$ satisfies generalized Weyl's theorem and $f(d^*_{A,B})$ satisfies a-Weyl's theorem for each function f analytic on a neighborhood of $\sigma(d_{A,B})$.
(4) $h\left(d_{f(A),g(B)}\right)$ satisfies generalized Weyl's theorem and $h\left(d^*_{f(A),g(B)}\right)$ satisfies generalized a-Weyl's theorem for every $h \in H\left(\sigma(d_{f(A),g(B)})\right)$, where $f \in H_c(\sigma(A))$ and $g \in H_c(\sigma(B))$.

Also, we establish some other related results.

1. PRELIMINARIES

In this section we will give some results concerning the class of (n,k)-quasi-$*$-paranormal operators.

Lemma 1.1 ([16, Theorem 2.2],[27]). *For each positive integer n, the operator T is n-$*$-paranormal if and only if*

$$T^{*(1+n)}T^{1+n} - (1+n)\lambda^n TT^* + n\lambda^{1+n} \geq 0 \textit{ for any } \lambda > 0.$$

Proposition 1.2. *Let T be a (n,k)-quasi-$*$-paranormal operator. If T^k has dense range, then T is n-$*$-paranormal.*

Proof. Assume that T^k has dense range. Let $y \in H = \overline{T^k(H)}$. Then there exists a sequence $\{x_m\}_{m=1}^{\infty}$ in H such that $T^k(x_m) \to y$ as $m \to \infty$. Since T

is a (n, k)-quasi-$*$-paranormal operator, then

$$\left\langle T^{*k}\left(T^{*(1+n)}T^{1+n}-(1+n)\lambda^{n}TT^{*}+n\lambda^{1+n}\right)T^{k}x_{m},x_{m}\right\rangle\geq 0,$$

for all $\lambda > 0$. Hence

$$\left\langle\left(T^{*(1+n)}T^{1+n}-(1+n)\lambda^{n}TT^{*}+n\lambda^{1+n}\right)T^{k}x_{m},T^{k}x_{m}\right\rangle\geq 0,$$

for all $\lambda > 0$ and for all $m \in \mathbb{N}$. By the continuity of the inner product, we have

$$\left\langle\left(T^{*(1+n)}T^{1+n}-(1+n)\lambda^{n}TT^{*}+n\lambda^{1+n}\right)y,y\right\rangle\geq 0\,\text{for all}\,\lambda>0.$$

Therefore, T is an n-$*$-paranormal operator. □

Proposition 1.3 ([29]). *Let $T \in B(H)$ be an (n, k)-quasi-$*$-paranormal operator, such that T^k does not have dense range, and*

$$T=\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}\ \textit{on}\,H=\overline{R(T^{k})}\oplus N(T^{*k}).$$

Then $A = T|_{\overline{R(T^k)}}$ is n-$*$-paranormal, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proposition 1.4 ([29, Theorem 2.1]). *If T is a (n, k)-quasi-$*$-paranormal operator and M is an invariant subspace of T, then $T|_M$ is also a (n, k)-quasi-$*$-paranormal operator.*

Lemma 1.5 ([29, Theorem 4.1]). *Let T be a (n, k)-quasi-$*$-paranormal operator and $0 \neq \lambda \in \mathbb{C}$. Then $N(T - \lambda) \subseteq N(T^* - \overline{\lambda})$.*

Lemma 1.6 ([30, Lemma 2.4]). *Let T be a (n, k)-quasi-$*$-paranormal operator. If T is quasinilpotent, then T is a nilpotent operator.*

Lemma 1.7. *Let T be an (n, k)-quasi-$*$-paranormal operator with $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$, and $T^{k+1} = 0$ if $\lambda = 0$.*

Proof. We can consider two cases:

Case I: If $\lambda \neq 0$, the range of T^k is dense. It follows from Proposition 1.2 that T is an n-$*$-paranormal operator and the assertion holds by [23, Lemma 3.10].

