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k-DOMINATOR COLORING OF GRAPHS

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Abstract. Coloring and domination in graphs are two well explored areas of research in graph theory. Blending these notions, the dominator coloring of graphs was introduced in the literature; following which several variants of domination related coloring patterns have been defined and studied. In this article, we investigate the k-dominator coloring of graphs, a generalisation of the concept of dominator coloring of graphs. Here, we discuss certain properties of the k-dominator coloring of graphs based on the structure of the graphs, and the k-dominator chromatic number of certain classes of graphs such as paths, cycles, complete multi-partite graphs, etc. are determined. Also, the structure of trees whose 2-dominator coloring is the trivial proper coloring are characterised in the study.

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1. INTRODUCTION

For terminology in graph theory, refer to [9], and for concepts pertaining to the theory of domination in graphs, see [6].

By G, we always mean a simple, undirected and a finite graph with its vertex set $V(G)$ and edge set $E(G)$. The *open neighbourhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and for each vertex $v \in V(G)$, the set $N[v] = N(v) \cup \{v\}$ is called its *closed neighbourhood.* The *degree* of a vertex $v \in V(G)$ is given by $d(v) = |N(v)|$ and the minimum degree $\delta(G)$ of the graph G is $\delta(G) = \min\limits_{v\in V(G)}\{d(v)\}$. A vertex $v \in V(G)$ for which $N[v] = V(G)$ is called a *universal vertex* of the graph G and a vertex $v \in V(G)$ having $N[v] = \{v\}$ is called an *isolated vertex.* A vertex $v \in V(G)$ with $|N(v)| = 1$ is called a *leaf* or a *pendant vertex* in G and the vertex $u \in N(v)$, where v is a leaf, is called its support or *support vertex* in G. A subset $S \subseteq V(G)$ is called an *independent* set of G if for every pair $u, v \in S$, $uv \notin E(G)$.

Graph coloring can be viewed as partitioning the entities of a graph such as its vertices or edges, according to certain pre-defined rules, where each partite set of the partition is called a *color class.* A *proper vertex coloring* of a graph G is partitioning its vertex set $V(G)$ into independent sets and the minimum

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number of partite sets in a proper vertex coloring of G is called the *chromatic number* of G, denoted by $\chi(G)$. Any chromatic partition of $V(G)$ with $\chi(G)$ partite sets is called a *χ-coloring* of G. Note that by a *trivial coloring* of G, we mean the partition of $V(G)$ into color classes that are singletons.

Domination in graphs can be seen as the process of selecting the graph entities; usually vertices, such that an entity of the graph is either selected or is related to the selected entities. In a graph G, if a vertex $v \in V(G)$ is adjacent to all vertices $u \in A$, for some $A \subseteq V(G)$ or $A = \{v\}$, we say that *v dominates* A and A is *dominated* by v. By convention, a vertex v always dominates itself (ref. [7]). A *dominating set* D of a graph G is a set of vertices such that $N[D] = V(G)$ and the minimum cardinality of a dominating set of G is the *domination number* of G, denoted by $\gamma(G)$.

Graph coloring and domination in graphs are two well-known research areas in graph theory, owing to their applications. As the applications of these areas similar in nature and coincide in many aspects, the notion of dominator coloring of graphs was introduced in [4], by blending the concepts of coloring and domination in graphs as follows.

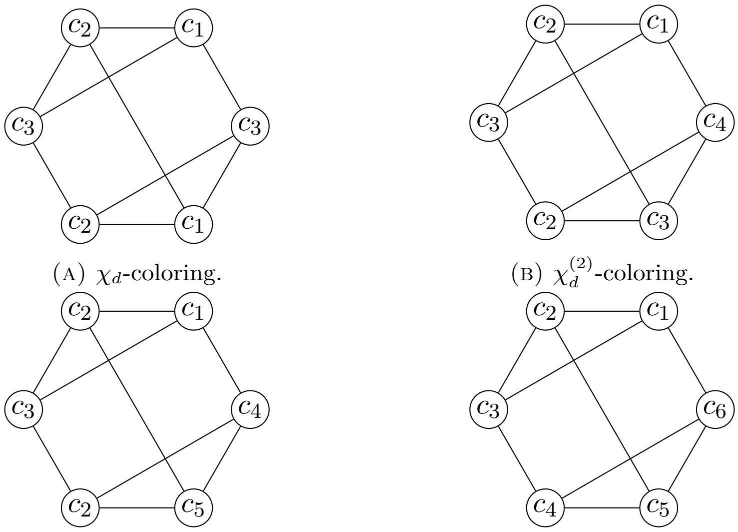
A *dominator coloring* of a graph G is a proper vertex coloring of G in which every vertex $v \in V(G)$ dominates at least one color class. The minimum number of colors used to obtain a dominator coloring of G is called the *dominator chromatic number* of G and it is denoted by $\chi_d(G)$.

Following the introduction of the concept of dominator coloring of graphs, many variants of these dominator colorings of graphs are introduced in the literature, infusing different types of domination and coloring schemes of graphs (*c.f.* [7]). One such variant, that was introduced in [8], is the *k-dominator coloring* of graphs, which is defined as follows.

