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STUDY OF AN EXTENDED TRIVARIATE HILBERT INTEGRAL INEQUALITY

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Abstract. In this paper, we establish a new integral inequality that extends the scope of the classical trivariate Hardy-Hilbert integral inequality. The main contributions are (i) to generalize a famous result in the literature with the use of additional adaptable functions and parameters, and (ii) to prove it using a well-calibrated spherical change of variables, which is quite unusual in this area of research. Several special cases are highlighted and other new trivariate integral inequalities are derived.

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1. INTRODUCTION

The Hardy-Hilbert integral inequality is a fundamental result in mathematical analysis, especially in the study of bivariate integral inequalities. Formally, it states that, for any $(a, b) \in (1, +\infty)^2$ such that $1/a + 1/b = 1$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} \mathrm{d}x\mathrm{d}y \leq \delta \left[\int_0^{+\infty} f^a(x)\mathrm{d}x\right]^{1/a} \left[\int_0^{+\infty} g^b(x)\mathrm{d}x\right]^{1/b},$$

where δ is a specific constant factor depending on a, given by $\delta := \frac{\pi}{\sin(\pi/a)}$, provided that all the integrals involved exist (or converge). It is shown that the inequality is indeed strict, and that the constant factor δ is the best possible. We can refer to [9, 26]. The Hardy-Hilbert integral inequality has been studied extensively and has led to many generalizations. Among the important references, we mention [20, 22, 23, 19, 24, 13, 25, 27, 8, 17, 16, 29, 18, 3, 4, 2, 1, 7]. The survey in [6] is also recommended. It provides valuable insights into some of these developments.

There are several studies on multivariate variants of the classical Hardy-Hilbert integral inequality. We refer to [12, 21, 14, 10, 5, 30, 15, 28, 11]. For the purposes of this paper, let us present a special trivariate integral inequality

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derived from the main result in [21]. It states that, for any $(a,b,c) \in (1,+\infty)^3$ such that $1/a+1/b+1/c=1$, $\sigma \in \mathbb{R}$, and $f,g,h : [0,+\infty) \mapsto [0,+\infty)$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^\sigma}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

(1)

$$\leq \epsilon \left[\int_0^{+\infty} x^{2-\sigma} f^a(x)\mathrm{d}x\right]^{1/a} \left[\int_0^{+\infty} x^{2-\sigma} g^b(x)\mathrm{d}x\right]^{1/b} \left[\int_0^{+\infty} x^{2-\sigma} h^c(x)\mathrm{d}x\right]^{1/c},$$

where ϵ is the constant factor given by

$$\epsilon := \frac{1}{\Gamma(\sigma)}\Gamma\left(\frac{\sigma-3}{a}+1\right)\Gamma\left(\frac{\sigma-3}{b}+1\right)\Gamma\left(\frac{\sigma-3}{c}+1\right) \tag{2}$$

and $\Gamma(x) := \int_0^{+\infty} t^{x-1}e^{-t}\mathrm{d}t$, with $x > 0$, is the standard gamma function, provided that all the integrals involved exist (implying, in particular, $\sigma > 3 - \min(a,b,c)$). It is shown that the inequality is indeed strict, and that the constant factor ϵ is the best possible. The proof is based on a thorough application of the generalized Hölder integral inequality, and several well-calibrated integral decompositions. This trivariate integral inequality is important because it shows how the third dimension and the parameter σ significantly affect the form of the classical Hardy-Hilbert integral inequality. It also shows that the constant factor adapts to the context. The gamma function plays a central role in defining it.

In this paper, we make two contributions to this topic. First, we extend the inequality in Equation (1) by introducing two intermediate and adaptable functions, k and ℓ. The general form of the triple integral to be bounded is as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^\sigma} k\left(\frac{x}{y}\right)\ell\left(\frac{x+y}{z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Thus, the presence of k and ℓ adds a higher level of generality and complexity compared to the triple integral in Equation (1). We therefore determine a sharp upper bound for this integral, depending on all the components involved. More precisely, following the spirit of Equation (1), we exhibit a constant ξ such that the upper bound obtained has the following form:

$$\xi \left[\int_0^{+\infty} x^{2-\sigma} f^a(x)\mathrm{d}x\right]^{1/a} \left[\int_0^{+\infty} x^{2-\sigma} g^b(x)\mathrm{d}x\right]^{1/b} \left[\int_0^{+\infty} x^{2-\sigma} h^c(x)\mathrm{d}x\right]^{1/c}.$$

The constant ξ can be defined as the product of special integral transformations of k and ℓ, with an indirect connection to the beta function, at least in construction. The constant in Equation (2) (and the overall upper bound) is expected to be rediscovered by taking k and ℓ as constant functions equal to 1. To the best of our knowledge, this is the first time that such a bound has been obtained for this general trivariate variant.

