MATHEMATICA, 68 (91), N° 1, 2026, pp. 34–45

FROM GENERALIZED GROUPS TO THE NEW VERSION OF GENERALIZED POLYGROUPS

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Abstract. Molaei's generalized groups are completely simple semigroups and the polygroups are generalization of groups. In this article, we review and study the generalized polyactions of polygroups. We define a new version of generalized polygroup (NVG-Polygroups) and study its properties. The relationship between the polygroups of generalization and NVG-Polygroups is investigated. We extend the classical results of the generalized actions of groups to polygroups.

MSC 2020. 20E34, 20E99, 20N20.

Key words. Polygroup, action, polyaction, NVG-polygroups.

1. INTRODUCTION

In [8, 9], Molaei has defined the concept of generalized groups. In fact, they are the completely simple semigroups [1, 2]. These groups have a very deep and detailed physical background in the unified gauge. The concept of hypergroups was defined by Marty[7] and the concept of polygroups was defined by Comer [4].

We remind that if $< P, \circ, e, ^{-1} >$ is a polygroup, the relation β^*_P on P, is actually the smallest equivalence relation, provided that the set of all equivalence classes, i.e. the quotient $\frac{P}{\beta^*_P}$, has the structure of a group. Also β^*_P is called the *fundamental equivalence* on P and $\frac{P}{\beta^*_P}$ is called the *fundamental group*.

Since a group can be made with a polygroup and fundamental equivalence, so the study of generalized polyactions for polygroups is interesting and useful.

2. GENERALIZED POLYACTION

Note that a generalization of group theory is polygroup theory. In groups, the combination of two members of the group is a member, but in polygroups, the result of the combination of two members is a set. The theory of polygroups is used in various fields, such as lattices, geometry, color scheme and combinatorics. For more study, see [6]. In hypergroup theory, a subclass is that of polygroups, which has been studied by Comer[4], also see [5, 6].

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DOI: 10.24193/mathcluj.2026.1.04

Definition 2.1 ([10]). A non-empty set H with a map $*: H \times H \longrightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H, is called *hyperstructure*.

According to [4], a *polygroup* is a system $\mathcal{M} =< P, \circ, e, {}^{-1} >$, with $e \in P$, ${}^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following three conditions are true for all r, s, t in P:

(i) $(r \circ s) \circ t = r \circ (s \circ t)$
(ii) $e \circ r = r \circ e = r$
(iii) $r \in s \circ t$ implies $s \in r \circ t^{-1}$ and $t \in s^{-1} \circ r$.

Of course, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and if $x \in P$ and A, B are non-empty subsets of P, then we have $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$.

From the principles of defining polygroups, the following are easily concluded: $e \in r \circ r^{-1} \cap r^{-1} \circ r$, $e^{-1} = e$ and $(r^{-1})^{-1} = r$.

Example 2.2. Suppose that $P = \{e, a, b\}$ is a set. Then $P =< P, \circ, e, {}^{-1} >$ by the multiplication table

$\circ$	e	a	b
e	e	a	s
a	a	$\{e, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{e, a\}$

is a polygroup.

To enter the discussion, first we need the definition of generalized groups according to [8, 10]:

Definition 2.3 ([8]). A non-empty set G an operation (called *multiplication*) is called a *generalized group*, if the following statements are true:

(i) $(rs)t = r(st)$, for all $r, s, t \in G$,
(ii) for each $r \in G$, there exists a unique $e(r) \in G$, such that $re(r) = e(r)r = r$,
(iii) for each $r \in G$, there exists $r^{-1} \in G$ such that $rr^{-1} = r^{-1}r = e(r)$.

Definition 2.4 ([10]). If $M =< P, \circ, e, {}^{-1} >$ is a polygroup and Ω is a none-empty set, then map $f : \Omega \times P \longrightarrow \mathcal{P}^*(\Omega)$ is called an *action of P on Ω*, if we have the following conditions:

(i) $f(\omega, e) = \{\omega\} = \omega$, for all $\omega \in \Omega$,
(ii) $f(f(\omega, g), h) = \cup_{\alpha \in g.h} f(\omega, \alpha)$; for all $g, h \in P$ and $\omega \in \Omega$,
(iii) $\cup_{\omega \in \Omega} f(\omega, g) = \Omega$ for all $g \in P$,
(iv) for all $g \in P$, $\alpha \in f(\beta, g) \Rightarrow \beta \in f(\alpha, g^{-1})$.