Definition 1.1. [8] A *k-dominator coloring* of a graph G is a proper vertex coloring of G in which every vertex $v \in V(G)$ dominates at least k color classes and the minimum number of colors used to obtain a k-dominator coloring of G is called the *k-dominator chromatic number* of G, denoted by $\chi_d^{(k)}(G)$.

This concept of k-dominator coloring of graphs can be viewed as a generalisation of the dominator coloring of graphs because when $k = 1$, the k-dominator coloring of a graph reduces to a dominator coloring and for our understanding, we can say that when $k = 0$, it reduces to just a proper coloring of the graph. Therefore, the interest of studying the k-dominator coloring of graphs lies in examining the coloring scheme when $k \geq 2$. An illustration of k-dominator coloring of a graph, for different values of k is given in Figure 1.1, where it can be seen that the graph G in Figure 1.1 has $\chi(G) = \chi_d(G) = 3$, $\chi_d^{(2)}(G) = 4$, $\chi_d^{(3)}(G) = 5$, and $\chi_d^{(4)}(G) = 6$.

On introducing the notion of k-dominator coloring of graphs in [3], certain preliminary results on the 2-dominator coloring of graphs were obtained in [3]; following which the 2-dominator coloring of certain graphs were discussed in



(A) χ_d-coloring. (B) $\chi_d^{(2)}$-coloring.

(C) $\chi_d^{(3)}$-coloring. (D) $\chi_d^{(4)}$-coloring.

FIG. 1.1. An example of k-dominator coloring(s) of a graph, for $1 \leq k \leq 4$.

[1] and [2], in which we observed that the values of the 2-dominator chromatic numbers determined for some of these graphs are inaccurate. These research gaps in the literature motivate us to study the k-dominator coloring of graphs and hence in this article, we obtain certain properties of the the k-dominator coloring of graphs, and analyse the coloring pattern of the k-dominator coloring of graphs, for different admissible values of k, for certain standard graph classes.

2. k-DOMINATOR COLORING OF GRAPHS

In this section, we discuss certain general properties of the k-dominator coloring of graphs, and determine the k-dominator coloring of some graphs, for all admissible values of k. By Definition 1.1, we can see that a graph G can admit a k-dominator coloring only for $1 \leq k \leq \delta(G)+1$, as any vertex $v \in V(G)$ with the minimum degree $\delta(G)$ of G can at most dominate $\delta(G)+1$ color classes. Therefore, we have the following proposition.

PROPOSITION 2.1. *All graphs are k-dominator colorable for any k such that $1 \leq k \leq \delta(G)+1$.*

Based on Proposition 2.1, we can deduce that for a graph G to have any k-dominator coloring, $k \geq 2$, it must not have any isolated vertices and hence we consider only such graphs for further discussions. In the following result, we obtain the relation between the k-dominator chromatic numbers of a graph G, for $1 \leq k \leq \delta(G)+1$.

Proposition 2.2. *For any graph G of order n, $\chi(G) \leq \chi_d(G) \leq \chi_d^{(2)}(G) \leq \ldots \leq \chi_d^{(\delta(G)+1)}(G) \leq n$.*

Proof. Let $1 \leq k \leq \delta(G)+1$ and consider a 0-dominator coloring of the graph G in this context as the χ-coloring of G. The proof is immediate from the fact that every k-dominator coloring of a graph G is a $(k-1)$-dominator coloring, but it need not be an optimal one; whereas, every $(k-1)$-dominator coloring of a graph G need not be a k-dominator coloring of G. □

The following results discuss the structures of graphs for which a k-dominator coloring of a graph G becomes a trivial coloring such that the equality $\chi_d^{(\delta(G)+1)}(G) = n$ holds.

Proposition 2.3. *For a graph G in which the vertices of degree $\delta(G)$ forms a dominating set, $\chi_d^{(\delta(G)+1)}(G) = n$.*

Proof. In a $(\delta(G)+1)$-dominator coloring of any graph G with the minimum degree $\delta(G)$, the closed neighbourhood of every vertex of G having the degree $\delta(G)$ is assigned a unique color. If G has a dominating set $S \subseteq V(G)$, where $S = \{v \in V(G) : d(v) = \delta(G)\}$, then by the definition of a dominating set, $N[S] = V(G)$ and hence every vertex $v \in V(G)$ is assigned a unique color. □

The converse of Proposition 2.3 is not true, as the graph G in Figure 2.2 in which the vertices having minimum degree does not form a dominating set; yet the $(\delta(G)+1)$-dominator coloring of G is the trivial coloring.

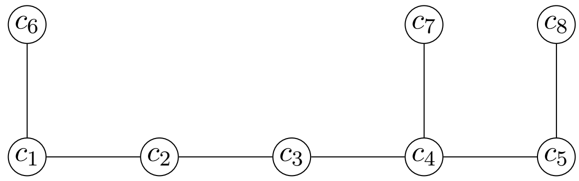


Fig. 2.2. An example of a graph for which the converse of Proposition 2.3 does not hold.