The second contribution is the proof techniques developed to establish this result. We mainly use a special spherical change of variables and the generalized Hölder integral inequality. In a sense, we thus extend the underexploited polar change of variables technique in [16, 7] to the trivariate case, with a more sophisticated integrated function. We believe that this technique can inspire further research in this area, beyond the Hardy-Hilbert-type integral inequalities.

The other three sections of the paper are as follows: Section 2 presents our main trivariate integral inequality and highlights some special cases. It is complemented by some related results in Section 3. Section 4 gives a conclusion.

2. RESULTS

Our extended trivariate Hardy-Hilbert integral inequality is presented in detail in the proposition below.

PROPOSITION 2.1. *For any* $f, g, h, k, \ell : [0, +\infty) \mapsto [0, +\infty)$, $\sigma \in \mathbb{R}$ *and* $(a, b, c) \in (1, +\infty)^3$ *such that* $1/a + 1/b + 1/c = 1$, *we have*

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^\sigma} k\left(\frac{x}{y}\right) \ell\left(\frac{x+y}{z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\leq \xi \left[\int_0^{+\infty} x^{2-\sigma} f^a(x)\mathrm{d}x\right]^{1/a} \left[\int_0^{+\infty} x^{2-\sigma} g^b(x)\mathrm{d}x\right]^{1/b} \left[\int_0^{+\infty} x^{2-\sigma} h^c(x)\mathrm{d}x\right]^{1/c},$$

where ξ *is the constant factor "product of two integrals" given by*

$$\begin{aligned} \xi := &\left[\int_0^1 k\left(\frac{1-x}{x}\right)(1-x)^{(\sigma-3)/a} x^{(\sigma-3)/b}\mathrm{d}x\right] \times \\ &\left[\int_0^1 \ell\left(\frac{x}{1-x}\right) x^{\sigma-2+(3-\sigma)/c}(1-x)^{(\sigma-3)/c}\mathrm{d}x\right], \end{aligned} \tag{3}$$

provided that all the integrals involved exist.

Proof. For this proof, let us consider the following triple integral:

$$A := \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^\sigma} k\left(\frac{x}{y}\right) \ell\left(\frac{x+y}{z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{4}$$

For the rest, we suggest an original line with multiple changes of variables. Applying first the change of variables $(x, y, z) = (r^2, s^2, t^2)$ with $(r, s, t) \in [0, +\infty)^3$ and the Jacobian given by $J(r, s, t) = 8rst$, we get

$$\begin{aligned} A &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(r^2)g(s^2)h(t^2)}{(r^2+s^2+t^2)^\sigma} k\left(\frac{r^2}{s^2}\right) \ell\left(\frac{r^2+s^2}{t^2}\right) |J(r,s,t)|\mathrm{d}r\mathrm{d}\mathrm{d}t \\ &= 8\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(r^2)g(s^2)h(t^2)}{(r^2+s^2+t^2)^\sigma} k\left(\frac{r^2}{s^2}\right) \ell\left(\frac{r^2+s^2}{t^2}\right) rst\ \mathrm{d}r\mathrm{d}s\mathrm{d}t. \end{aligned}$$

On the basis of this new expression, we now consider the spherical change of variables (r, s, t), classically given by $r = \rho\sin(\varphi)\cos(\theta)$, $s = \rho\sin(\varphi)\sin(\theta)$,

$t = \rho\cos(\varphi)$, with $(\rho,\theta,\varphi) \in [0,+\infty) \times [0,\pi/2]^2$ and the Jacobian given by $J_\star(\rho,\theta,\varphi) = -\rho^2\sin(\varphi)$. With this change and some standard simplifications, including $\cos^2(x)+\sin^2(x) = 1$, $\tan(x) = \sin(x)/\cos(x)$ and $\mathrm{cotan}(x) = 1/\tan(x)$, we get