Definition 2.5 ([10]). A hyperstructure $(H, *)$ is called a *generalized polygroup* if the following conditions hold:

GP_1 : The operation $*$ is associative,
GP_2 : for all $r \in H$, there exists $e(r) \in H$ such that $r * e(r) = e(r) * r = \{r\}$,

GP_3 : for all $r \in H$, there exists $r' \in H$ such that $e(r) \in r * r' \cap r' * r$,
GP_4 : for all r, s, and $t \in H$, $t \in r * s \Rightarrow r * e(s) \subseteq t * s' \Rightarrow e(r) * s \subseteq y' * z$.

Definition 2.6. Consider generalized groups G and S. The function $^\circ$: $G \times S \to S$, $[(g, x) \mapsto g^\circ x$, is called a *generalized action of G on S*, if the following conditions are true:

(i) $(g_1g_2)^\circ x = g_1^\circ(g_2^\circ x)$, for all $g_1, g_2 \in G$ and $x \in S$,
(ii) $g^\circ(x_1x_2) = (g^\circ x_1)(g^\circ x_2)$, for all $g \in G$ and $x_1, x_2 \in S$,
(iii) for all $x \in S$, there exists an element $e(g) \in G$ such that $e(g)^\circ x = x$. Also, the following condition which can be easily obtained from the previous condition can be added to the previous condition.
(iv) $g^\circ e(x) = e(x)$ for all $g \in G$ *and* $x \in S$.

Example 2.7. The set G with the product $g.h = h$, is a generalized group.

Definition 2.8 (Generalized Polyaction). Let P be generalized polygroup and X be a non-empty set. A map $f : P \times X \to \mathcal{P}^*(X)$ is called a *generalized polyaction of P on X*, if the following axioms hold for $f : P \times X \to \mathcal{P}^*(X)$, $(p, x) \mapsto f(p, x) = p^\circ x$:

(i) $f(e, x) = \{x\} = x$ for all $x \in X$,
(ii) $f(p', f(p, x)) = \bigcup_{\alpha \in p'.p} f(p, \alpha)$, for all $x \in X$ and $p, p' \in P$,
(iii)) $\bigcup_{x \in X} f(p, x) = X$, for all $p \in P$,
(iv) $\forall p \in P,\ \alpha \in f(p, p') \Rightarrow \beta \in f(p^{-1}, \alpha)$.

Also, by the second condition, we have $\bigcup_{x_0 \in f(p', x)} f(p, x_0) = \bigcup_{q \in p'.p} f(q, x)$. But the second condition gives $\bigcup_{\omega_0 \in f(\omega, g)} f(\omega_0, h) = \bigcup_{\alpha \in g.h} f(\omega, \alpha)$.

If for $\omega \in \Omega$, we write $\omega^g := f(\omega, g)$, then we have

(i) $\omega^e = \omega$,
(ii) $(\omega^g)^h = \omega^{g.h}$, where $A^g = \bigcup_{\alpha \in A} \alpha^g$ and $\omega^B = \bigcup_{g \in B} \omega^g$ for all $A \subseteq \Omega, B \subseteq P$,
(iii) $\bigcup_{\alpha \in \omega} \omega^g = \Omega$,
(iv) for all $g \in P$, $\alpha \in \beta^g \Rightarrow \beta \in \alpha^{g^{-1}}$.

Now, if for $x \in X$, we write $x^\circ p = f(p, x)$, then we have:

(i) $x^\circ e = x$
(ii) $(x^\circ p)^\circ p' = x^\circ(p.p')$, where $A^\circ p = \bigcup_{\alpha \in A} \alpha^\circ p$, $B^\circ X = \bigcup_{\beta \in B} p^\circ X$ for all $A \subseteq X, B \subseteq P$,
(iii) $\bigcup_{x \in X} x^\circ p = X$,
(iv) for all $p \in P$, $\alpha \in \beta^\circ p \Rightarrow \beta \in \alpha^\circ p^{-1}$.

Example 2.9. If P is a generalized polygroup, then P acts on itself with $p^\circ x = \{x\} = x$. Because:

(i) For all $p_1, p_2, p_3 \in P$, we have $(p_1p_2)^\circ p_3 = \{p_3\} = p_3$ and $p_1^\circ(p_2^\circ p_3) = p_1^\circ p_3 = \{p_3\} = p_3$. Therefore $(p_1p_2)^\circ p_3 = p_1^\circ(p_2^\circ p_3)$.
(ii) For all $p_1, p_2, p_3 \in P$, we have $p_1^\circ(p_2p_3) = p_2p_3$ and $(p_1^\circ p_2)(p_1^\circ p_3) = p_2p_3$. Therefore $p_1^\circ(p_2p_3) = (p_1^\circ p_2)(p_1^\circ p_3)$.