Corollary 2.4. *For a k-regular graph G of order n, $\chi_d^{(k+1)}(G) = n$.*

Theorem 2.5. *If G is a graph with a universal vertex v and $G' = G - \{v\}$, then for any $1 \leq k \leq \delta(G)+1$, any $\chi_d^{(k-1)}$-coloring of G' is a $\chi_d^{(k)}$-coloring of G and $\chi_d^{(k)}(G) = \chi_d^{(k-1)}(G') + 1$.*

Proof. In any proper coloring c of G, the universal vertex v is always assigned a unique color and it dominates all the color classes of G with respect to c. If every vertex in G' already dominates at least k color classes in its $\chi_d^{(k-1)}$-coloring, we are done. Otherwise, there exists at least one vertex, say u, in G' that dominates exactly $k-1$ color classes in its $\chi_d^{(k-1)}$-coloring. In this case, the vertex u dominates k color classes in G, as the color class $\{v\}$ is also dominated by u and hence the result. $\square$

The following corollary is obtained as a generalisation of the above result by extending it to graphs having r universal vertices.

Corollary 2.6. *If G is a graph with r universal vertices, then $\chi_d^{(k)}(G) = \chi(G)$, for all $1 \leq k \leq r$.*

Based on the above discussions, the k-dominator coloring of certain graphs are determined in the following results. These results depict both static and dynamic nature of k-dominator chromatic number, as the value k changes, authenticating Proposition 2.2.

Corollary 2.7. *A graph G of order n has $\chi_d^{(k)}(G) = n$, for all $1 \leq k \leq \delta(G) + 1$, if and only if $G = K_n$.*

Theorem 2.8. *For $a \geq 2$, $b \geq 3$ and $b \geq a$,*

$$\chi_d^{(k)}(K_{a,b}) = \begin{cases} 2k, & 1 \leq k \leq a; \\ a+b, & k = a+1. \end{cases}$$

Proof. For the complete bipartite graph $K_{a,b}$; $a \geq 2$, $b \geq 3$, $b \geq a$, let $V(K_{a,b}) = U \cup V$, where $|U| = a$ and $|V| = b$. If a vertex $v \in V$ (resp. $u \in U$) dominates k color classes that are all subsets of U (resp. V), then k colors, say $c_1, c_2, \ldots, c_k$, must be assigned to the vertices of V and k colors, say $c_{k+1}, c_{k+2}, \ldots, c_{2k}$, must be assigned to the vertices of U, yielding $\chi_d^{(k)}(K_{a,b}) \leq 2k$; $1 \leq k \leq a$.

The other possibility for a vertex $u \in U$ (resp. $v \in V$) to dominate k color classes is when if it is assigned a unique color and the remaining $k-1$ colors are assigned to the vertices of V (resp. U). In this manner, if every vertex $u \in U$ has to dominate its own color class, we need at least $a + k - 1$ colors in the k-dominator coloring of the complete bipartite graph $K_{a,b}$. If this assignment is done by assigning unique colors to the vertices of V and $(k-1)$ colors to all the vertices of U, we require $b + k - 1$ colors such a k-dominator coloring of $K_{a,b}$, which is an optimal coloring if and only if $k = a + 1$. Therefore, $\chi_d^{(a+1)}(K_{a,b}) = a + b$.

Here, if we assign unique colors to $k-1$ vertices of V and $k-1$ vertices of U, all these $2k-2$ vertices of $K_{a,b}$ will dominate k color classes; whereas, the remaining $a-(k-1)$ and $b-(k-1)$ vertices of U and V should be colored with two additional colors, one assigned to the vertices of each partite; in which case also we require $2k$ colors in the k-dominator coloring of $K_{a,b}$. Therefore, we obtain $\chi_d^{(k)}(K_{a,b}) = 2k$; $1 \leq k \leq a$. $\square$

Let $K_{t,t}$ be a complete bipartite graph with $V(K_{t,t}) = V \cup U$ such that $V = \{v_i : 1 \leq i \leq t\}$ and $U = \{u_i : 1 \leq i \leq t\}$. A *crown graph* Cr_t is obtained by removing the edges v_iu_i, for all $1 \leq i \leq t$, from the graph $K_{t,t}$.

THEOREM 2.9. *For $t \geq 3$, $\chi_d^{(k)}(Cr_t) = 2(k+1)$, for all $1 \leq k \leq t$.*

Proof. Consider a coloring $c : V(Cr_t) \to \{c_i : 1 \leq i \leq 2(k+1)\}$, for any $1 \leq k \leq t$, of crown graph with $V(Cr_t) = V \cup U$, where $V = \{v_i : 1 \leq i \leq t\} \cup U = \{u_i : 1 \leq i \leq t\}$, such that $c(v_i) = c_i$, $c(u_i) = c_{k+1+i}$; $1 \leq i \leq k$, and $c(v_i) = c_{k+1}$, $c(u_i) = c_{2k+2}$; $k+1 \leq i \leq t$. In this coloring c of Cr_t with $2k+2$ colors, for any $1 \leq k \leq t$, every vertex $v_i \in V$, for all $1 \leq i \leq k$, dominates the color classes $\{v_i\}$, $\{u_i\}$, for $1 \leq i \neq j \leq k$ and $\{u_i : k+1 \leq i \leq t\}$ and $v_i \in V$, for all $k+1 \leq i \leq t$, dominates the k-color classes $\{u_i\}$; $1 \leq i \neq j \leq k$. Similarly, each vertex $u_i \in U$, for all $1 \leq i \leq k$, dominates the color classes $\{u_i\}$, $\{v_i\}$, for $1 \leq i \neq j \leq k$ and $\{v_i : k+1 \leq i \leq t\}$ and $u_i \in U$, for all $k+1 \leq i \leq t$, dominates the k-color classes $\{v_i\}$; $1 \leq i \neq j \leq k$. Therefore, c is a k-dominator coloring of the graph Cr_t and $\chi_d^{(k)}(Cr_t) \leq 2(k+1)$; $1 \leq k \leq t$, and $t \geq 3$.

