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# EXISTENCE RESULTS FOR FRACTIONAL BOUNDARY VALUE PROBLEM IN A PARTICULAR BANACH SPACE

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**Abstract.** In this paper, we study a differential equation of fractional order in which the nonlinear term depends on the Riemann-Liouville integral of an unknown function. By using both Banach contraction mapping and Schaefer fixed point theorem, an existence and uniqueness result is obtained. We close this work with an illustrative example.

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**Key words.** Reimann-Liouville integral, Green's function, Caputo fractional derivative, fixed-point theorem.

## 1. INTRODUCTION

Differential equations of fractional order became more and more a very attractive area of research, it has attracted the attention of several authors (see [1, 8, 16] and so on). In fact, it has arisen in various fields of science and engineering such as electromagnetism, control, electrochemistry, viscoelasticity, porous media, fluid electricity, probability and statistics etc. For more information, see [6, 7, 9, 12, 13, 14, 15]. We also refer to [10] for additional information on the fractional differential calculus's properties.

The paper [4] discusses the solvability of the nonlinear fractional differential equation in the ordinary Banach space

$$\begin{cases} {}^{c}\mathrm{D}_0^{\alpha} y(t) = f(t, y(t)), \;\; 0 \le t \le T, \;\; 1 < \alpha \le 2 \\ y(0) - y'(0) = \displaystyle\int_0^1 g(s, y(s))\mathrm{d}s, \\ y(1) + y'(1) = \displaystyle\int_0^1 h(s, y(s))\mathrm{d}s. \end{cases}$$

It is worth mentioning that the nonlinear term in this paper is independent of the Riemann-Liouville integral.

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In [11], the aim of their work is to study the existence and uniqueness of solution for nonlinear relation integro-differential equation with boundary condition of the following problem

$$\begin{cases} {}^{\beta}\mathrm{D}^{Lc}\mathrm{D}^{\alpha}x(t) + \lambda x(t) = f\left(t, x(t), I^{\gamma}x(t)\right), \quad t \in [0,T], \quad \lambda \in \mathbb{R}, \\ {}^{Lc}\mathrm{D}^{\alpha}x(0) =^{Lc} \mathrm{D}^{\alpha}x(T) = 0, x(0) = a\displaystyle\int_0^T x(s)\mathrm{d}s + b, \end{cases}$$

where ${}^{\beta}\mathrm{D}$ and ${}^{Lc}\mathrm{D}^{\alpha}$ are Riemann-Liouville fractional derivative and Liouville-Caputo fractional derivative of order $\beta$ and $\alpha$ respectively. Similar existence results are studied in [2, 3, 5].

Motivated by the above works, in this work we investigate the following boundary value problem (BVP):

$$(1) \qquad \begin{cases} {}^{c}\mathrm{D}_0^{\alpha}y(t) = f(t, y(t), \mathrm{I}^{\mu}(\mathrm{y(t)})), \quad 0 \leq \mathrm{t} \leq 1, \\ y(0) + y'(0) = \displaystyle\int_0^1 g(s, y(s))\mathrm{d}s, \\ y(1) + y'(1) = \displaystyle\int_0^1 h(s, y(s))\mathrm{d}s, \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \mu \leq 1$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, ${}^{c}\mathrm{D}_0^{\alpha}$ is the standard Caputo derivative, $\mathrm{I}^{\mu}$ is the Riemann-Liouville integral.

Our work plan is as follows: In section 2 we recall some tools needed to demonstrate our results. The section 3 is devoted to the main results. Illustrative example is given in the last section.

In order to start, let us review the background information from the theory of fractional calculus.

Definition 1.1 ([10]). The Riemann-Liouville fractional integral of order $\alpha$ of the function $f : [0,\infty] \longrightarrow \mathbb{R}$, locally integrable, is defined as

$$I_0^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s,$$

provided that the integral exists.

Definition 1.2 ([10]). The Caputo fractional derivative of order $\alpha$ of the function $f : [0,\infty] \longrightarrow \mathbb{R}$, twice differentiable and its second derivative belongs to $L^1[0,t]$, is defined as

$${}^{c}\mathrm{D}_0^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\mathrm{d}s = \mathrm{I}^{n-\alpha}f^{(n)}(t),$$

such that $n-1 < \alpha \leq n$ and $n = [\alpha]+1$, where $[\alpha]$ denotes the integer part of the real number $\alpha$.