(iii) For all $p_1 \in P$, we have $e(p_1)^\circ h = h$. So we conclude that the element $e(p_1) \in P$ exists. That is, there is element $e(p_1)$ which is directly obtained from the definition of the action.

(iv) For all $p_1, p_2 \in P$, since $p_1^\circ e(p_2) = e(p_2)$ the fourth condition is verified.

THEOREM 2.10. *Every abelian generalized polygroup is a polygroup.*

Proof. If P is an abelian generalized polygroup, then, for every $p_1, p_2 \in P$, we show that $e(p_1) = e(p_2)$. Let $p_1, p_2 \in P$, we have $(p_1p_2)e(p_2) = p_1p_2$ and $e(p_2)(p_1p_2) = e(p_2)(p_2p_1) = p_2p_1$. So, $e(p_1p_2) = e(p_2)$, and by the similar method, we have $e(p_1) = e(p_1p_2)$. Therefore, $e(p_1) = e(p_2)$. □

EXAMPLE 2.11. Suppose that $X_n = \{1, 2, ..., n\}$, $n \in \aleph$, S_n is the symmetric group of order n, and $P = S_n$. Then $f : X_n \times P \longrightarrow \mathcal{P}^*(X_n)$, by $f(k, \alpha) = \{\alpha(k)\}$ is an action, which we call it trivial action P on X_n.

PROPOSITION 2.12. *Every polygroup is a generalized polygroup.*

Proof. The proof of the statement is clear according to the definition. □

DEFINITION 2.13. Consider a polygroup of P. If $P/\beta^* \cong S_n$, then P is called special polygroup.

EXAMPLE 2.14. If $P = \{a, b, c, d, e, f, g\}$, and $< P, \circ, a,^{-1} >$ is the polygroup, where $\circ$ is defined as follows:

$\circ$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	$\{a,b\}$	c	d	e	f	g
c	c	c	$\{a,b\}$	g	f	e	d
d	d	d	f	$\{a,b\}$	g	c	e
e	e	e	g	f	$\{a,b\}$	d	c
f	f	f	d	e	c	g	$\{a,b\}$
g	g	g	e	c	d	$\{a,b\}$	f

then P is a special polygroup.

DEFINITION 2.15. Consider a special polygroup P with an action of P on $X_n = \{1, 2, ..., n\}$. Then S_{X_n} is called the *generalized symmetric polygroup* and $S'_{X_n} = A_{X_n}$ is called the *generalized alternative polygroup.*

EXAMPLE 2.16. Suppose that P and S_n are the special polygroup of example 2.14, and $X_7 = \{a, b, c, d, e, f, g\}$. Consider the action $kp = k \circ p$,, then $S_{X_7} = P$ and $A_{X_7} = P'$ and also $P' =< a, b, f, g >= \{a, b, f, g\} \underline{\triangle} P$.

THEOREM 2.17. *Let P be a generalized polygroup and X be a non-empty set. Also $\alpha : P \times X \longrightarrow \mathcal{P}^*(X)$ be a generalized polyaction, and P be an abelian generalized polygroup. Then P is a polygroup and α is an action.*

Proof. By Theorem 2.10, P is a polygroup. So α also is an action. □

The following proposition is straightforward to obtain.

Proposition 2.18. *Suppose that P is a generalized polygroup, X is a set, and P acts on the set X. Then the relation $\sim$ defined by*

$$x_1 \sim x_2 \Leftrightarrow p_1^\circ x_1 = \{x_2\} \ \text{ and } p_2^\circ x_2 = \{x_1\} \ \text{ for some } p_1, p_2 \in P,$$

is an equivalence relation.

Now we state the following definition:

Definition 2.19. With the symbols of Proposition2.18, if $x \in X$ then,

$$\bar{x} = [x] = \{y \mid y \in X, x \sim y\}$$

is called the *generalized orbit of x.*

Theorem 2.20. *If a generalized polygroup P polyact on a set X, then for every $x \in X$, the set I_x, defined by $I_x = \{p \mid p \in P, p^\circ x = \{x\}\}$ is a generalized subpolygroup of P.*

Proof. If $p \in I_x$, then we have:

$$\begin{aligned} p^\circ x = \{x\} &\Rightarrow (e(p)p)^\circ x = \{x\} \Rightarrow e(p)(p^\circ x) = \{x\} \\ &\Rightarrow e(p)^\circ x = \{x\} \Rightarrow e(p) \in I_x. \end{aligned}$$