If possible, assume that $\chi_d^{(k)}(Cr_t) < 2(k+1)$. Let $c' : V(Cr_t) \to \{c_i : 1 \leq i \leq 2k+1\}$, for any $1 \leq k \leq t$, be a k-dominator coloring of Cr_t with $2k+1$ colors. As any color in a proper coloring of a crown graph can either be assigned to two vertices v_i and u_i, for some $1 \leq i \leq t$, or to t vertices of the same partite set; in a k-dominator coloring of Cr_t, there are two possibilities for a vertex $v_i \in V$ or $u_i \in U$ to dominate k-color classes, based on which we have the following cases.

Case 1: A vertex $v_i \in V$ (resp. $u_i \in U$) can dominate the k-color classes of the colors assigned to the vertices of $U - \{u_i\}$ (resp. $V - \{v_i\}$), for each $1 \leq i \leq t$. Therefore, in this case, we must assign at least $k+1$ colors to the vertices of each partite set, leading to a contradiction.

Case 2: A vertex $v_i \in V$ (resp. $u_i \in U$) can dominate its own color class and $k-1$ color classes of the colors assigned to the vertices of U (resp. V). Therefore, in view of this, the vertices of each partite set must be assigned at least $k-1$ colors and this yields two ways to partition the $2k+1$ colors used in the $\chi_d^{(k)}$-coloring c' of Cr_t which are considered in the following cases. Note that as the properties of both the partite sets are similar and the exchange of these colors also yield the same result, without loss of generality, we consider only two subcases.

Subcase 2a: Here, $k-1$ colors are assigned to the vertices of V and the remaining $k+2$ colors are assigned to the vertices of U. In this assignment, each vertex $v_i \in V$; $1 \leq i \leq t$, dominates at least $k+1$ color classes; whereas, every vertex $u_i \in U$ can only dominate at most $k-1$ color classes as each $v_i \in V$ cannot dominate the color class to which the vertex u_i belongs to and vice versa. Hence, this assignment of $2k+1$ colors such that $k-1$ and $k+2$ colors are assigned to the partite sets is not possible.

Subcase 2b: In this case, k colors are assigned to the vertices of V and $k+1$ colors are assigned to the vertices of U. Here, the vertices of V dominate at least k color classes; whereas, the vertices of U dominate k color classes if and only if $t = k+1$. If $t \neq k+1$, the only possibility is $t > k+1$ as $t < k+1$

yields a contradiction to the hypothesis. If $t > k+1$, then there exists at least two vertices $u_i, u_j \in U$; $1 \leq i \neq j \leq t$, which are assigned the same color and dominates only $k-1$ color classes; yielding a contradiction to our assumption. Therefore, for $t \geq 3$, $\chi_d^{(k)}(Cr_t) = 2(k+1)$; $1 \leq k \leq t$. □

THEOREM 2.10. *The k-dominator chromatic number of a complete multi-partite graph $K_{s_1,s_2,\dots,s_t}$; $s_i \geq 3$, for all $1 \leq i \leq t$,*

$$\chi_d^{(k)}(K_{s_1,s_2,\dots,s_t}) = \begin{cases} t, & 1 \leq k \leq t-1; \\ t+2k, & t+1 \leq k \leq \delta(K_{s_1,s_2,\dots,s_t})+1. \end{cases}$$

Proof. Consider a complete multi-partite graph $K_{s_1,s_2,\dots,s_t}$; $s_i \geq 3$, such that $s_i \leq s_j$, when $1 \leq i < j \leq t$, and $V(K_{s_1,s_2,\dots,s_t}) = \bigcup\limits_{i=1}^{t} V_{s_i}$, where $|V_i| = s_i$; $1 \leq i \leq t$. As each vertex of every partite set in a complete t-partite graph is adjacent to all the vertices of the $t-1$ other partite sets, $\chi_d^{(k)}(K_{s_1,s_2,\dots,s_t}) = t$, for any $1 \leq k \leq t-1$.