$$\begin{aligned}
A &= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{+\infty} \frac{f[\rho^2\sin^2(\varphi)\cos^2(\theta)]g[\rho^2\sin^2(\varphi)\sin^2(\theta)]h[\rho^2\cos^2(\varphi)]}{[\rho^2\sin^2(\varphi)\cos^2(\theta)+\rho^2\sin^2(\varphi)\sin^2(\theta)+\rho^2\cos^2(\varphi)]^\sigma}\times\\
&\quad k\left[\frac{\rho^2\sin^2(\varphi)\cos^2(\theta)}{\rho^2\sin^2(\varphi)\sin^2(\theta)}\right]\ell\left[\frac{\rho^2\sin^2(\varphi)\cos^2(\theta)+\rho^2\sin^2(\varphi)\sin^2(\theta)}{\rho^2\cos^2(\varphi)}\right]\times\\
&\quad \rho\sin(\varphi)\cos(\theta)\rho\sin(\varphi)\sin(\theta)\rho\cos(\varphi)|J_\star(\rho,\theta,\varphi)|\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\
&= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{+\infty} f[\rho^2\sin^2(\varphi)\cos^2(\theta)]g[\rho^2\sin^2(\varphi)\sin^2(\theta)]h[\rho^2\cos^2(\varphi)]\times\\
&\quad k\left[\mathrm{cotan}^2(\theta)\right]\ell\left[\tan^2(\varphi)\right]\rho^{5-2\sigma}\sin^2(\varphi)\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)\ \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\
&= 8\int_0^{\pi/2}\int_0^{\pi/2} k\left[\mathrm{cotan}^2(\theta)\right]\ell\left[\tan^2(\varphi)\right]\sin^2(\varphi)\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)\times\\
&\quad \left[\int_0^{+\infty}\rho^{5-2\sigma}f[\rho^2\sin^2(\varphi)\cos^2(\theta)]g[\rho^2\sin^2(\varphi)\sin^2(\theta)]h[\rho^2\cos^2(\varphi)]\ \mathrm{d}\rho\right]\mathrm{d}\theta\mathrm{d}\varphi.
\end{aligned}$$

Note that we voluntarily keep the form $\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)$ in the integral to prepare for future changes of variables with respect to θ and φ.

Taking into account the last expression of A, using $1/a+1/b+1/c = 1$, the generalized Hölder integral inequality applied with the parameters a, b and c gives

$$\begin{aligned}
A &\leq 8\int_0^{\pi/2}\int_0^{\pi/2} k\left[\mathrm{cotan}^2(\theta)\right]\ell\left[\tan^2(\varphi)\right]\sin^2(\varphi)\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)\times\\
&\quad \left\{\int_0^{+\infty}\rho^{5-2\sigma}f^a[\rho^2\sin^2(\varphi)\cos^2(\theta)]\mathrm{d}\rho\right\}^{1/a}\times\\
&\quad \left\{\int_0^{+\infty}\rho^{5-2\sigma}g^b[\rho^2\sin^2(\varphi)\sin^2(\theta)]\mathrm{d}\rho\right\}^{1/b}\times\\
&\quad \left\{\int_0^{+\infty}\rho^{5-2\sigma}h^c[\rho^2\cos^2(\varphi)]\mathrm{d}\rho\right\}^{1/c}\mathrm{d}\theta\mathrm{d}\varphi\\
&= 8\int_0^{\pi/2}\int_0^{\pi/2} k\left[\mathrm{cotan}^2(\theta)\right]\ell\left[\tan^2(\varphi)\right]\sin^2(\varphi)\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)\times\\
&\quad B(\theta,\varphi)\mathrm{d}\theta\mathrm{d}\varphi,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
B(\theta,\varphi) := &\left\{\int_0^{+\infty}\rho_1^{5-2\sigma}f^a[\rho_1^2\sin^2(\varphi)\cos^2(\theta)]\mathrm{d}\rho_1\right\}^{1/a}\times\\
&\left\{\int_0^{+\infty}\rho_2^{5-2\sigma}g^b[\rho_2^2\sin^2(\varphi)\sin^2(\theta)]\mathrm{d}\rho_2\right\}^{1/b}\times
\end{aligned}$$

$$\left\{\int_0^{+\infty} \rho_3^{5-2\sigma} h^c[\rho_3^2 \cos^2(\varphi)]\mathrm{d}\rho_3\right\}^{1/c},$$

with three different notations for ρ to examine the integrals independently. Applying the three (independent) changes of variables $\mu_1 = \rho_1^2 \sin^2(\varphi)\cos^2(\theta)$, $\mu_2 = \rho_2^2 \sin^2(\varphi)\sin^2(\theta)$ and $\mu_3 = \rho_3^2\cos^2(\varphi)$ with $(\mu_1, \mu_2, \mu_3) \in [0, +\infty)^3$, we have

$$B(\theta,\varphi) = \left\{\int_0^{+\infty} \left[\frac{\mu_1}{\sin^2(\varphi)\cos^2(\theta)}\right]^{2-\sigma} f^a(\mu_1)\frac{1}{2\sin^2(\varphi)\cos^2(\theta)}\mathrm{d}\mu_1\right\}^{1/a} \times$$
$$\left\{\int_0^{+\infty} \left[\frac{\mu_2}{\sin^2(\varphi)\sin^2(\theta)}\right]^{2-\sigma} g^b(\mu_2)\frac{1}{2\sin^2(\varphi)\sin^2(\theta)}\mathrm{d}\mu_2\right\}^{1/b} \times$$
$$\left\{\int_0^{+\infty} \left[\frac{\mu_3}{\cos^2(\varphi)}\right]^{2-\sigma} h^c(\mu_3)\frac{1}{2\cos^2(\varphi)}\mathrm{d}\mu_3\right\}^{1/c}$$
$$= \frac{1}{2^{1/a+1/b+1/c}}[\sin^2(\varphi)]^{(\sigma-3)(1/a+1/b)}[\cos^2(\theta)]^{(\sigma-3)/a}[\sin^2(\theta)]^{(\sigma-3)/b}\times$$
$$[\cos^2(\varphi)]^{(\sigma-3)/c}\left[\int_0^{+\infty} \mu_1^{2-\sigma} f^a(\mu_1)\mathrm{d}\mu_1\right]^{1/a}\left[\int_0^{+\infty} \mu_2^{2-\sigma} g^b(\mu_2)\mathrm{d}\mu_2\right]^{1/b}\times$$
$$\left[\int_0^{+\infty} \mu_3^{2-\sigma} h^c(\mu_3)\mathrm{d}\mu_3\right]^{1/c}.$$