Also, $p^{-1\circ}x = p^{-1\circ}(p^\circ x) = (p^{-1}p)^\circ x = e(p)^\circ x = \{x\}$. So $p^{-1} \in I_x$. Now, if $p_1, p_2 \in I_x$, then $(p_1p_2)^\circ x = p_1^\circ(p_2^\circ x) = p_1^\circ x = \{x\}$. Therefore $p_1p_2 \in I_x$. Therefore, I_x is a generalized subpolygroup of P. $\square$

Definition 2.21. Suppose that $(P, \circ)$ and $(P_2, *)$ are the generalized polygroups, and $f : P_1 \to P_2$. Then f is called an *inclusion generalized polygroup homomorphism*, if $f(p_1 \circ p_2) \subseteq f(p_1) * f(p_2)$, for all of $p_1, p_2 \in P_1$. The map f is called a *strong generalized polygroup homomorphism* (a good generalized polygroup or generalized polygroup homomorphism), if $f(p_1 \circ P_2) = f(p_1) * f(p_2)$.

Example 2.22. Let

$$P = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mid \ x, y \in R, y \neq 0 \right\}.$$

It is clear that

$$e\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \dfrac{x}{y} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ \dfrac{x}{y^2} & \dfrac{1}{y} \end{pmatrix}.$$

So P is a generalized polygroup. Also, if $Q = R \times R - \{0\}$, then Q is a generalized polygroup with multiplication $(x, y)(r, s) = (yr, ys)$.

Now, $f : P \to Q$, $\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mapsto (x, y)$ is a generalized polygroup homomorphism.

Note that we have:

$$f\left(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ x' & y' \end{pmatrix}\right) = f\begin{pmatrix} 0 & 0 \\ yx' & yy' \end{pmatrix} = (yx', yy')$$

and

$$f\begin{pmatrix}0 & 0\\ x & y\end{pmatrix} = (x,y) \quad , \quad f\begin{pmatrix}0 & 0\\ x' & y'\end{pmatrix} = (x',y').$$

In addition $(x,y)(x',y') = (yx', yy')$.

THEOREM 2.23. *If P, Q are generalized polygroup, and $f : P \to Q$ is a generalized polygroup homomorphism, then*

$$\alpha : P \times Q \longrightarrow \mathcal{P}^*(E) \qquad \textit{defined by} \qquad (p,q) \mapsto f(p)q$$

is a generalized polyaction.

Proof. If $p, p' \in P$ and $q \in Q$, then

$$\alpha(p, \alpha(p',q)) = f(p)\alpha(p',q) = f(p)(f(p')q) = f(pp')q = \bigcup_{x\in f(pp')} xq = \alpha(pp',q).$$

Also, if $p \in f^{-1}(\{e(q)\})$, then,

$$\alpha(e(p))h = f(e(p))h = e(f(p))q = e(e(q))q = e(q)q = \{q\} = q.$$

So, α is a generalized polyaction. □

EXAMPLE 2.24. If P, Q and f are as defined in Example 2.22, then

$$\alpha : P \times Q \longrightarrow \mathcal{P}^*(Q)$$

$$\left(\begin{pmatrix}0 & 0\\ x & y\end{pmatrix}, \ (x',y')\right) \longmapsto f\begin{pmatrix}0 & 0\\ x & y\end{pmatrix}(x',y') = (x,y)(x',y') = (yx', yy')$$

is a generalized polyaction.

THEOREM 2.25. *If P is a generalized polygroup, then for each element p in P, there exists a unique $p^{-1} \in P$.*

Proof. If $p \in P$, then

$$p = \{p\} = pe(p) = p(p^{-1}p) = p(p^{-1}e(p^{-1}))p = (e(p)e(p^{-1}))p.$$

With a similar method, $p = \{p\} = p(e(p^{-1})e(p)$. But the uniqueness of $e(p)$ shows that $e(p)e(p^{-1}) = e(p^{-1}) = e(p) = \{e(p)\}$. So, $e(p^{-1}) = e(p)$. But, if $p'p = pp' = e(p) = \{e(p)\}$, and $p''p = pp'' = e(p) = \{e(p)\}$, then $p'(pp'') = p'(e(p))$. So $e(p)p'' = p'e(p)$. So $p'' = p'$. □

THEOREM 2.26 (Generalized Subpolygroup Benchmark Theorem). *If P is a generalized polygroup, and Q is a non-empty subset P, then Q is a generalized subpolygroup of P, if and only if, for $q, q' \in Q$, $qq' \in \mathcal{P}^*(Q)$.*

Proof. If Q is a generalized subpolygroup of P, and $q, q' \in Q$, then $q^{-1} = \{q^{-1}\} \in \mathcal{P}^*(Q)$, and $qq'^{-1} \in \mathcal{P}^*(Q)$.