When $k = t$, we can observe that $\chi_d^{(t)}(K_{s_1,s_2,\dots,s_t}) \geq t+1$. Assume that $\chi_d^{(t)}(K_{s_1,s_2,\dots,s_t}) = t+1$. This implies that exactly one V_i, for some $1 \leq i \leq t$, is again partitioned into two color classes. In this coloring, all the vertices in $V(K_{s_1,s_2,\dots,s_t}) - V_i$ dominate t color classes, but it can be observed that there exists at least one vertex in the V_i which is partitioned into two color classes that dominates only $t-1$ color classes; yielding a contradiction. Therefore, $\chi_d^{(t)}(K_{s_1,s_2,\dots,s_t}) \geq t+2$. From a χ-coloring $c : V(K_{s_1,s_2,\dots,s_t}) \to \{c_i : 1 \leq i \leq t\}$ such that $c(v) = i$, for $v \in V_i$; $1 \leq i \leq t$, color exactly one vertex in V_{s_t} and $V_{s_{t-1}}$ with the colors c_{t+1} and c_{t+2}, respectively. This gives a t-dominator coloring of $K_{s_1,s_2,\dots,s_t}$, as every vertex in V_{s_t} and $V_{s_{t-1}}$ dominates at least t color classes. Therefore, $\chi_d^{(t)}(K_{s_1,s_2,\dots,s_t}) = t+2$. Here, it can be seen that for every vertex in the graph $K_{s_1,s_2,\dots,s_t}$ to dominate one additional color class, we require two additional unique colors, according to the structure of the complete multi-partite graph and based on this, the result can be established in a similar fashion for all k-dominator colorings of the graph $K_{s_1,s_2,\dots,s_t}$, where $t+1 \leq k \leq \delta(K_{s_1,s_2,\dots,s_t})+1$. □

Following the investigation of the k-dominator coloring of the standard graph classes like complete bipartite and complete multi-partite graphs, we proceed to discuss the k-dominator coloring of paths and cycles in the following results. As we know that paths have pendant vertices, it admits only k-dominator coloring for $k \leq 2$, and this 2-dominator coloring pattern of paths is given in the following theorem.

THEOREM 2.11. *For a path P_n,*

$$\chi_d^{(2)}(P_n) = \begin{cases} n, & 1 \leq n \leq 6; \\ 6, & n = 7; \\ n - \lfloor \frac{n}{5} \rfloor, & n \geq 8 \ \textit{and} \ n \equiv 3, 4 \pmod 5; \\ n - \lfloor \frac{n}{5} \rfloor - 1, & n \geq 8; \ \textit{and} \ n \equiv 0, 1, 2 \pmod 5). \end{cases}$$

Proof. For any path $P_n = v_1 - v_2 - \ldots - v_n$ of order n, the pendant vertices v_1, v_n and the support vertices v_2, v_{n-1} must be assigned unique colors in any of its 2-dominator coloring, so that the pendant vertices v_1 and v_n dominate two color classes. Note that here the support vertices also dominate two color classes. Also, in any 2-dominator coloring of P_n, any color class can have cardinality either 1 or 2 and hence to obtain any optimal 2-dominator coloring, we increase the number of colors that can be assigned to two vertices. As this can be done only to the graph $P'_n = P_n - \{v_1, v_2, v_{n-1}, v_n\}$, we focus on discussing the 2-dominator coloring pattern of the same.

In this regard, $\chi_d^{(2)}(P_n) = n$; $1 \leq n \leq 4$, as every vertex of P_n in this case is either a pendant vertex or a support vertex. When $n = 5, 6$, only the vertices v_3 and v_4 are neither support nor the pendant vertices and remains to be colored. Since they are adjacent to the support vertices, they must receive unique colors and we get $\chi_d^{(2)}(P_n) = n$, when $n = 5, 6$. When $n = 7$, we have a $P'_7 = v_3 - v_4 - v_5$ induced by the vertices which are neither a support nor a leaf of P_7. The vertices v_3 and v_5 are assigned the same colors and v_4 is assigned a unique color. It can be seen that this coloring is a 2-dominator coloring of P_7, where the vertices v_3 and v_5 dominate the color classes $\{v_2\}, \{v_4\}$ and $\{v_4\}, \{v_6\}$, respectively. Also, the vertex v_4 dominates it own color class and the color class $\{v_3, v_5\}$. The optimality of this coloring is also ensured as we need at least two colors to properly color any P_3. Therefore, $\chi_d^{(2)}(P_7) = 6$.

Consider the paths P_n; $n \geq 8$. If a color class has to be assigned to two alternate vertices, say v_i and v_{i+2}, then, the vertices v_{i-2}, v_{i-1}, v_i and v_{i+3}, v_{i+4} must be assigned unique colors so that the vertices v_j; $i - 2 \leq j \leq i + 4$; for some $3 \leq i \leq n - 2$, dominate two color classes. That is, v_i dominates the color classes $\{v_{i+1}\}$ and $\{v_{i-1}\}$, v_{i+2} dominates the color classes $\{v_{i+1}\}$ and $\{v_{i+3}\}$, as these vertices cannot dominate its own color class. Also, the vertex v_{i+1} dominates the color classes $\{v_{i+1}\}$ and $\{v_i, v_{i+2}\}$, and the vertices v_{i-1} and v_{i+3} dominate its own color class and the color classes $\{v_{i-2}\}$ and $\{v_{i+4}\}$, respectively. Here, we can note that the vertices v_{i-2} and v_{i+4} also dominate 2-color classes; that is, itself and the color classes $\{v_{i-1}\}$ and $\{v_{i+3}\}$, respectively. Therefore, for $n \geq 8$, we begin by assigning the vertices v_3 and v_5 the same color as the vertices v_1 and v_2 are already assigned unique colors.