Case II: If $\lambda = 0$, T^k does not have dense range. Hence by Proposition 1.3, we can write

$$T=\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}\ \text{on}\,H=\overline{R(T^{k})}\oplus N(T^{*k}).$$

From the assumption, $\sigma(T) = \{0\}$ and from Proposition 1.3 we have $\sigma(A) = \{0\}$. Since A is an n-$*$-paranormal operator, $A = 0$ and we have

$$T^{k+1}=\begin{pmatrix}0 & BC^{k}\\ 0 & C^{k+1}\end{pmatrix}=0.$$

□

2. MAIN RESULTS

Recall that the ascent of an operator T is defined as the smallest non-negative integer $p = p(T)$ such that $N(T^p) = N(T^{p+1})$, and the descent of T is defined as the smallest positive integer $q = q(T)$ such that $R(T^q) = R(T^{q+1})$. It is well known that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$. Moreover, if $\lambda \in \mathbb{C}$, the condition $0 < p(\lambda - T) = q(\lambda - T) < \infty$ is equivalent to saying that λ is a pole of the resolvent. In this case λ is an eigenvalue of T and an isolated point of the spectrum $\sigma(T)$.

Theorem 2.1. *Let T be an (n,k)-quasi-$*$-paranormal operator. Then T is polaroid.*

Proof. Let $\lambda \in \mathbb{C}$ be an isolated point of the spectrum of T. Then the Riesz idempotent E of T with respect to λ is defined by

$$E_\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} (z - T)^{-1} \mathrm{d}z,$$

where D is a closed disk centered at λ which contains no other points of the spectrum of T. Then T can be represented as follows

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ on } H = R(E_\lambda) \oplus N(E_\lambda),$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. According to Proposition 1.4, T_1 is (n,k)-quasi-$*$-paranormal. It follows from Lemma 1.6 that $T_1 - \lambda$ is nilpotent. Therefore, $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, it has finite ascent and descent. Therefore, $T - \lambda$ has finite ascent and descent, and λ is a pole of the resolvent of T. Thus $\operatorname{iso}\sigma(T) \subset \pi(T)$. So T is polaroid. $\square$

Corollary 2.2. *An (n,k)-quasi-$*$-paranormal operator is isoloid.*

Theorem 2.3. *Let T be a (n,k)-quasi-$*$-paranormal operator. Then T has Bishop's property (β).*

Proof. If the range of T^k is dense, then T is n-$*$-paranormal by Proposition 1.2. It follows from [9, Theorem 1] that T has Bishop's property (β). Suppose T^k does not have dense range. According to Proposition 1.3, we have

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Let U be an open subset of $\mathbb{C}$ and $f_n(z)$ be analytic functions on U to H such that $(T - z) f_n(z) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Put $f_n(z) = f_{n1}(z) \oplus f_{n2}(z)$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$. Then we have

$$\begin{pmatrix} A - z & B \\ 0 & C - z \end{pmatrix} \begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (A - z) f_{n1}(z) + B f_{n2}(z) \\ (C - z) f_{n2}(z) \end{pmatrix} \to 0.$$

Since C is nilpotent, C has Bishop's property (β). Hence $f_{n2}(z) \to 0$ uniformly on every compact subset of U. It follows that $(A-z)f_{n1}(z) \to 0$. Since A is n-$*$-paranormal, then A has Bishop's property (β). Hence $f_{n1}(z) \to 0$ uniformly on every compact subset of U. Thus T has Bishop's property (β). $\square$

Two important subspaces in local spectral theory are defined in the sequel. The quasinilpotent part of $T-\lambda$ is the subspace

$$H_0\left(T-\lambda\right) = \left\{x \in H : \lim_{n\to\infty} \left\|\left(T-\lambda\right)^n x\right\|^{\frac{1}{n}} = 0\right\}.$$

The analytic core of $T-\lambda$ is the subspace

$$\begin{aligned} K\left(T-\lambda\right) = \{&x \in H : \text{there is a sequence } \{x_n\} \subseteq H \text{ and } \gamma > 0 \text{ for which} \\ &x = x_0, \left(T-\lambda\right) x_{n+1} = x_n \text{ and } \|x_n\| \leq \gamma^n \|x\| \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