Conversely, if $Q \neq \emptyset$ and $q, q' \in Q$, then $qq^{-1} = \{e(q)\} = e(q) \in Q$, $e(q)q^{-1} = \{q^{-1}\} = q^{-1} \in Q$, $qq' = q(q'^{-1})^{-1} \in \mathcal{P}^*(Q)$. □

Theorem 2.27. *If $\{P_i \mid i \in I\}$ is a family of generalized subpolygroup of P and $Q = \bigcap_{i\in I} P_i \neq \emptyset$, then subscription $Q = \bigcap_{i\in I} P_i$ is a generalized subpolygroup of P.*

Proof. If $p, p' \in Q$, then $p, p' \in P_i$, for every $i \in I$, and thus $pp'^{-1} \in \mathcal{P}^*(P_i)$, for every $i \in I$. So $pp'^{-1} \in \mathcal{P}^*(Q)$, and by generalized subpolygroup benchmark Theorem, the result is correct. □

The following result is easy to obtain.

Theorem 2.28. *If P is a generalized polygroup, and $p \in P$, then*

$$P_p = \{p' \mid p' \in P, e(p') = e(p)\},$$

is a generalized subpolygroup of P. Also P_p is a polygroup.

If P is a set and $(P, \circ)$ is a polygroup, and $\mathcal{P}^*(P)$, is the power set of P without the empty set, also if

$$H = \{(A, B) \mid A,\, B \in \mathcal{P}^*(P),\, A \subseteq B\},$$

and f is a singleton map from P into $\mathcal{P}^*(P)$, (i. e, $f(x) = \{x\}$, for all $x \in P$), and $g : H \to \mathcal{P}^{*2}(P)$ is the inclusion map, then every polygroup is a polygroup object in sets.

Example 2.29. Let $P = \{e, a, b\}$, and $\circ : P \times P \to \mathcal{P}^*(P)$, be given by the following table:

$\circ$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	a
b	b	a	e

If $i : P \to P$ be identity and $e_P, e'_P : P \to P$ defined such that $e_P : e, a, b \longmapsto e$ and $e'_P : e, a, b \longmapsto e, b, e$, then i is the unique inverse for e_P and e'_P.

3. NVG-POLYGROUPS

If $(S, *)$ is a semigroup, such that for any $s \in S$, there exist $e, y \in S$ such that $e * s = s * e = s$, and $s * y = y * s = e$, then it is said that semigroup admit relative inverses. Also, note that such semigroups are sometimes called Clifford semigroups. Molaei in[8] expressed the concept of generalized groups for the first time. They are interesting and special class of algebras. In fact, he obtained results from this class of algebras, that were used in Physics. We recall that a generalized group is actually a semigroup $(S, *)$, so that for any $s \in S$, there are unique members such as $e(s)$, $s^{-1} \in S$, such that $e(s) * s = s * e(s) = s$, and $s * s^{-1} = s^{-1} * s = e(s)$.

Now we give the new version of generalized polygroups (NVG-Polygroups).

Definition 3.1. Let P be an arbitrary non-empty set, E be an equivalence relation on P, and $*$ be a function from $P \times P$ to set $\mathcal{P}^*(X)$, containing $\mathcal{P}^*(P)$. Then a triple $(P, E, *)$, is a *new version of generalized polygroup (NVG-Polygroup)*, if the following conditions are true:

(i) (Closure): For any $p_1, p_2 \in P$, $\ p_1 * p_2 \subseteq P$.
(ii) (Associativity): For any $p_1, p_2, p_3 \in P$, $\ p_1 * (p_2 * p_3) = (p_1 * p_2) * p_3$.
(iii) (Identity): For any $p \in P$, there exist $e(p) \in [p]$ such that

$$e(p) * q = q * e(p) = \{q\}, \text{for any} q \in [p].$$

(iv) (Inverses): For any $p \in P$ and $q \in [p]$, there exist $q^{-1} \in [p]$, such that $q * q^{-1} = \{e(p)\} = e(p)$, where $[p]$ denotes the equivalence class that contains $p \in P$.

The following proposition is easy to obtain by Definition 3.1.

Proposition 3.2. (i) *If $(P, E, *)$ is a NVG-polygroup, where E is the universal equivalence relation on P, then P has an algebraic polygroup structure.*
(ii) *If $\{(P_i, E, *)\}_{i\in I}$ is a family of doubly distinct polygroups, $P_i \bigcap P_j = \emptyset$, for any $i, j \in I$, and $i \neq j$. Also E is an equivalence relation on $\cap_{i\in I} P_i \times \cap_{i\in I} P_i$, then $\cap_{i\in I} P_i$, is an NVG-polygroup.*

Remark 3.3. An NVG-polygroup is not just a union of a collection of pairwise disjoint polygroups.