Lemma 1.3 ([10]). *For $\alpha > 0$, the general solution of the fractional differential equation ${}^{c}\mathrm{D}_0^{\alpha}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 0, 1, ..., n-1.$$

In light of Lemma 1.3, it follows that

$$\mathrm{I}^{\alpha}\, {}^{c}\mathrm{D}_0^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$ and $n = [\alpha] + 1$. Further, the Riemann-Liouville fractional differential equation $\mathrm{D}^{\alpha} h(t) = 0$ has a general solution:

$$h(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2} + a_3 t^{\alpha-3} + ... + a_m t^{\alpha-m}, \ \ a_i \in \mathbb{R}, \ \ i = 1, 2, ..., m.$$

THEOREM 1.4 ([18, Banach fixed point theorem]). *Let $\Omega$ be a nonempty closed convex subset of a Banach space $(S, \|.\|)$. Then any contraction mapping* $\mathrm{T}$ *of $\Omega$ into itself has a unique fixed point.*

THEOREM 1.5 ([17, Schaefer fixed point theorem]). *Let $X$ be a Banach space and let $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ be a completely continuous operator. Then either*

i) *the operator equation $x = \lambda \mathrm{Tx}$ has a solution for $\lambda = 1$ or*
ii) *the set $\varepsilon = \{u \in X, u = \lambda \mathrm{Tu}\}$ is unbounded for $\lambda \in (0, 1)$.*

## 2. MAIN RESULTS

In this section, we apply the Banach fixed point theorem to prove an existence and uniqueness result for solution for problem (1).

First of all, we consider a linear problem, and its solution is found in terms of a Green function.

LEMMA 2.1. *Assume $\varphi \in C([0, 1])$ and $1 < \alpha \leq 2$. Then, the solution to the boundary value problem*

$$\begin{cases} {}^{c}\mathrm{D}_0^{\alpha} y(t) = \varphi(t), \ \ 0 \leq t \leq 1, \\ y(0) + y'(0) = \displaystyle\int_0^1 \sigma_1(s) \mathrm{d}s, \\ y(1) + y'(1) = \displaystyle\int_0^1 \sigma_2(s) \mathrm{d}s, \end{cases} \tag{2}$$

*is given by*

$$y(t) = \int_0^1 G(t, s) \varphi(s) \mathrm{d}s + P(t),$$

*where $G(t, s)$ is the Green's function given by the expression*

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} + (1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \textit{if } 0 \leq s \leq t \leq 1, \\[2ex] \dfrac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \textit{if } 0 \leq t \leq s \leq 1. \end{cases}$$

*and*

$$P(t) = (2-t) \int_0^1 \sigma_1(s) \mathrm{d}s + (t-1) \int_0^1 \sigma_2(s) \mathrm{d}s.$$

*Proof.* For the proof of this lemma, we use the same approach as in [4, Lemma 3.4].

Remark 2.2. (i) We can get the estimation:

$$\int_0^1 G(t,s)\varphi(s)\mathrm{d}s = \mathrm{I}^{\alpha}\left(\varphi(\mathrm{s})\right) + (1-\mathrm{t})\left[\mathrm{I}^{\alpha}\left(\varphi(1)\right) + \mathrm{I}^{\alpha-1}\left(\varphi(1)\right)\right].$$

(ii) For each $1<\alpha\leq 2$, one has $|G(s,t)| \leq \dfrac{3}{\Gamma(\alpha)}$.

(iii)
- $\mathrm{I}^{\mu}(1-\mathrm{t}) = \dfrac{1}{\Gamma(\mu+1)}\mathrm{t}^{\mu} - \dfrac{\Gamma(2)}{\Gamma(\mu+2)}\mathrm{t}^{\mu+1}$.
- $\mathrm{I}^{\mu}(2-\mathrm{t}) = \dfrac{2}{\Gamma(\mu+1)}\mathrm{t}^{\mu} - \dfrac{\Gamma(2)}{\Gamma(\mu+2)}\mathrm{t}^{\mu+1}$.