It is known that $H_0(T-\lambda)$ and $K(T-\lambda)$ are generally non-closed hyperinvariant subspaces of $T-\lambda$ such that $N(T-\lambda)^q \subseteq H_0(T-\lambda)$ for all $q \in \mathbb{N}$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$ [20]. Also if $\lambda \in \operatorname{iso}\sigma(T)$, then $K(T-\lambda)$ and $H_0(T-\lambda)$ are closed and $H = H_0(T-\lambda) \oplus K(T-\lambda)$ [1, Theorem 3.74]. Also the global spectral subspace is defined by

$$\chi_T(F) = \{x \in H \mid \exists \text{ analytic } f(z) : (T-\lambda)f(z) = x \text{ on } \mathbb{C} \setminus F\}.$$

THEOREM 2.4. *Let T be an (n,k)-quasi-$*$-paranormal operator. Then*

$$H_0(T-\lambda) = \begin{cases} N(T-\lambda), & \text{if } \lambda \neq 0 \\ N(T^{k+1}), & \text{if } \lambda = 0. \end{cases}$$

Proof. Since T has Bishop's property (β) by Theorem 2.3 and $H_0(T-\lambda) = \chi_T(\{\lambda\})$ by [1, Theorem 2.20]. It follows from [17, Proposition 1.2.19] that $H_0(T-\lambda)$ is closed and $\sigma(T|_{H_0(T-\lambda)}) \subset \{\lambda\}$. Let $S = T|_{H_0(T-\lambda)}$. Then S is an (n,k)-quasi-$*$-paranormal operator by Proposition 1.4. If $\sigma(S) = \emptyset$, then $H_0(T-\lambda) = \{0\}$, and so $N(T-\lambda) = \{0\}$. If $\sigma(S) = \{\lambda\}$ and $\lambda \neq 0$, then $S = \lambda$ by Lemma 1.7, and $H_0(T-\lambda) = N(S-\lambda) \subset N(T-\lambda)$. If $\sigma(S) = \{0\}$, applying Lemma 1.7, we get $S^{k+1} = 0$. Thus $H_0(T) = N(S^{k+1}) \subset N(T^{k+1})$. $\square$

The following theorem has been proved in [30, Theorem 2.1]. We give a new proof here.

THEOREM 2.5. *Let T be an (n,k)-quasi-$*$-paranormal operator. If μ is a non-zero isolated point of $\sigma(T)$ and E_μ is the Riesz idempotent of T with respect to μ, then E_μ is self-adjoint and*

$$R(E_\mu) = H_0(T-\mu) = N(T-\mu) = N(T^*-\overline{\mu}).$$

Moreover, if $\mu = 0$, then $R(E_\mu) = H_0(T) = N(T^{k+1})$.

Proof. If the range $R(T^k)$ is dense, then T is n-$*$-paranormal. So the result follows from [30, Theorem 2.1]. Assume T^k does not have dense range. Let

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}),$$

where A is n-$*$-paranormal, $C^k=0$ and $\sigma(T)=\sigma(A)\cup\{0\}$. If $0\neq\mu\in\operatorname{iso}\sigma(T)$, then $\mu\in\operatorname{iso}\sigma(A)$, because $\sigma(T)=\sigma(A)\cup\{0\}$. Then

$$\begin{aligned}E_\mu &= \frac{1}{2\pi\mathrm{i}}\int_{\partial D}\begin{pmatrix} z-A & -B\\ 0 & z-C\end{pmatrix}^{-1}\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\partial D}\begin{pmatrix} (z-A)^{-1} & (z-A)^{-1}B(z-C)^{-1}\\ 0 & (z-C)^{-1}\end{pmatrix}\mathrm{d}z.\end{aligned}$$

Let $E_\mu^A=\frac{1}{2\pi\mathrm{i}}\int_{\partial D}(z-A)^{-1}\mathrm{d}z$ be the Riesz idempotent of A for μ. Then E_μ^A is self-adjoint and $R(E_\mu^A)=N(A-\mu)=N(A-\mu)^*$. Let us show that $Z=\frac{1}{2\pi\mathrm{i}}\int_{\partial D}(z-A)^{-1}B(z-C)^{-1}\mathrm{d}z=0$. Since $(z-C)^{-1}=\sum_{j=1}^k\frac{C^{j-1}}{z^j}$, we have:

$$Z=\frac{1}{2\pi\mathrm{i}}\sum_{j=1}^k\int_{\partial D}(z-A)^{-1}B\frac{C^{j-1}}{z^j}\mathrm{d}z=\sum_{j=1}^k Z_j.$$