Though this coloring pattern is greedy, its optimality can be proved immediately using mathematical induction. Also, in this assignment of colors, as exactly one color repeats twice when coloring five consecutive vertices and until the third of the next five consecutive vertices are colored, there is no repetition of a new color possible, we obtain the values $n - \lfloor \frac{n}{5} \rfloor$ and $n - \lfloor \frac{n}{5} \rfloor - 1$ as the 2-dominator chromatic number of P_n when $n \geq 8$; $n \equiv 3, 4 \pmod 5$ and $n \geq 8$; $n \equiv 0, 1, 2 \pmod 5$, respectively. □

Next, we determine the 2 and 3-dominator chromatic numbers of the cycles C_n, for any $n \geq 3$, as they admit a k-dominator coloring only for $k \leq 3$.

THEOREM 2.12. *For a cycle C_n; $n \geq 3$,*

$$\chi_d^{(k)}(C_n) = \begin{cases} 3, & n = 4 \text{ and } k = 2; \\ n - \lfloor \frac{n}{5} \rfloor, & n \neq 4 \text{ and } k = 2 \\ n, & k = 3. \end{cases}$$

Proof. Consider a cycle $C_n = v_1 - v_2 - \ldots - v_n - v_1$ of order n. Similar to the paths P_n, the color classes in any 2-dominator coloring of C_n can have at most 2 vertices and hence to obtain any optimal 2-dominator coloring of a cycle, we increase the number of color classes having cardinality two. When $n = 3$, it is evident that the trivial coloring is the 2-dominator as well as the 3-dominator coloring. When $n = 4$, we obtain a $\chi_d^{(2)}$-coloring of C_4 from the χ-coloring or χ_d-coloring with 2 colors by replacing 2 color classes of cardinality 2 with 3 color classes such that exactly one among them has cardinality 2. Therefore, $\chi_d^{(2)}(C_4) = 3$.

When $n \geq 5$, if a color has to be assigned to two vertices, then we must assign unique colors to two consecutive vertices on the either sides of the vertices which forms a color class of cardinality 2, similar to the 2-dominator coloring pattern of paths. To ensure optimality in such a 2-dominator coloring of C_n, as n increases, we assign a unique color to the vertex v_1 and the vertices v_2 and v_n are given the same color. Therefore, the vertices $v_3, v_4, v_{n-1}, v_{n-2}$ must be given unique colors, so that all of these vertices dominate two color classes. As the next color class having cardinality 2 will be $\{v_5, v_7\}$ or $\{v_{n-3}, v_{n-5}\}$, we obtain, $\chi_d^{(2)}(C_n) = n - 1$; $5 \leq n \leq 9$.

As the seven vertices v_i; $1 \leq i \leq 4$, and $n-2 \leq i \leq n$, form the base for the pattern for a 2-dominator coloring of any C_n; $n \geq 7$, by having two uniquely colored vertices at either ends, the coloring pattern for a cycle C_n; $n \geq 7$, repeats either from the vertex v_5 or v_{n-3} in quintuples. The optimality of this 2-dominator coloring follows from Theorem 2.11 and hence we deduce that $\chi_d^{(2)}(C_n) = n - \lfloor \frac{n}{5} \rfloor$; $n \geq 5$. Also, as an immediate consequence of Corollary 2.4, we have $\chi_d^{(3)}(C_n) = n$, for all $n \geq 3$. □

A *wheel graph* of order $n = t + 1$, denoted by $W_{1,t}$, is a graph obtained by making a vertex, say v, adjacent to all the vertices of C_t. Based on Theorem 2.5 and Theorem 2.12, we obtain the k-dominator coloring of wheel graphs, for all $1 \leq k \leq 4$, as follows.

Corollary 2.13. *For a wheel graph $W_{1,t}$; $t \geq 3$,*

$$\chi_d^{(k)}(W_{1,t}) = \begin{cases} \chi(W_{1,t}), & k=1; \\ \chi_d(C_t), & k=2; \\ \chi_d^{(2)}(C_t), & k=3; \\ t+1, & k=4. \end{cases}$$

A *gear graph* $G_{1,t,t}$; $t \geq 3$, of order $n = 2t+1$ is obtained by making a vertex, say v, adjacent to the vertices v_i; $i \equiv 0 \pmod 2$, of the cycle C_{2t}. As $\delta(G_{1,t,t}) = 2$; $t \geq 3$, we obtain the 2 and 3 dominator coloring of the gear graphs in the following result.