Note that we voluntarily keep the forms $[\cos^2(\theta)]^*$, $[\sin^2(\theta)]^*$, $[\cos^2(\varphi)]^*$ and $[\sin^2(\varphi)]^*$, where $*$ denotes a certain exponent, in the integral to prepare for future changes of variables with respect to θ and φ.

If we use $1/a + 1/b + 1/c = 1$ and standardize the notation, we can write

$$(6)\quad B(\theta,\varphi) = \frac{1}{2}[\sin^2(\varphi)]^{(\sigma-3)(1-1/c)}[\cos^2(\theta)]^{(\sigma-3)/a}[\sin^2(\theta)]^{(\sigma-3)/b}[\cos^2(\varphi)]^{(\sigma-3)/c}\times$$
$$\left[\int_0^{+\infty} x^{2-\sigma} f^a(x)\mathrm{d}x\right]^{1/a}\left[\int_0^{+\infty} x^{2-\sigma} g^b(x)\mathrm{d}x\right]^{1/b}\left[\int_0^{+\infty} x^{2-\sigma} h^c(x)\mathrm{d}x\right]^{1/c}.$$

Combining Equations (5) and (6), we obtain:

(7)
$$A \le 8\int_0^{\pi/2}\int_0^{\pi/2} k\left[\mathrm{cotan}^2(\theta)\right]\ell\left[\tan^2(\varphi)\right]\sin^2(\varphi)\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)\times$$
$$\frac{1}{2}[\sin^2(\varphi)]^{(\sigma-3)(1-1/c)}[\cos^2(\theta)]^{(\sigma-3)/a}[\sin^2(\theta)]^{(\sigma-3)/b}[\cos^2(\varphi)]^{(\sigma-3)/c}\times$$
$$\left[\int_0^{+\infty} x^{2-\sigma} f^a(x)\mathrm{d}x\right]^{1/a}\left[\int_0^{+\infty} x^{2-\sigma} g^b(x)\mathrm{d}x\right]^{1/b}\left[\int_0^{+\infty} x^{2-\sigma} h^c(x)\mathrm{d}x\right]^{1/c}\mathrm{d}\theta\mathrm{d}\varphi$$
$$= \left[\int_0^{+\infty} x^{2-\sigma} f^a(x)\mathrm{d}x\right]^{1/a}\left[\int_0^{+\infty} x^{2-\sigma} g^b(x)\mathrm{d}x\right]^{1/b}\left[\int_0^{+\infty} x^{2-\sigma} h^c(x)\mathrm{d}x\right]^{1/c}\times$$
$$C \times D,$$

where

$$C := \int_0^{\pi/2} k\left[\text{cotan}^2(\theta)\right][\cos^2(\theta)]^{(\sigma-3)/a}[\sin^2(\theta)]^{(\sigma-3)/b}[2\cos(\theta)\sin(\theta)]\mathrm{d}\theta$$

and

$$D := \int_0^{\pi/2} \ell\left[\tan^2(\varphi)\right][\sin^2(\varphi)]^{\sigma-2+(3-\sigma)/c}[\cos^2(\varphi)]^{(\sigma-3)/c}[2\cos(\varphi)\sin(\varphi)]\mathrm{d}\varphi.$$

Applying the change of variables $u = \sin^2(\theta)$ with $u \in [0,1]$ and $\mathrm{d}u = 2\cos(\theta)\sin(\theta)\mathrm{d}\theta$, we obtain

$$\begin{aligned} C &= \int_0^{\pi/2} k\left[\frac{1-\sin^2(\theta)}{\sin^2(\theta)}\right][1-\sin^2(\theta)]^{(\sigma-3)/a}[\sin^2(\theta)]^{(\sigma-3)/b}[2\cos(\theta)\sin(\theta)]\mathrm{d}\theta \\ &= \int_0^1 k\left(\frac{1-u}{u}\right)(1-u)^{(\sigma-3)/a}u^{(\sigma-3)/b}\mathrm{d}u. \end{aligned}$$