Now, we define the space $X = \{y|\ y \in C([0,1]),\ \mathrm{I}^{\mu}(\mathrm{y}) \in \mathrm{C}([0,1])\}$ endowed with the norm $\|y\|_X = \max\limits_{t\in[0,1]}|y(t)| + \max\limits_{t\in[0,1]}|\mathrm{I}^{\mu}(\mathrm{y}(\mathrm{t}))|$.

Lemma 2.3. $(X, \|, \|_X)$ *has a structure of Banach space.*

*Proof.* Let $\{y_n\}_{n\geq 1}$ be a Cauchy sequence in $(X, \|, \|_X)$. Immediately, we have $\{y_n\}_{n\geq 1}$ and $\{\mathrm{I}^{\mu}(\mathrm{y_n})\}_{\mathrm{n}\geq 1}$ are Cauchy sequences in $C([0,1])$ converge to some $y$ and $v$ on $[0,1]$ uniformly respectively. We need to establish that $v = \mathrm{I}^{\mu}(\mathrm{y})$. Indeed,

$$\begin{aligned} |\mathrm{I}^{\mu}(\mathrm{y_n}(\mathrm{t})) - \mathrm{I}^{\mu}(\mathrm{y}(\mathrm{t}))| &= \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}|y_n(s)-y(s)|\,\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\mu+1)}\|y_n - y\|_{C([0,1])}, \end{aligned}$$

which means that $\mathrm{I}^{\mu}(\mathrm{y_n}) \to \mathrm{I}^{\mu}(\mathrm{y})$ in $C([0,1])$. By uniqueness of limit, we conclude that $v = \mathrm{I}^{\mu}(\mathrm{y})$.

### 2.1. Existence and uniqueness results via Banach's fixed point.

Theorem 2.4. *Assume that there exist constants* $k$, $k^*$ *and* $k^{**}$ *such that*

$(H_1)$ $|f(t,x,y)) - f(t,\overline{x},\overline{y})| \leq k\left[|x-\overline{x}| + |y-\overline{y}|\right]$.
$(H_2)$ $|g(t,x)) - g(t,\overline{x})| \leq k^*|x-\overline{x}|$.
$(H_3)$ $|h(t,x)) - h(t,\overline{x})| \leq k^{**}|x-\overline{x}|$.

*For each* $t\in[0,1]$, $(x,\overline{x},y,\overline{y}) \in \mathbb{R}^4$. *Then, if*

$$\begin{aligned} &\left[\frac{3}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+\mu+1)} + \frac{1}{\Gamma(\mu+1)}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right]\right]k \\ &+ \left(1 + \frac{1}{\Gamma(\mu+1)}\right)\left[2k^* + k^{**}\right] < 1, \end{aligned} \tag{3}$$

*then the boundary value problem (1) has a unique solution.*

*Proof.* We will transform the problem (1) into a fixed point problem. Consider the operator $\mathrm{F} : (X, \|.\|_X) \to (X, \|.\|_X)$, defined as

$$\mathrm{F}(y(t)) = \int_0^1 G(t,s) f(s, y(s), \mathrm{I}^\mu(y(s))\mathrm{d}s + Q(t),$$

where $Q(t) = (2-t)\int_0^1 g(s,y(s))\mathrm{d}s + (t-1)\int_0^1 h(s,y(s))\mathrm{d}s$. Then, by Lemma 2.1, $y$ is a solution of BVP (1) if, and only if, $y$ is a fixed point of the operator F. We prove that F is a contraction.

Let $y, \overline{y} \in X$. Then, for any $t \in [0,1]$, one has:

$$\begin{aligned}
&|\mathrm{F}(y(t)) - \mathrm{F}(\overline{y}(t))| \le \int_0^1 |G(t,s)| \left| f(s, y(s), \mathrm{I}^\mu(y(s)) - f(s, \overline{y}(s), \mathrm{I}^\mu(\overline{\mathrm{y}}(\mathrm{s}))\right| \mathrm{d}s \\
&+(2-t)\int_0^1 |g(s,y(s)) - g(s,\overline{y}(s))| \,\mathrm{d}s \\
&+(1-t)\int_0^1 |h(s,y(s)) - h(s,\overline{y}(s))| \,\mathrm{d}s \\
&\le \frac{3}{\Gamma(\alpha)} k \left[\|y-\overline{y}\|_{C([0,1])} + \|\mathrm{I}^\mu(y) - \mathrm{I}^\mu(\overline{y})\|_{C([0,1])}\right] + 2k^* \|y-\overline{y}\|_{C([0,1])} \\
&+k^{**}\|y-\overline{y}\|_{C([0,1])} \le \left[\frac{3}{\Gamma(\alpha)}k + 2k^* + k^{**}\right] \|y-\overline{y}\|_X.
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
&|\mathrm{I}^\mu(\mathrm{F}(\mathrm{y}(\mathrm{t}))) - \mathrm{I}^\mu(\mathrm{F}(\overline{\mathrm{y}}(\mathrm{t})))| \le \mathrm{I}^\mu\left[\mathrm{I}^\alpha \left| f(t,y(t), \mathrm{I}^\mu(y(t)) - f(t,\overline{y}(t), \mathrm{I}^\mu(\overline{y}(t))\right|\right] \\
&+ \mathrm{I}^\mu(1-t) I^\alpha \left| f(1,y(1), I^\mu(y(1)) - f(1,\overline{y}(1), \mathrm{I}^\mu(\overline{y}(1))\right| \\
&+ \mathrm{I}^\mu(1-t)\mathrm{I}^{\alpha-1} \left| f(1,y(1), I^\mu(y(1)) - f(1,\overline{y}(1), \mathrm{I}^\mu(\overline{y}(1))\right| \\
&+ \mathrm{I}^\mu(2-t)\int_0^1 |g(s,y(s)) - g(s,\overline{y}(s))| \,\mathrm{d}s \\
&+ \mathrm{I}^\mu(1-t)\int_0^1 |h(s,y(s)) - h(s,\overline{y}(s))| \,\mathrm{d}s.
\end{aligned}$$

Thus, after making some calculations and using Remark 2.2, we obtain:

$$\begin{aligned}
&|\mathrm{I}^\mu(\mathrm{F}(y(t))) - \mathrm{I}^\mu(\mathrm{F}(\overline{y}(t)))| \\
&\le \left[\frac{k}{\Gamma(\mu+\alpha+1)} + \frac{1}{\Gamma(\mu+1)}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right] k \right. \\
&\left. + \frac{2}{\Gamma(\mu+1)}k^* + \frac{1}{\Gamma(\mu+1)}k^{**}\right] \|y-\overline{y}\|_X.
\end{aligned}$$

In view of (3), F is a contraction, which ends the proof.

**2.2. Existence result via Schaefer's fixed point theorem.** Our second result is based on Schaefer's fixed point theorem.

THEOREM 2.5. *Assume* $(H_1)$-$(H_3)$ *are fulfilled. Then, the boundary value problem (1) admits at least one solution on* $[0,1]$.

*Proof.* We divide the proof in several steps.

**Step 1.** F is continuous. Let $\{y_n\}_{n\geq 1}$ be a sequence that converges to $y$ in $(X, \|, \|_X)$. Then for any $t \in [0,1]$, one has:

$$\begin{aligned}
&|\mathrm{F}(y_n(t)) - \mathrm{F}(y(t))| \\
&\leq \int_0^1 |G(t,s)|\, |f(s, y_n(s), \mathrm{I}^\mu(y_n(s)) - f(s, y(s), \mathrm{I}^\mu((s))|\, \mathrm{d}s \\
&+(2-t)\int_0^1 |g(s, y_n(s)) - g(s, y(s))|\, \mathrm{d}s \\
&+(1-t)\int_0^1 |h(s, y_n(s)) - h(s, y(s))|\, \mathrm{d}s \\
&\leq \frac{3}{\Gamma(\alpha)}\int_0^1 |f(s, y_n(s), \mathrm{I}^\mu(y_n(s)) - f(s, y(s), \mathrm{I}^\mu(y(s))|\, \mathrm{d}s \\
&+2\int_0^1 |g(s, y_n(s)) - g(s, y(s))|\, \mathrm{d}s + \int_0^1 |h(s, y_n(s)) - h(s, y(s))|\, \mathrm{d}s.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|\mathrm{I}^\mu(\mathrm{F}(y_n(t))) - \mathrm{I}^\mu(\mathrm{F}(y(t)))| \\
&\leq \mathrm{I}^\mu\left[\mathrm{I}^\alpha |f(s, y_n(s), \mathrm{I}^\mu(y_n(s)) - f(s, y(s), \mathrm{I}^\mu(y(s))|\right] \\
&+ \mathrm{I}^\mu(1-t)\mathrm{I}^\alpha |f(1, y_n(1), \mathrm{I}^\mu(y_n(1)) - f(1, y(1), \mathrm{I}^\mu(y(1))| \\
&+ \mathrm{I}^\mu(1-t)\mathrm{I}^{\alpha-1} |f(1, y_n(1), \mathrm{I}^\mu(y_n(1)) - f(1, y(1), \mathrm{I}^\mu(y(1))| \\
&+ \mathrm{I}^\mu(2-t)\int_0^1 |g(s, y_n(s)) - g(s, y(s))|\, \mathrm{d}s \\
&+ \mathrm{I}^\mu(1-)\int_0^1 |h(s, y_n(s)) - h(s, y(s))|\, \mathrm{d}s.
\end{aligned}$$