Theorem 2.14. *For a gear graph $G_{1,t,t}$; $t \geq 3$, $\chi_d^{(2)}(G_{1,t,t}) = t+2$ and $\chi_d^{(3)}(G_{1,t,t}) = 2t+1$.*

Proof. Let $G_{1,t,t}$ be a gear graph with $V(G_{1,t,t}) = \{v\} \cup \{v_i : 1 \leq i \leq 2t\}$ and $E(G_{1,t,t}) = \{v_iv_{i+1}, : 1 \leq i \leq 2t\} \cup \{vv_j : j \equiv 0 \pmod 2, 1 \leq j \leq 2t\}$, where the suffixes are taken modulo $2t$. Consider the coloring $c : V(G_{1,t,t}) \to \{c_i : 1 \leq i \leq t+2\}$ such that $c(v_i) = c_{\frac{i}{2}}$; $i \equiv 0 \pmod 2$, $1 \leq i \leq 2t$, $c(v_i) = c_{t+1}$; $i \equiv 1 \pmod 2$, $1 \leq i \leq 2t$, and $c(v) = c_{t+2}$. This coloring c is a 2-dominator coloring of the gear graph $G_{1,t,t}$ as every vertex of degree 3 in $G_{1,t,t}$ dominates its own color class and the color class $\{v\}$, every vertex of degree 2 in $G_{1,t,t}$ dominates the color classes of both of its degree 3 neighbours, as every degree 3 vertex is a assigned a unique color by c and the vertex v dominates all the color classes except $\{v_i : i \equiv 0 \pmod 2, 1 \leq i \leq 2t\}$. Therefore, $\chi_d^{(2)}(G_{1,t,t}) \leq t+2$.

To prove its optimality, we analyse all the possible 2-dominator coloring patterns of a gear graph $G_{1,t,t}$. The graph $G_{1,t,t}$ has t vertices, v_i; $1 \leq i \leq 2t$, $i \equiv 1 \pmod 2$, of degree 2, which are adjacent to the vertices v_{i-1} and v_{i+1}. The following are the only possibilities of assigning colors to the vertices of the gear graph such that every vertex of degree 2 dominates two color classes.

(i) A vertex v_i of degree 2 dominates the color classes $\{v_{i-1}\}$ and $\{v_{i+1}\}$. That is, v_i dominates the color classes of both its neighbours and not its own.

(ii) A vertex v_i of degree 2 dominates color classes $\{v_i\}$ and $\{v_{i+1}, v_{i-1}\}$. That is, the vertex is assigned a unique color and only both its neighbours are assigned the a single unique color.

(iii) A vertex v_i of degree 2 dominates the color classes $\{v_i\}$ and $\{v_{i+1}\}$ or $\{v_i\}$ and $\{v_{i-1}\}$. That is, v_i dominates its own color class and the color class of one its neighbours.

The 2-dominator coloring in which every vertex of degree 2 in a gear graph dominating the two color classes in its neighbourhood gives $\chi_d^{(2)}(G_{1,t,t}) = t+2$. The second possibility in which the vertex of degree 2 in $G_{1,t,t}$ has to be assigned a unique color and only both its neighbours are assigned the a single unique color cannot happen as every vertex of degree 2 in a gear graph shares a vertex of degree 3 in its neighbour.

Consider the third possibility, where a vertex of degree 2 in the graph $G_{1,t,t}$ dominates its own color class and the color class of one its neighbours. Here, to minimise the number of colors, we consider that a vertex v_i; $1 \leq i \leq 2t$, $i \equiv 1 \pmod 2$, of degree 2 dominates the color class $\{v_{i+1}\}$, when $i \equiv 1 \pmod 4$, and the color class $\{v_{i-1}\}$, when $i \equiv 3 \pmod 4$. Therefore, in this case, all vertices v_i; $1 \leq i \leq 2t$, $i \equiv 0 \pmod 4$ are assigned the same color and only the vertices v_i; $1 \leq i \leq 2t$ and $i \equiv 1, 2, 3 \pmod 4$, are assigned unique colors. Therefore, we require at least $3\lfloor \frac{2t}{4} \rfloor + 1$ colors in this 2-dominator coloring of the gear graph $G_{1,t,t}$; $t \geq 4$, and 6 colors when $t = 3$. As $3\lfloor \frac{2t}{4} \rfloor + 1 \geq t + 2$, for all $t \geq 4$ and $6 > t + 2$, when $t = 3$, this cannot be the optimal 2-dominator coloring of the gear graph. Hence, $\chi_d^{(2)}(G_{1,t,t}) = t + 2$.

In a 3-dominator coloring of a gear graph $G_{1,t,t}$ every vertex of degree 2 must dominate its own color class and the two color classes of its neighbours. Therefore, all the vertices of the a gear graph $G_{1,t,t}$ must be assigned unique colors in its optimal 3-dominator coloring, thereby proving the result. □

As trees admit only dominator and 2-dominator colorings, the investigation of when a 2-dominator coloring of a tree is the trivial coloring is an inevitable topic of study. In this regard, we obtain the following characterisation of trees whose 2-dominator coloring is the trivial coloring.

A *caterpillar* is a tree in which the removal of the pendant vertices yields a path and this path is known as the *spine* of the caterpillar. We define a *full spine caterpillar* as a tree T in which every vertex is either a leaf or a support.

THEOREM 2.15. *For a tree T of order n, $\chi_d^{(2)}(T) = n$ if and only if T has one of the following structures.*

(i) T is a full spine caterpillar.

(ii) T is a caterpillar in which at most two consecutive vertices are neither support vertices nor leavess.