Example 3.4. If we consider the set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, and also we define the equivalence relation E on $\mathbb{Z}_4$ by rEs, if and only if, there exists $k \in \mathbb{Z}$, such that $r = s +_4 2k$ for $r, s \in \mathbb{Z}_4$, then the equivalence classes of this relation are $[0] = \{0, 2\}$ and $[1] = \{1, 3\}$. But, if we let $\bar{t} = min[t]$, for $t \in \mathbb{Z}_4$, and $k_t = \frac{t-\bar{t}}{2}$, for any $t \in \mathbb{Z}_4$, then $\bar{0} = 0$, $\bar{1} = 1$, $\bar{2} = 0$, $\bar{3} = 1$, and $k_0 = 0$, $k_1 = 0$, $k_2 = 1$, $k_3 = 1$.

If on $\mathbb{Z}_4$ the binary operation $*$, is defined by $r * s = \bar{r} +_4 2(k_r + k_s)$, for $r, s \in Z_4$, then it follows that:

$*$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{2\}$	$\{2\}$
1	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{1\}$	$\{1\}$

So, the triple $(Z_4, E, *)$ is an NVG-polygroup, with $e(0) = \bar{0}$, $e(1) = \bar{1}$, $e(2) = \bar{2}$, $e(3) = \bar{3}$. But one can check that $(Z_4, *)$ is not a polygroup.

Theorem 3.5. *Suppose that $\{(P_i, E, *)\}_{i\in I}$ is a family of doubly distinct polygroups, $P_i \bigcap P_j = \emptyset$, for any $i, j \in I$, and $i \neq j$. Also E is an equivalence relation on $\bigcup_{i\in I} P_i$, defined by pEq, if and only if $p, q \in P_i$, for some $i \in I$, and $*$, is defined on $\bigcup_{i\in I} P_i \times \bigcup_{i\in I} P_i$, by:*

$$p * q = \begin{cases} p *_i q & if\, p, q \in P_i\, for\, some\, i \in I \\ \{p\} & if\, p \in P_i, q \in P_j,\, i \neq j\, for\, some\, i, j \in I \end{cases}$$

then the triple $(\bigcup_{i\in I} P_i, E, *)$ *is an NVG-polygroup.*

Proof. For the proof, we check the quadruple properties in Definition 3.1.

(i) For any $p_1, p_2 \in \bigcup_{i\in I} P_i$, we have $r, s \in I$ such that $p_1 \in P_r$, and $p_2 \in P_s$. If $r = s$, then $p_1 * p_2 = p_1 *_r p_2 \subseteq \bigcup_{i\in I} P_i$, and if $r \neq s$, then $p_1 * p_2 = \{p_1\} \subseteq \bigcup_{i\in I} P_i$.

(ii) The proof is obvious.

(iii) For any $p \in \bigcup_{i\in I} P_i$, there is $r \in I$, such that $p \in P_r$. But P_r is an NVG-polygroup, and so there is $e(p) \in [p]$ such that $e(p) * q = q * e(p) = \{q\}$, for any $q \in [p]$.

(iv) For any $p \in \bigcup_{i\in I} P_i$, there is $r \in I$, such that $p \in P_r$, and P_r is an NVG-polygroup. So for any $p \in P$, $q \in [p]$ there is $q^{-1} \in [p]$ such that $q * q^{-1} = \{e(p)\} = e(p)$. □

The following theorem shows that the generalization defined by Dehghanizadeh is a special and interesting mode of NVG-polygroups.

Theorem 3.6. *If* $(P, *)$ *is a generalized polygroup, with an equivalence relation of* E*, which is defined on* P*, by* pEq*, if and only if* $e(p) = e(q)$*, for any* $p, q \in P$*, then* $(P, E, *)$ *is an NVG-polygroup.*

Proof. We have to check the correctness of the conditions of Definition 3.1.

(i) and (ii) are clear.

(iii) By $GP2$, for all $p \in P$ we have, $e(p) \in P$ such that $p * e(p) = e(p) * p = \{p\}$. But, for any $q \in [p]$, we have $e(p) = e(q)$, therefore $p*e(q) = e(p)*p = \{p\}$.