Since $\frac{1}{z}=\sum_{j=0}^\infty(-1)^j\frac{(z-\mu)^j}{\mu^{j+1}}$, we have:

$$\begin{aligned}Z_1 &= \frac{1}{2\pi\mathrm{i}}\int_{\partial D}(z-A)^{-1}B\frac{1}{z}\mathrm{d}z\\ &= \sum_{j=0}^\infty(-1)^j\frac{1}{2\pi\mathrm{i}}\int_{\partial D}(z-A)^{-1}B\frac{(z-\mu)^j}{\mu^{j+1}}\mathrm{d}z\\ &= \sum_{j=0}^\infty\frac{(-1)^j}{\mu^{j+1}}(A-\mu)^jE_\mu^AB.\end{aligned}$$

Let $y=E_\mu^Ax$ for $x\in\overline{R(T)}$. Then $y\in N(A-\mu)=N(A-\mu)^*$. Hence

$$0=(T-\mu)^*\begin{pmatrix} y\\ 0\end{pmatrix}=\begin{pmatrix}(A-\mu)^*y\\ B^*y\end{pmatrix}.$$

Thus $B^*y=B^*E_\mu^Ax=0$ for $x\in\overline{R(T)}$. This implies that $E_\mu^AB=0$ because E_μ^A is self-adjoint. Hence $Z_1=0$. Since

$$\frac{1}{z^2}=\sum_{j=0}^\infty(-1)^j\frac{(j+1)(z-\mu)^j}{\mu^{j+2}},$$

we have:

$$\begin{aligned}Z_2 &= \frac{1}{2\pi\mathrm{i}}\int_{\partial D}(z-A)^{-1}B\frac{C}{z^2}\mathrm{d}z\\ &= \sum_{j=0}^\infty(-1)^j\frac{1}{2\pi\mathrm{i}}\int_{\partial D}(z-A)^{-1}BC\frac{(j+1)(z-\mu)^j}{\mu^{j+2}}\mathrm{d}z\\ &= \sum_{j=0}^\infty\frac{(-1)^j(j+1)}{\mu^{j+2}}(A-\mu)^jE_\mu^ABC=0.\end{aligned}$$

By induction, we get $Z_1 = Z_2 = \cdots = Z_{k-1} = 0$, and $Z = 0$. This implies that $E_\mu = \begin{pmatrix} E_\mu^A & 0 \\ 0 & 0 \end{pmatrix}$ is self-adjoint as well as E_μ^A. Also we have

$$R(E_\mu) = H_0(T-\mu) = N(T-\mu) \subset N(T-\mu)^*,$$

by Theorem 2.4 and Lemma 1.5. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N(T-\mu)^*$. Then

$$0 = N(T-\mu)^* x = \begin{pmatrix} (A-\mu)^* x_1 \\ B^* x_1 + (C-\mu)^* x_2 \end{pmatrix}.$$

Therefore $x_1 \in N(A-\mu)^* = N(A-\mu) \subset N(T-\mu)$. Then $(T-\mu)\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which implies

$$(T-\mu)^* \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (A-\mu)^* x_1 \\ B^* x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence we have $B^* x_1 = 0$. Then $(C-\mu)^* x_2 = 0$ and $x_2 = 0$, because C is nilpotent. Thus $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \in N(A-\mu) \subset N(T-\mu)$. It follows that $N(T-\mu)^* \subset N(T-\mu)$.