Proof. If T is a full spine caterpillar of order n, $\chi_d^{(2)}(T) = n$ because every leaf can dominate exactly two color classes; its own color class and the color class of its support. Therefore, every leaf and support of a tree is assigned a unique color. If T is a caterpillar in which at most two consecutive vertices are neither support or leaf, all the pendant and support vertices will have unique color in their 2-dominator coloring according to the previous case and in view of this, the two consecutive vertices that are neither support nor leaves of T also receive unique colors. Hence, the caterpillar T considered with the given properties also has a trivial coloring.

Conversely, assume that a tree T has a trivial 2-dominator coloring and T is not any of the structures mentioned above. This implies that T can have at least two non-consecutive vertices which are neither leaves nor the support vertices of T or at least three vertices which are neither leaves nor the support vertices of T.

Case 1: First, consider a tree T of order n having two non-consecutive vertices v and u which are neither leaves nor the support vertices of T. Here, it can be observed that in any 2-dominator coloring of the tree T these vertices v and u are given the same color these vertices dominate the two color classes of its neighbours, which are the support vertices of some pendant vertices in T. Hence, here we obtain $\chi_d^{(2)}(T) = n-1$; which is not a trivial coloring of T.

Case 2: Consider the case when the tree T of order n contains three vertices, say v_1, v_2 and v_3, which are neither leaves nor the support vertices of T. Based on the position of these vertices, we consider the following subcases.

Subcase 2a: Let v_1, v_2 and v_3 be consecutive vertices in T. Therefore, T can be viewed as a tree obtained by adjoining a P_3 between a support vertex of the tree T_1 and a support vertex of the tree T_2, where T_1 and T_2 are full spine caterpillars. In the tree T, the vertices v_1 and v_3 can be assigned the same color and the vertex v_2 can be assigned a unique color in any of its optimal 2-dominator coloring, as the vertex v_2 will dominate the color classes $\{v_1, v_3\}$ and $\{v_2\}$, and the vertices v_1 and v_3 dominate the color class $\{v_2\}$ and the color class of other neighbour from the subtree T_1 and T_2, respectively. Therefore, we obtain, $\chi_d^{(2)}(T) = n-1$; a contradiction.

Subcase 2b: Let v_1, v_2 and v_3 be non-consecutive vertices in T. Here, either all 3 are non-consecutive or we have 2 consecutive and one non-consecutive. If all 3 are non-consecutive, then all of them can be assigned the same color as they dominate the color classes of its neighbours, which gives that the 2-dominator coloring of such a tree will have only $n-3$ colors. If only two of them, say v_1 and v_2, are consecutive, these two vertices must be unique colors in the 2-dominator coloring of T and the other vertex v_3 can be assigned the color assigned to v_1, as v_3 dominates the color classes of its neighbours and the vertex v_1 in such a coloring will dominate the color class $\{v_2\}$ and the color class of the other neighbour, which is a support vertex in T. Hence, we see that the 2-dominator coloring of T in this case is also non-trivial and thereby completing the proof. □

COROLLARY 2.16. *For a* bi-star $S_{a,b}$ *which is obtained by making the universal vertices of the stars* $K_{1,a}$ *and* $K_{1,b}$ *adjacent,* $\chi_d^{(2)}(S_{a,b}) = a+b+2$.

As any graph with a pendant vertex admits at most a 2-dominator coloring, we deduce the following from Theorem 2.15, for graphs having $\delta(G) = 1$.

COROLLARY 2.17. *For any graph* G, $\chi_d^{(2)}(G) \geq l+s$ *and* $\chi_d^{(2)}(G) \geq 2s$*, where* l *and* s *denote the number of pendant and support vertices of* G*, respectively.*

The *corona* of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2 (see [5]).

Corollary 2.18. *For any graph G of order n, $\chi_d^{(2)}(G \odot rK_1) = (r+1)n$.*

A *helm graph* of order $n = 2t+1$, denoted by $H_{1,t,t}$, is obtained by attaching a leaf to each of the vertex of degree 3 in a wheel graph.

Proposition 2.19. *For a helm graph $H_{1,t,t}$; $t \geq 3$, $\chi_d^{(2)}(H_{1,t,t}) = 2t+1$.*

Proof. All the vertices of a helm graph $H_{1,t,t}$, except the vertex of degree t, say v, are either pendant vertices or support vertices, which are assigned unique colors in any 2-dominator coloring of the graph $H_{1,t,t}$. Hence, the vertex v is also assigned a unique color, thereby yielding the 2-dominator chromatic number of $H_{1,t,t}$; $t \geq 3$, $\chi_d^{(2)}(H_{1,t,t}) = 2t+1$. □

Note that the characterisation in Theorem 2.15 need not hold for all graphs with pendant vertices, as the 2-dominator coloring of pineapple graph $K_{t'}^t$ is an example for which the converse of Theorem 2.15. This is because any 2-dominator coloring of the graph $K_{t'}^t$ is a trivial coloring. Note that a *pineapple graph* of order $n = t + t'$, denoted by $K_{t'}^t$, is obtained by adjoining t pendant vertices to exactly one vertex of a complete graph $K_{t'}$.

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