We deduce from above that

$$\begin{aligned}
&|\mathrm{I}^\mu(\mathrm{F}(y_n(t))) - \mathrm{I}^\mu(\mathrm{F}(y(t)))| \leq \mathrm{I}^{\mu+\alpha}\left[|f(s, y_n(s), \mathrm{I}^\mu(y_n(s)) - f(s, y(s), \mathrm{I}^\mu(y(s))|\right] \\
&+ \left[\frac{1}{\Gamma(\mu+1)}t^\mu - \frac{\Gamma(2)}{\Gamma(\mu+2)}t^{\mu+1}\right]\left[\mathrm{I}^\alpha |f(1, y_n(1), \mathrm{I}^\mu(y_n(1)) - f(1, y(1), \mathrm{I}^\mu(y(1))|\right. \\
&+\mathrm{I}^{\alpha-1} |f(1, y_n(1), \mathrm{I}^\mu(y_n(1)) - f(1, y(1), \mathrm{I}^\mu(y(1))|\Big] \\
&+\mathrm{I}^{\alpha-1} |f(1, y_n(1), \mathrm{I}^\mu(y_n(1)) - f(1, y(1), \mathrm{I}^\mu(y(1))|\Big] \\
&+ \left[\frac{2}{\Gamma(\mu+1)}t^\mu - \frac{\Gamma(2)}{\Gamma(\mu+2)}t^{\mu+1}\right]\int_0^1 |g(s, y_n(s)) - g(s, y(s))|\, \mathrm{d}s \\
&+ \left[\frac{1}{\Gamma(\mu+1)}t^\mu - \frac{\Gamma(2)}{\Gamma(\mu+2)}t^{\mu+1}\right]\int_0^1 |h(s, y_n(s)) - h(s, y(s))|\, \mathrm{d}s.
\end{aligned}$$

Since $f, g$ and $h$ are continuous functions, we deduce that $\|\mathrm{F}y_n - \mathrm{F}y\|_X$ tends to zero when $n$ goes to $\infty$.

**Step 2.** F maps bounded sets into bounded sets in $X$. Denote $N = \sup_{s\in[0,1]} |f(s,0,0)| < \infty$, $N^* = \sup_{s\in[0,1]} |g(s,0,0)| < \infty$, $N^{**} = \sup_{s\in[0,1]} |h(s,0,0)| <$

$\infty$ and $B_\eta = \{y \in X; \|y\|_X \leq \eta\}$. We have to show that there exists $l$ such that $\|\mathrm{Fy}\|_X \leq l$.

By $(H_1)$-$(H_3)$, one has:

$$\begin{aligned} |\mathrm{F}(y)(t)| &\leq \int_0^1 |G(s,t)||f(s,y(s),\mathrm{I}^\mu(y(s))|\mathrm{d}s + (2-t)\int_0^1 |g(s,y(s))|\,\mathrm{d}s \\ &+ (1-t)\int_0^1 |h(s,y(s))|\,\mathrm{d}s \\ &\leq \frac{3}{\Gamma(\alpha)}\left[k\|y\|_X + N\right] + 2\left[k^*\|y\|_X + N^*\right] + \left[k^{**}\|y\|_X + N^{**}\right] \\ &\leq \frac{3}{\Gamma(\alpha)}\left[k\eta + N\right] + 2\left[k^*\eta + N^*\right] + \left[k^{**}\eta + N^{**}\right] = l_1. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\mathrm{I}^\mu\left(\mathrm{F}(y(t))\right)| &\leq \mathrm{I}^\mu\left[\mathrm{I}^\alpha\left|f(t,y(t),\mathrm{I}^\mu(y(t))\right|\right] + \mathrm{I}^\mu(1-t)\mathrm{I}^\alpha\left|f(1,y(1),\mathrm{I}^\mu(y(1))\right| \\ &+ \mathrm{I}^\mu(1-t)\mathrm{I}^{\alpha-1}\left|f(1,y(1),\mathrm{I}^\mu(y(1))\right| + \mathrm{I}^\mu(2-t) \\ &\times \int_0^1 |g(s,y(s))|\,\mathrm{d}s + \mathrm{I}^\mu(1-t)\int_0^1 |h(s,y(s))|\,\mathrm{d}s \\ &\leq \left[\frac{1}{\Gamma(\mu+\alpha+1)} + \frac{1}{\Gamma(\mu+1)}\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right]\right] \\ &\left[k\eta + N\right] + \frac{2}{\Gamma(\mu+1)}\left[k^*\eta + N^*\right] \\ &+ \frac{1}{\Gamma(\mu+1)}\left[k^{**}\eta + N^{**}\right] = l_2. \end{aligned}$$

Hence, $\|\mathrm{F}(y)\|_X \leq l$, where $l = l_1 + l_2$.

**Step 3.** F maps bounded sets into equicontinuous sets in $X$. For $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$ and for $y \in B_\eta$, we get:

$$\begin{aligned} |\mathrm{F}(y(t_2)) - \mathrm{F}(y(t_1))| &\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s),\mathrm{I}^\mu(y(s))\mathrm{d}s\right. \\ &\left.-\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s),\mathrm{I}^\mu(\mathrm{y(s)})\mathrm{d}s\right| + (t_2-t_1)\left|\mathrm{I}^\alpha\mathrm{f}(1,\mathrm{y}(1),\mathrm{I}^\mu(\mathrm{y}(1))\right| \\ &+ (t_2-t_1)\left|\mathrm{I}^{\alpha-1}\mathrm{f}(1,\mathrm{y}(1),\mathrm{I}^\mu(\mathrm{y}(1))\right| + (t_2-t_1)\int_0^1 |g(s,y(s))|\,\mathrm{d}s \\ &+ (t_2-t_1)\int_0^1 |h(s,y(s))|\,\mathrm{d}s, \end{aligned}$$

which yields

$$|\mathrm{F}(y(t_2)) - \mathrm{F}(y(t_1))| \leq \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_2-s)^{\alpha-1}f(s,y(s),\mathrm{I}^\mu(y(s))\mathrm{d}s\right.$$

$$
\begin{aligned}
&- \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, y(s), \mathrm{I}^{\mu}(y(s)) \mathrm{d}s \Big| \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} \left| f(s, y(s), \mathrm{I}^{\mu}(y(s)) \right| \mathrm{d}s \\
&+ (t_2 - t_1) \left| \mathrm{I}^{\alpha} f(1, y(1), \mathrm{I}^{\mu}(y(1)) \right| + (t_2 - t_1) \left| \mathrm{I}^{\alpha-1} f(1, y(1), \mathrm{I}^{\mu}(y(1)) \right| \\
&+ (t_2 - t_1) \int_0^1 |g(s, y(s))| \, \mathrm{d}s + (t_2 - t_1) \int_0^1 |h(s, y(s))| \, \mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left( (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) [k \|y\|_X + N] + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} \\
&[k \|y\|_X + N] + (t_2 - t_1) \left| \mathrm{I}^{\alpha} f(1, y(1), \mathrm{I}^{\mu}(y(1)) \right| \\
&+ (t_2 - t_1) \left| \mathrm{I}^{\alpha-1} f(1, y(1), \mathrm{I}^{\mu}(y(1)) \right| \\
&+ (t_2 - t_1) [k^* \|y\|_X + N^*] + (t_2 - t_1) [k^{**} \|y\|_X + N^{**}] \,.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Moreover, we also have:

$$
\begin{aligned}
&|\mathrm{I}^{\mu}(\mathrm{F}(y(t_2))) - \mathrm{I}^{\mu}\mathrm{F}(y(t_1))| \leq \\
&\left[ \mathrm{I}^{\alpha+\mu} f(t_2, y(t_2), \mathrm{I}^{\mu} y(t_2)) - \mathrm{I}^{\alpha+\mu} f(t_1, y(t_1), \mathrm{I}^{\mu} y(t_1)) \right] \\
&+ \left[ \left| \frac{1}{\Gamma(\mu+1)} t_2^{\mu} - \frac{1}{\Gamma(\mu+1)} t_1^{\mu} \right| + \left| \frac{\Gamma(2)}{\Gamma(\mu+2)} t_2^{\mu+1} - \frac{\Gamma(2)}{\Gamma(\mu+2)} t_1^{\mu+1} \right| \right] \\
&\times \left[ \mathrm{I}^{\alpha} f(1, y(1), \mathrm{I}^{\mu} y(1)) + \mathrm{I}^{\alpha-1} f(1, y(1), \mathrm{I}^{\mu} y(1)) \right] \\
&+ \left[ \left| \frac{2}{\Gamma(\mu+1)} t_2^{\mu} - \frac{2}{\Gamma(\mu+1)} t_1^{\mu} \right| + \left| \frac{\Gamma(2)}{\Gamma(\mu+2)} t_2^{\mu+1} - \frac{\Gamma(2)}{\Gamma(\mu+2)} t_1^{\mu+1} \right| \right] \\
&\times \int_0^1 |g(s, y(s))| \, \mathrm{d}s \\
&+ \left[ \left| \frac{1}{\Gamma(\mu+1)} t_2^{\mu} - \frac{1}{\Gamma(\mu+1)} t_1^{\mu} \right| + \left| \frac{\Gamma(2)}{\Gamma(\mu+2)} t_2^{\mu+1} - \frac{\Gamma(2)}{\Gamma(\mu+2)} t_1^{\mu+1} \right| \right] \\
&\times \int_0^1 |h(s, y(s))| \, \mathrm{d}s.
\end{aligned}
$$

Clearly, the right-hand side of the above estimation goes to zero when $t_1 \to t_2$.

**Step 4.** A priori bounds. It remains to show that the set $\varepsilon = \{y \in X, y = \lambda \mathrm{Fy}\}$ is bounded for $\lambda \in (0, 1)$. By the same manner in step 2, we find $\|y\|_X \leq C$. As a result of Schaefer's fixed point theorem, F has at least one fixed point which is the solution of problem (1) on $[0, 1]$, this ends the proof.

## 3. EXAMPLE

Let us consider the following fractional boundary problem:

$$(4)\qquad \begin{cases} {}^{c}\mathrm{D}_0^{\frac{3}{2}}y(t) &= \dfrac{1}{t^2+10}\left[\sin y(t) + \mathrm{I}^{\frac{1}{4}}\mathrm{y(t)}\right], \quad 0 \le t \le 1, \\ y(0)+y'(0) &= \dfrac{1}{2000}\displaystyle\int_0^1 \arctan y(s)\mathrm{d}s, \\ y(1)+y'(1) &= \displaystyle\int_0^1 \frac{e^{-s}}{e^s+3000}\cos y(s)ds. \end{cases}$$

Here, we have:

$$\alpha = 1,5, \quad \mu = 0,25$$
$$f(t,x,y) = \frac{1}{t^2+10}\left[\sin x + y\right],$$
$$g(t,x) = \frac{1}{2000}\arctan x,$$
$$h(t,x) = \frac{\mathrm{e}^{-s}}{\mathrm{e}^s+3000}\cos x.$$

Clearly, $(H_1)$-$(H_3)$ hold with $k = \dfrac{1}{10}$, $k^* = \dfrac{1}{2000}$ and $k^{**} = \dfrac{1}{3001}$. Moreover, we have:

$$\left[\frac{3}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+\mu+1)} + \frac{1}{\Gamma(\mu+1)}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right]\right]k$$
$$+\left(1+\frac{1}{\Gamma(\mu+1)}\right)\left[2k^* + k^{**}\right] \simeq 0,41 < 1.$$

Hence, Theorem 2.4 implies that problem (4) has a unique solution.

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