With the standardization of the notation, we can write

$$C = \int_0^1 k\left(\frac{1-x}{x}\right)(1-x)^{(\sigma-3)/a}x^{(\sigma-3)/b}\mathrm{d}x. \tag{8}$$

Similarly, with the change of variables $v = \sin^2(\varphi)$ with $v \in [0,1]$ and $\mathrm{d}v = 2\cos(\varphi)\sin(\varphi)\mathrm{d}\varphi$, we get

$$\begin{aligned} D = &\int_0^{\pi/2} \ell\left[\frac{\sin^2(\varphi)}{1-\sin^2(\varphi)}\right][\sin^2(\varphi)]^{\sigma-2+(3-\sigma)/c}[1-\sin^2(\varphi)]^{(\sigma-3)/c}\times \\ &[2\cos(\varphi)\sin(\varphi)]\mathrm{d}\varphi \\ = &\int_0^1 \ell\left(\frac{v}{1-v}\right)v^{\sigma-2+(3-\sigma)/c}(1-v)^{(\sigma-3)/c}\mathrm{d}v. \end{aligned}$$

With the standardization of the notation, we can write

$$D = \int_0^1 \ell\left(\frac{x}{1-x}\right)x^{\sigma-2+(3-\sigma)/c}(1-x)^{(\sigma-3)/c}\mathrm{d}x. \tag{9}$$

Combining Equations (4), (7), (8) and (9), we finally obtain the desired result, the constant ξ corresponding to the product $C \times D$. This concludes the proof of Proposition 2.1. □

Proposition 2.1 thus provides a very general trivariate Hardy-Hilbert integral inequality that can be modulated with different functions and parameters. We can see that ξ depends on the product of special integral transformations of k and ℓ, similar to what we can see for modified beta transformations. We recall that the standard beta function is given by $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

Obviously, by taking k and ℓ as constant functions equal to 1, we must find the inequality in Equation (1). To check this, Proposition 2.1 gives

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z)^{\sigma}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$\leq \xi\left[\int_0^{+\infty}x^{2-\sigma}f^a(x)\mathrm{d}x\right]^{1/a}\left[\int_0^{+\infty}x^{2-\sigma}g^b(x)\mathrm{d}x\right]^{1/b}\left[\int_0^{+\infty}x^{2-\sigma}h^c(x)\mathrm{d}x\right]^{1/c},$$

where, by Equation (3),

$$\begin{aligned}\xi &= \left[\int_0^1(1-x)^{(\sigma-3)/a}x^{(\sigma-3)/b}\mathrm{d}x\right]\left[\int_0^1 x^{\sigma-2+(3-\sigma)/c}(1-x)^{(\sigma-3)/c}\mathrm{d}x\right]\\
&= B\left[\frac{\sigma-3}{b}+1,\frac{\sigma-3}{a}+1\right]B\left[\sigma-1+\frac{3-\sigma}{c},\frac{\sigma-3}{c}+1\right]\\
&= \frac{\Gamma\left[(\sigma-3)/b+1\right]\Gamma\left[(\sigma-3)/a+1\right]}{\Gamma\left[(\sigma-3)(1-1/c)+2\right]}\frac{\Gamma\left[\sigma-1+(3-\sigma)/c\right]\Gamma\left[(\sigma-3)/c+1\right]}{\Gamma(\sigma)}\\
&= \frac{1}{\Gamma(\sigma)}\Gamma\left(\frac{\sigma-3}{a}+1\right)\Gamma\left(\frac{\sigma-3}{b}+1\right)\Gamma\left(\frac{\sigma-3}{c}+1\right).\end{aligned}$$

The constant ϵ defined in Equation (2) is obtained as expected. In this case, we know that it is optimal (see [21]).

Note that, in the general and much more complex setting of Proposition 2.1, where two more functions are involved, such an optimality result seems very difficult to establish. It may be possible for some simple and fixed functions k and ℓ. We do not develop this claim further, it requires additional work, which we leave for the future.

As another example, more original and sophisticated than the constant case for k and ℓ, let us consider the power functions $k(x)=x^{\theta}$ and $\ell(x)=x^{\nu}$. It follows from Proposition 2.1 that