In the case $\mu = 0$: since $N(T^{k+1}) \subset R(E_\mu)$, we have to show that $R(E_\mu) \subset N(T^{k+1})$. It is known that $R(E_\mu)$ is an invariant subspace of T and we have $\sigma\left(T|_{R(E_\mu)}\right) = \{0\}$. It follows from Theorem 2.4, that $(T|_{R(E_\mu)})^{k+1} = T^{k+1}|_{R(E_\mu)} = 0$. This implies $R(E_\mu) \subset N(T^{k+1})$. □

Theorem 2.6. *If A, B^* are (n,k)-quasi-$*$-paranormal operators, then*

$$N(d_{A,B} - \lambda) \subseteq N(d_{A,B}^* - \overline{\lambda}),$$

for all non null complex number λ.

Proof. We have from Theorem 2.1 that (n,k)-quasi-$*$-paranormal operators are polaroid and from Theorem 2.3 that (n,k)-quasi-$*$-paranormal operators have property (β). Since by Lemma 1.5, A and B^* are reduced by each of its eigenspaces, then the result follows from [19], and Theorems 2.2 and 2.3. □

Theorem 2.7. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators such that $N(A) \subseteq N(A^*)$ and $N(B^*) \subseteq N(B)$, then $N(d_{A,B}) \subseteq N(d_{A^*,B^*})$.*

Proof. The conditions $N(A) \subseteq N(A^*)$ and $N(B^*) \subseteq N(B)$ imply that $N(A)$ reduces A and $N(B^*)$ reduces B^*(i.e., 0 is normal eigenvalue of both A and B^*). It follows from [19], and Theorem 2.2 and 2.3, that $N(d_{A,B}) \subseteq N(d_{A^*,B^*})$. □

Theorem 2.8. *If A, B^* are (n,k)-quasi-$*$-paranormal operators, then*

$$H_0(d_{A,B} - \lambda) = N(d_{A,B} - \lambda) \textit{ for all non zero } \lambda \in \operatorname{iso}\sigma(d_{A,B}).$$

Proof. The proof is similar to that given in the proof of [10, Lemma 3.2] and [25, Theorem 2.10]. □

Lemma 2.9. *If A, B^* are (n,k)-quasi-$*$-paranormal operators, then $R(d_{A,B}-\lambda)$ is closed for all $\lambda \in \operatorname{iso}\sigma(d_{A,B})$.*

Proof. Let $\lambda \in \operatorname{iso}\sigma(d_{A,B})$. By the previous theorem, $H_0(d_{A,B}-\lambda) = N(d_{A,B}-\lambda)$. Then

$$B(H) = H_0(d_{A,B}-\lambda) \oplus K(d_{A,B}-\lambda) = N(d_{A,B}-\lambda) \oplus K(d_{A,B}-\lambda).$$

Hence

$$R(d_{A,B}-\lambda) = 0 \oplus (d_{A,B}-\lambda)(K(d_{A,B}-\lambda)) = K(d_{A,B}-\lambda)$$

and $B(H) = N(d_{A,B}-\lambda) \oplus R(d_{A,B}-\lambda)$. □

Lemma 2.10. *If A, B^* are (n,k)-quasi-$*$-paranormal operators. Then $d_{A,B}$ has SVEP.*

Proof. It follows from [32, Lemma 4.3] that $d_{A,B}$ has SVEP. □

Proposition 2.11. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. If $\lambda \in \operatorname{iso}\sigma(d_{A,B})$. Then $H_0(d_{A,B}) = E_\lambda H$, where E_λ denotes the Riesz idempotent for λ.*

Proof. It follows from Lemma 2.10 that $d_{A,B}$ has SVEP. We get the equality from [24, p. 424]. □

Lemma 2.12. *If A, B^* are (n,k)-quasi-$*$-paranormal operators. Then $d_{A,B}$ is polaroid.*

Proof. Since A, B are polaroid operators, it follows from [18, Lemma 2.2] that $d_{A,B}$ is polaroid. □

Corollary 2.13. *If A, B^* are (n,k)-quasi-$*$-paranormal operators. Then $d_{A,B}$ is isoloid and $d^*_{A,B}$ is a-isoloid.*