(iv) By $GP3$, for all $p \in P$, there exists $p' \in P$, such that $e(p) \in p*p' \cap p'*p$. But, for any $p \in P$ and $q \in [p]$, we have $e(p) = e(q)$. Therefore, $e(p) = e(q) \in p * p' \cap p' * p$ and by $GP4$, we have:

$$\begin{gathered} p * e(p') \subseteq e(p) * p = \{p\} \\ \Rightarrow e(p) * p' \subseteq p * e(p) = \{p\} \\ \Rightarrow p * e(p') = e(p) * p' = \{p\}, \end{gathered}$$

so the result follows. □

Remark 3.7. However, it has been shown that every abelian generalized polygroup, is a polygroup, but this is not necessarily true in NVG-polygroups.

Example 3.8. If $P = \mathbb{Z}_3 = \{0, 1, 2\}$, then we defined the equivalence relation E on $\mathbb{Z}_3$, by xEy, if and only if, there exists $k \in \mathbb{Z}$ such that $x = y +_3 2k$, or $y = x +_3 2k$. The equivalence classes are $\bar{0} = \{0, 2\}$, and $\bar{1} = \{1\}$.

Now, if the operation $*$, on $\mathbb{Z}_3$ is defined by:

$*$	0	1	2
0	$\{0\}$	$\{0\}$	$\{2\}$
1	$\{0\}$	$\{1\}$	$\{0\}$
2	$\{2\}$	$\{0\}$	$\{1\}$

then the triple $(Z_3, E, *)$ is an abelian NVG-polygroup, and $(Z_3, *)$ is not a polygroup.

THEOREM 3.9. *If* $(P, E, *)$ *is a NVG-polygroup, and* $p \in P$, *then:*

(i) $[p] = [p^{-1}] = [e(p)]$,
(ii) *For* $p \in P$, *the identity element* $e(p)$ *of* $[p]$, *is unique,*
(iii) *For* $p \in P$, *the inverse* p^{-1}, *is unique,*
(iv) *For any* $p, q \in P$, *we have* $e(p) = e(q)$, *if and only if* $[p] = [q]$,
(v) *For any* $p \in P$, *we have* $e(e(p)) = e(p^{-1}) = e(p)$.

Proof. (i) For all $p \in P$, we have $e(p)$ and p^{-1} in $[p]$. So (i) is clear.

(ii) If $e_1(p)$ and $e_2(p)$ are both identities elements of p, then for $q \in [p]$, $q = \{q\} = qe_1(p) = e_1(p)q$, and $q = \{q\} = qe_2(p) = e_2(p)q$. But $e_1(p)$ and $e_2(p) \in [p]$. Also $e_1(p)$ and $e_2(p)$ are identities of $[p]$. In addition $e_1(p) = e_1(p)e_2(p) = \{e_1(p)\} = e_2(p)e_1(p)$, and $e_2(p) = e_2(p)e_1(p) = \{e_2(p)\} = e_1(p)e_2(p)$, which implies that $e_1(p) = e_2(p)$.

(iii) If p has two inverses $p', p'' \in [p]$, then

$$p' = \{p'\} = p'e(p') = p'e(p) = p'pp'' = e(p)p'' = e(p'')p'' = p'' = \{p''\}.$$

(iv) If $e(p) = e(p')$, then $e(p) \in [p]$ and $e(p') \in [p']$, $e(p) = e(p') \in [p] \cap [p']$. So we have $[p] = [p']$.

Conversely, if $[p] = [p']$, then, due to the uniqueness of the identity member, we conclude that $e(p) = e(p')$.

(v) By (iv), we have $e(p), p^{-1} \in [p]$. So the result follows. $\square$

DEFINITION 3.10. Suppose that $(P, E, *)$ is an NVG-polygroup, and H is a non-empty subset of P, then $(H, E, *)$ is said to be an *NVG-subpolygroup* if $(H, E, *)$ is a NVG-polygroup.

The following theorem is easy to obtain by checking the quadruple properties in Definition 3.1.

THEOREM 3.11. *If* $(P, E, *)$ *is an NVG-polygroup, and* $\emptyset \neq H \subseteq P$, *then* $(H, E, *)$ *is an NVG-subpolygroup, if the following conditions are true:*

(i) *For any* $q, q' \in H,\ q * q' \subseteq H$,
(ii) *For any* $q \in H,\ e(q) \in H$,
(iii) *For any* $q \in H,\ q^{-1} \in H$.

THEOREM 3.12. *If* $(P, E, *)$ *is a NVG-polygroup, and* $\emptyset \neq H \subseteq P$, *then* $(H, E, *)$ *is an NVG-subpolygroup if and only if* $q * q^{-1} \subseteq H$, *for each* $q, q' \in H$.

Proof. Let $(H, E, *)$ be an NVG-subpolygroup, and $q, q' \in H$. Therefore H is an NVG-subpolygroup and $q'^{-1} \in H$, so $q, q'^{-1} \in H$, which implies that $q * q'^{-1} \subseteq H$.