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z)^{\sigma}}\left(\frac{x}{y}\right)^{\theta}\left(\frac{x+y}{z}\right)^{\nu}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$\leq \xi\left[\int_0^{+\infty}x^{2-\sigma}f^a(x)\mathrm{d}x\right]^{1/a}\left[\int_0^{+\infty}x^{2-\sigma}g^b(x)\mathrm{d}x\right]^{1/b}\left[\int_0^{+\infty}x^{2-\sigma}h^c(x)\mathrm{d}x\right]^{1/c},$$

where, by Equation (3), using the beta function and its relation to the gamma function,

$$\begin{aligned}\xi &= \left[\int_0^1\left(\frac{1-x}{x}\right)^{\theta}(1-x)^{(\sigma-3)/a}x^{(\sigma-3)/b}\mathrm{d}x\right]\times\\
&\quad\left[\int_0^1\left(\frac{x}{1-x}\right)^{\nu}x^{\sigma-2+(3-\sigma)/c}(1-x)^{(\sigma-3)/c}\mathrm{d}x\right]\\
&= \left[\int_0^1(1-x)^{(\sigma-3)/a+\theta}x^{(\sigma-3)/b-\theta}\mathrm{d}x\right]\left[\int_0^1 x^{\sigma-2+\nu+(3-\sigma)/c}(1-x)^{(\sigma-3)/c-\nu}\mathrm{d}x\right]\end{aligned}$$

$$= B\left[\frac{\sigma-3}{b}-\theta+1, \frac{\sigma-3}{a}+\theta+1\right] B\left[\sigma-1+\nu+\frac{3-\sigma}{c}, \frac{\sigma-3}{c}-\nu+1\right]$$
$$= \frac{\Gamma\left[(\sigma-3)/b-\theta+1\right]\Gamma\left[(\sigma-3)/a+\theta+1\right]}{\Gamma\left[(\sigma-3)(1-1/c)+2\right]}\times$$
$$\frac{\Gamma\left[\sigma-1+\nu+(3-\sigma)/c\right]\Gamma\left[(\sigma-3)/c-\nu+1\right]}{\Gamma(\sigma)}$$
$$= \frac{1}{\Gamma(\sigma)\Gamma\left[\sigma-1+(3-\sigma)/c\right]}\times$$
$$\Gamma\left(\frac{\sigma-3}{a}+\theta+1\right)\Gamma\left(\frac{\sigma-3}{b}-\theta+1\right)\Gamma\left(\frac{\sigma-3}{c}-\nu+1\right)\times$$
$$\Gamma\left(\sigma-1+\nu+\frac{3-\sigma}{c}\right),$$

provided that all linear combinations of the parameters involved in the gamma function are strictly positive. As a more concrete numerical example, for the case $\sigma = 1$, $a = b = c = 3$, and $\theta = \nu = 1/6$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{x+y+z}\left[\frac{x(x+y)}{yz}\right]^{1/6}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$\leq \xi\left[\int_0^{+\infty} xf^3(x)\mathrm{d}x\right]^{1/3}\left[\int_0^{+\infty} xg^3(x)\mathrm{d}x\right]^{1/3}\left[\int_0^{+\infty} xh^3(x)\mathrm{d}x\right]^{1/3},$$

where

$$\xi = \frac{1}{\Gamma(1)\Gamma\left[1-1+(3-1)/3\right]}\times$$
$$\Gamma\left(\frac{1-3}{3}+\frac{1}{6}+1\right)\Gamma\left(\frac{1-3}{3}-\frac{1}{6}+1\right)\Gamma\left(\frac{1-3}{3}-\frac{1}{6}+1\right)\times$$
$$\Gamma\left(1-1+\frac{1}{6}+\frac{3-1}{3}\right) = \frac{1}{\Gamma(2/3)}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{6}\right)^2\Gamma\left(\frac{5}{6}\right)\approx 45.778986.$$

An infinite number of other examples of functions k and ℓ can be examined in a similar way, beyond power functions and the parameter setting above.

In the next section, we show some consequences of our main result. In particular, we offer new trivariate integral inequalities of independent interest.

3. ADDITIONAL RESULTS

The result below is about some triple integral inequalities, which can be derived from Proposition 2.1, under special conditions.

PROPOSITION 3.1. *For any $f, g, h, k, \ell : [0, +\infty) \mapsto [0, +\infty)$ and $(a, b, c) \in (1, +\infty)^3$ such that*

- $1/a+1/b=1/2$, *we have*

$$\int_0^{+\infty}\left[\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y+z)^2}k\left(\frac{x}{y}\right)\ell\left(\frac{x+y}{z}\right)\mathrm{d}x\mathrm{d}y\right]^2\mathrm{d}z$$
$$\leq \xi^2\left[\int_0^{+\infty}f^a(x)\mathrm{d}x\right]^{2/a}\left[\int_0^{+\infty}g^b(x)\mathrm{d}x\right]^{2/b},$$

- $1/b+1/c=1/2$, *we have*

$$\int_0^{+\infty}\left[\int_0^{+\infty}\int_0^{+\infty}\frac{g(y)h(z)}{(x+y+z)^2}k\left(\frac{x}{y}\right)\ell\left(\frac{x+y}{z}\right)\mathrm{d}y\mathrm{d}z\right]^2\mathrm{d}x$$
$$\leq \xi^2\left[\int_0^{+\infty}g^b(x)\mathrm{d}x\right]^{2/b}\left[\int_0^{+\infty}h^c(x)\mathrm{d}x\right]^{2/c},$$