Proof. Recall that a bounded linear operator $T \in B\,(B(H))$ is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Let λ be an isolated point of $\sigma_a(d^*_{A,B})$. Since $d_{A,B}$ has SVEP by Lemma 2.10, it follows from [14], Corollary 7, that $\sigma_a(d^*_{A,B}) = \sigma(d^*_{A,B})$. Then λ is an isolated point of $\sigma(d^*_{A,B})$. Since $d_{A,B}$ is polaroid implies $d^*_{A,B}$ is polaroid, $d^*_{A,B}$ is isoloid. Hence λ is an eigenvalue of $d^*_{A,B}$. Thus $d^*_{A,B}$ is a-isoloid. □

Proposition 2.14. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. Then generalized Weyl's theorem holds for $d_{A,B}$ and generalized a-Weyl's theorem holds for $d^*_{A,B}$.*

Proof. Since A and B^* are (n,k)-quasi-$*$-paranormal operators, $d_{A,B}$ has $SVEP$. By Lemma 2.12, the operator $d_{A,B}$ is polaroid. Then by applying [2, Theorem 3.10], it follows that generalized Weyl's theorem holds for $d_{A,B}$ and generalized a-Weyl's theorem holds for $d^*_{A,B}$. □

Proposition 2.15. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. Then $f(d_{A,B})$ satisfies generalized Weyl's theorem and $f(d^*_{A,B})$ satisfies generalized a-Weyl's theorem for every $f \in H(\sigma(d_{A,B}))$.*

Proof. Since A and B^* are (n,k)-quasi-$*$-paranormal operators, $d_{A,B}$ has $SVEP$ by Lemma 2.10. According to Lemma 2.12, the operator $d_{A,B}$ is polaroid and so $d_{A,B}$ is isoloid. Then by applying [31, Theorem 2.2], it follows that generalized Weyl's theorem holds for $f(d_{A,B})$ for every $f \in H(\sigma(d_{A,B}))$. To prove that $f(d^*_{A,B})$ satisfies generalized a-Weyl's theorem, we have $d_{A,B}$ has SVEP, $d^*_{A,B}$ is a-isoloid and generalized a-Weyl's theorem holds for $d^*_{A,B}$. Then the result follows from [31, Theorem 2.4]. $\square$

Proposition 2.16. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. Then $d_{f(A),g(B)}$ has SVEP for $f \in H_c(\sigma(A))$ and $g \in H_c(\sigma(B))$.*

Proof. By [17, Lemma 3.3.6], if a Banach space operator T satisfies property (β), then so does $h(T)$ for every $h \in H_c(\sigma(T))$. So by [32, Lemma 4.9], $d_{f(A),g(B)}$ has SVEP. $\square$

Proposition 2.17. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. Then $d_{f(A),g(B)}$ is polaroid for $f \in H_c(\sigma(A))$ and $g \in H_c(\sigma(B))$.*

Proof. Since A, B^* are (n,k)-quasi-$*$-paranormal operators, A and B are polaroid. Then by applying [2, Lemma 3.11], we obtain that $f(A)$ and $g(B)$ are polaroid. It follows from [18], Lemma 2.2, that $d_{f(A),g(B)}$ is polaroid for $f \in H_c(\sigma(A))$ and $g \in H_c(\sigma(B))$. $\square$

Corollary 2.18. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. Then $d^*_{f(A),g(B)}$ is a-isoloid.*

Proposition 2.19. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. Then $d_{f(A),g(B)}$ satisfies generalized Weyl's theorem and $d^*_{f(A),g(B)}$ satisfies generalized a-Weyl's theorem for every $f \in H_c(\sigma(A))$ and $g \in H_c(\sigma(B))$.*

Proof. The proof is is similar to that given in the proof of Proposition 2.14. $\square$

Proposition 2.20. *Let A, B^* be (n,k)-quasi-$*$-paranormal operators. Then $h\left(d_{f(A),g(B)}\right)$ satisfies generalized Weyl's theorem and $h\left(d^*_{f(A),g(B)}\right)$ satisfies generalized a-Weyl's theorem for every $h \in H\left(\sigma(d_{f(A)g(B)})\right)$, $f \in H_c(\sigma(A))$ and $g \in H_c(\sigma(B))$.*

Proof. The proof is similar to that given in the proof of Proposition 2.15. $\square$

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