Conversely, if $q * q'^{-1} \subseteq H$, for $q, q' \in H$, then since $\emptyset \neq H$, so exists $q'' \in H$ and by assumption, $e(q'') = \{e(q'')\} = q'' * q''^{-1} \subseteq H$. But if $q' \in H$, then $e(q') \in H$, and so $q'^{-1} = \{q'^{-1}\} = e(q') * q' \subseteq H$. So for $q, q' \in H$, we have

$q, q'^{-1} \in H$, and $q * q' = q * (q'^{-1})^{-1} \subseteq H$. This show that $(H, E, *)$ is an NVG-subpolygroup. $\square$

Definition 3.13. Assume that P and P' are NVG-polygroups. A map $f : P \to P'$ is called a *homomorphism* if for any $p, q \in P$, we have $f(pq) = f(p)f(q)$, and $[f(p)] = [f(q)]$, if $[p] = [q]$. Also, the set $\mathrm{Ker}_p(f) = \{q \mid q \in P,\ f(q) = e(p)\}$, is called the *kernel of* f with respect to $p \in P$.

Theorem 3.14. *If P and P' are NVG-polygroups, and $f : P \to P'$ is a homomorphism, then:*

(i) $f(e(p)) = e(f(p))$,
(ii) $f(p^{-1}) = (f(p))^{-1}$,
(iii) *$f(H)$ is an NVG-subpolygroup of P' for any NVG-subpolygroup H of P,*
(iv) *$\mathrm{Ker}_p(f)$ is an NVG-subpolygroup of P, for any $p \in P$,*
(v) *f is injective if and only if $\mathrm{Ker}_p(f) = \{e(p)\}$, for all $p \in P$.*

Proof. (i) If $q = \{q\} = f(e(p))$ for $p \in P$, then $q = \{q\} = f(e(p)) = f(e(p)e(p)) = f(e(p))f(e(p)) = qq$. So $e(q) = q$, that is, $e(f(p)) = f(e(p))$.

(ii) If $q = p^{-1}$, then $e(f(p)) = f(e(p)) = f(pp^{-1}) = f(pq) = f(p)f(q)$, also $e(f(p)) = f(e(p)) = f(p^{-1}p) = f(p^{-1})f(p) = f(q)f(p)$. So $f(q)$ is the inverse of $f(p)$.

(iii) If $K = f(H)$, then K is non-empty and $pq^{-1} \subseteq K$ for any $p, q \in K$. But $H \neq \emptyset$, so there is $p \in H$ which gives that $f(p) \in K$, and so $K \neq \emptyset$. Now if $p, q \in K$, then $f^{-1}(p), f^{-1}(q) \in H$ and $f^{-1}(p)(f^{-1}(q))^{-1} \subseteq H$. But $f(f^{-1}(p))f((f^{-1}(q))^{-1}) \subseteq f(H)$. So $f(f^{-1}(p))f((f^{-1}(q))^{-1}) \subseteq K$, and hence $pq^{-1} \subseteq K$.

(iv) If $p \in P$, then we have $e(p) \in \mathrm{Ker}_p(f)$, and so $\mathrm{Ker}_p(f) \neq \emptyset$. Now, suppose that $p', q' \in \mathrm{Ker}_p(f)$, then $f(p') = f(q') = f(e(p))$. But,

$$f(p'q'^{-1}) = f(p')f(q'^{-1}) = f(e(p))(f(q'))^{-1}$$
$$= f(e(p))(f(e(p)))^{-1} = f(e(p))f(e(p)) = f(e(p)).$$

So $p'q'^{-1} \subseteq K$. Hence K is NVG-subpolygroup.

(v) In the case that f is injective, for each member $p \in P$,, there is at most one member, such that it is sent to the identity $e(f(p))$. But $e(f(p)) = f(e(p))$, hence $Kre_p(f) = \{e(p)\}$. Conversely, if $Kre_p(f) = \{e(p)\}$, for all $p \in P$ and $f(q) = f(q')$, for some $q, q' \in P$, then

$$f(q^{-1}q') = f(q^{-1})f(q') = f(q^{-1})f(q) = f(q^{-1}q) = f(e(q)) = e(f(q)).$$

So $q^{-1}q' \subseteq Ker_q(f)$, and $q^{-1}q' = \{e(q)\} = e(q)$. Hence $q^{-1} = q'^{-1}$, and since the inverse is unique, so $q = q'$, and f is injective. $\square$

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Received July 24, 2024
Accepted December 31, 2024

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