- $1/a+1/c=1/2$, *we have*

$$\int_0^{+\infty}\left[\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)h(z)}{(x+y+z)^2}k\left(\frac{x}{y}\right)\ell\left(\frac{x+y}{z}\right)\mathrm{d}x\mathrm{d}z\right]^2\mathrm{d}y$$
$$\leq \xi^2\left[\int_0^{+\infty}f^a(x)\mathrm{d}x\right]^{2/a}\left[\int_0^{+\infty}h^c(x)\mathrm{d}x\right]^{2/c},$$

where, for each case, ξ is (exactly) given in Equation (3), *provided that all the integrals involved exist.*

Proof. Let us prove only the first point (the others can be proved in a similar way, more details will be given later). We introduce the following operator depending on f and g (assuming that k and ℓ are fixed in advance):

$$T[f,g](z):=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y+z)^2}k\left(\frac{x}{y}\right)\ell\left(\frac{x+y}{z}\right)\mathrm{d}x\mathrm{d}y.$$

On the one hand, we can immediately see that

$$\begin{aligned}&\int_0^{+\infty}\left[\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y+z)^2}k\left(\frac{x}{y}\right)\ell\left(\frac{x+y}{z}\right)\mathrm{d}x\mathrm{d}y\right]^2\mathrm{d}z\\&=\int_0^{+\infty}T^2[f,g](z)\mathrm{d}z=\int_0^{+\infty}T^2[f,g](x)\mathrm{d}x,\end{aligned}\tag{10}$$

where the last expression is a standardization of the notation.

On the other hand, the following triple integral is obtained naturally by using the expression $T[f,g](z)$:

$$\begin{aligned}&\int_0^{+\infty}T^2[f,g](z)\mathrm{d}z=\int_0^{+\infty}T[f,g](z)T[f,g](z)\mathrm{d}z\\&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)T[f,g](z)}{(x+y+z)^2}k\left(\frac{x}{y}\right)\ell\left(\frac{x+y}{z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned}\tag{11}$$

Applying Proposition 2.1 to the functions f, g and $T[f,g]$, and parameters $\sigma = 2$ and $c = 2$, justifying that $1/a + 1/b = 1 - 1/c = 1/2$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)T[f,g](z)}{(x+y+z)^2} k\left(\frac{x}{y}\right) \ell\left(\frac{x+y}{z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\leq \xi \left[\int_0^{+\infty} f^a(x)\mathrm{d}x\right]^{1/a} \left[\int_0^{+\infty} g^b(x)\mathrm{d}x\right]^{1/b} \left\{\int_0^{+\infty} T^2[f,g](x)\mathrm{d}x\right\}^{1/2},
\end{aligned} \tag{12}$$

where ξ is given in Equation (3). From Equations (11) and (12) and a simplification of the exponent of the main integral term, we get

$$\left\{\int_0^{+\infty} T^2[f,g](x)\mathrm{d}x\right\}^{1/2} \leq \xi \left[\int_0^{+\infty} f^a(x)\mathrm{d}x\right]^{1/a} \left[\int_0^{+\infty} g^b(x)\mathrm{d}x\right]^{1/b},$$

so that

$$\int_0^{+\infty} T^2[f,g](x)\mathrm{d}x \leq \xi^2 \left[\int_0^{+\infty} f^a(x)\mathrm{d}x\right]^{2/a} \left[\int_0^{+\infty} g^b(x)\mathrm{d}x\right]^{2/b}.$$

Using Equation (10), the desired inequality is obtained.

For the second and third points, the proofs are similar; we just need to consider the following operators with respect of x and y instead of $T[f,g](z)$, respectively:

$$U[g,h](x) := \int_0^{+\infty}\int_0^{+\infty} \frac{g(y)h(z)}{(x+y+z)^2} k\left(\frac{x}{y}\right) \ell\left(\frac{x+y}{z}\right) \mathrm{d}y\mathrm{d}z$$

and

$$V[f,h](y) := \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)h(z)}{(x+y+z)^2} k\left(\frac{x}{y}\right) \ell\left(\frac{x+y}{z}\right) \mathrm{d}x\mathrm{d}z.$$

The details are omitted for the sake of redundancy. This ends the proof of Proposition 3.1. $\square$

Proposition 3.1 can be used from an operator theory perspective, some special cases of trivariate integral inequalities being clearly less developed than the bivariate cases.

The result below is a variant of Proposition 2.1, inspired by the original trivariate integral inequality in [21]. It allows us to consider more general functions f, g and h in terms of support.

Proposition 3.2. *For any $\alpha \in \mathbb{R}$, $f, g, h : [\alpha, +\infty) \mapsto [0, +\infty)$, $k, \ell : [0, +\infty) \mapsto [0, +\infty)$, $\sigma \in \mathbb{R}$ and $(a, b, c) \in (1, +\infty)^3$ such that $1/a + 1/b + 1/c = 1$, we have*

$$\begin{aligned}
&\int_\alpha^{+\infty}\int_\alpha^{+\infty}\int_\alpha^{+\infty} \frac{f(x)g(y)h(z)}{(x+y+z-3\alpha)^\sigma} k\left(\frac{x-\alpha}{y-\alpha}\right) \ell\left(\frac{x+y-2\alpha}{z-\alpha}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\leq \xi \left[\int_\alpha^{+\infty} (x-\alpha)^{2-\sigma} f^a(x)\mathrm{d}x\right]^{1/a} \left[\int_\alpha^{+\infty} (x-\alpha)^{2-\sigma} g^b(x)\mathrm{d}x\right]^{1/b} \times
\end{aligned}$$

$$\left[\int_{\alpha}^{+\infty}(x-\alpha)^{2-\sigma}h^{c}(x)\mathrm{d}x\right]^{1/c},$$

where ξ is given by Equation (3), provided that all the integrals involved exist.

Proof. Applying the scaled change of variables $(u,v,w)=(x-\alpha,y-\alpha,z-\alpha)$ with $(u,v,w)\in[0,+\infty)^3$ and the Jacobian $J_{\triangle}(u,v,w)=1$, we have:

$$\begin{aligned}
&\int_{\alpha}^{+\infty}\int_{\alpha}^{+\infty}\int_{\alpha}^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z-3\alpha)^{\sigma}}k\left(\frac{x-\alpha}{y-\alpha}\right)\ell\left(\frac{x+y-2\alpha}{z-\alpha}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&=\int_{0}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{f_*(u)g_*(v)h_*(w)}{(u+v+w)^{\sigma}}k\left(\frac{u}{v}\right)\ell\left(\frac{u+v}{w}\right)|J_{\triangle}(u,v,w)|\mathrm{d}u\mathrm{d}v\mathrm{d}w\\
&=\int_{0}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{f_*(u)g_*(v)h_*(w)}{(u+v+w)^{\sigma}}k\left(\frac{u}{v}\right)\ell\left(\frac{u+v}{w}\right)\mathrm{d}u\mathrm{d}v\mathrm{d}w,
\end{aligned}\tag{13}$$

with $f_*(u)=f(u+\alpha)$, $g_*(v)=g(v+\alpha)$ and $h_*(w)=h(w+\alpha)$. Note that these functions satisfy $f_*,g_*,h_*:[0,+\infty)\mapsto[0,+\infty)$. It follows from Proposition 2.1 applied to these functions and the "backward" scaled change of variables $x=y-\alpha$ in the upper bound that

$$\begin{aligned}
&\int_{0}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{f_*(u)g_*(v)h_*(w)}{(u+v+w)^{\sigma}}k\left(\frac{u}{v}\right)\ell\left(\frac{u+v}{w}\right)\mathrm{d}u\mathrm{d}v\mathrm{d}w\\
&\leq\xi\left[\int_{0}^{+\infty}x^{2-\sigma}f_*^{a}(x)\mathrm{d}x\right]^{1/a}\left[\int_{0}^{+\infty}x^{2-\sigma}g_*^{b}(x)\mathrm{d}x\right]^{1/b}\left[\int_{0}^{+\infty}x^{2-\sigma}h_*^{c}(x)\mathrm{d}x\right]^{1/c}\\
&=\xi\left[\int_{\alpha}^{+\infty}(y-\alpha)^{2-\sigma}f^{a}(y)\mathrm{d}y\right]^{1/a}\left[\int_{\alpha}^{+\infty}(y-\alpha)^{2-\sigma}g^{b}(y)\mathrm{d}x\right]^{1/b}\times\\
&\left[\int_{\alpha}^{+\infty}(y-\alpha)^{2-\sigma}h^{c}(y)\mathrm{d}y\right]^{1/c},
\end{aligned}\tag{14}$$

where ξ is given in Equation (3). Combining Equations (13) and (14) and standardizing the notation, we get the desired inequality. This completes the proof of Proposition 3.2. □

In this form, Proposition 3.2 extends the main result in [21] for the trivariate case, still thanks to the consideration of k and ℓ. We thus gain another dimension with the presence of α in the support of f, g and h, making the main result in Proposition 3.2 more adaptable.

4. CONCLUSION

In this paper, we contribute to the mathematical analysis by establishing a very general trivariate Hardy-Hilbert integral inequality. This generality is mainly characterized by the presence of five adaptable functions and four tuning parameters. A famous result in the literature is thus extended to the trivariate setting. The corresponding proof is also innovative, centered on a special spherical change of variables, which remains an underexplored technique in this area of research. Several examples are given, as well as new triple integral inequalities. Some theoretical results and proof techniques may thus inspire further integral inequalities, including those of dimension greater than or equal to 4.

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