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# STUDY OF THE STRUCTURE OF NEAR-RINGS WITH ADDITIVE MAPS

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**Abstract.** In this article, we study the structure of 3-prime near-rings through the action of derivations and left multipliers. It should be noted that the combination of these types of maps allows us to obtain more precise results. Thus, and under appropriate additional assumptions, the commutativity and properties of a near-ring are discussed. Furthermore, an example is given to illustrate that the 3-primeness hypothesis cannot be omitted.

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**Key words.** Prime near-rings, derivations, commutativity.

## 1. INTRODUCTION

Throughout this paper, $\mathcal{N}$ represents a left near-ring and $Z(\mathcal{N})$ its multiplicative center, that is, $Z(\mathcal{N}) = \{x \in \mathcal{N} \mid xy = yx \text{ for all } y \in \mathcal{N}\}$ and $\mathcal{N}$ will be 3-prime, if $\mathcal{N}$ verifies the following property: $x\mathcal{N}y = \{0\}$ for $x, y \in \mathcal{N}$ implies $x = 0$ or $y = 0$. Also, $\mathcal{N}$ is said to be 2-torsion free if $(\mathcal{N}, +)$ has no elements of order 2. For any pair of elements $x, y \in \mathcal{N}$, $[x, y]$ denotes the commutator $xy - yx$, while the symbol $x \circ y$ denotes the anticommutator $xy + yx$. A near-ring $\mathcal{N}$ is called a zero-symmetric if $0.x = 0$ for all $x \in \mathcal{N}$ (recall that left distributivity yields that $x.0 = 0$). Note that in the case of the left near-ring, $0.x$ is not necessarily zero, and $-(xy) = x(-y)$ not $(-x)y$. Therefore, near-rings are generalized rings, need not be commutative, and most importantly, only one distributive law is postulated. As for terminologies used here without mention, we refer to Pilz [9].

An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently , as noted in Wang [11], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. An additive mapping $H : \mathcal{N} \longrightarrow \mathcal{N}$ is called a left multiplier if $H(xy) = H(x)y$ holds for all $x, y \in \mathcal{N}$. Under the same assumption, $H$ is said to be a right multiplier if it verifies $H(xy) = xH(y)$ for all $x, y \in \mathcal{N}$ and if $H$ is both a left multiplier and a right multiplier, then $H$ is said to be a multiplier on $\mathcal{N}$.

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During the last few decades, several authors have studied the relationship between the commutativity of a near-ring $\mathcal{N}$ and certain special types of maps defined on $\mathcal{N}$ (see, [1–3, 6, 7, 10], where further references can be found). In this context, the first demonstrated result is due to Bell and Mason [4], when they proved that a 2-torsion-free 3-prime near-ring $\mathcal{N}$ must be a commutative ring if $\mathcal{N}$ admits a non-trivial derivation $d$ for which $d(\mathcal{N}) \subseteq Z(\mathcal{N})$ or $d$ is a commuting derivation on $\mathcal{N}$.

In 1992, Daif and Bell [8] showed that a prime ring $\mathcal{R}$ must be commutative if it admits a derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y \in \mathcal{R}$. Later, in 2004, Ashraf et al. [1] studied the commutativity of a 2-torsion free prime near-ring $\mathcal{N}$ equipped with a derivation $d$ verifying $d([x, y]) = 0$ for all $x, y \in \mathcal{N}$. Also, in 2011, A. Boua and L.Oukhtite [5] proved that a prime near-ring $\mathcal{N}$ must be a commutative ring if $\mathcal{N}$ admits a derivation $d$ satisfying $d([x, y]) = [x, y]$ for all $x, y \in \mathcal{N}$. Since prime rings are the setting in which derivations and multipliers have been most fruitfully studied, our goal is to investigate some identities by considering prime near-rings with derivations and left multipliers without the "2-torsion freeness" hypothesis.

## 2. PRELIMINARY LEMMAS

In this section, we present some well-known results and add a new lemma that is very important for developing the proofs of our main results.

LEMMA 2.1. *Let $\mathcal{N}$ be a prime near-ring.*

(i) [8, Lemma 2] *Let $d$ be a derivation on $\mathcal{N}$. If $x \in Z(\mathcal{N})$, then $d(x) \in Z(\mathcal{N})$.*

(ii) [3, Theorem 2.1] *If $\mathcal{N}$ admits a nonzero derivation $d$ for which $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(iii) [3, Lemma 1.5] *If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

LEMMA 2.2 ([3, Theorem 2.1]). *Let $\mathcal{N}$ be a prime near-ring. If $\mathcal{N}$ admits a nonzero derivation $d$ for which $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

We proceed by proving the key lemma.

LEMMA 2.3. *Let $\mathcal{N}$ be a 3-prime near-ring, $d$ be a derivation of $\mathcal{N}$ and $H$ be a left multiplier of $\mathcal{N}$.*

(i) *If $a \in Z(\mathcal{N})$ and $d([x, y]) - H([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring or $d(a) = 0$.*

(ii) *If $a \in Z(\mathcal{N})$ and $d(x \circ y) - H(x \circ y) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring of characteristic 2 or $d(a) = 0$.*

*Proof.* (i) Let $a \in Z(\mathcal{N})$ and suppose that

$$d([x, y]) - H([x, y]) = 0 \text{ for all } x, y \in \mathcal{N}. \tag{1}$$

Taking $x = ax$ in the preceding equation, we find that

$$d(a[x, y]) = H(a[x, y]) = H([x, y]a) \text{ for all } x, y \in \mathcal{N}.$$

that is,

$$d(a)[x,y]+ad([x,y])=H([x,y])a \text{ for all } x,y\in\mathcal{N}.$$

In view of (1), the last result shows that

$$d(a)[x,y]=0 \text{ for all } x,y\in\mathcal{N}. \tag{2}$$

Left multiplying (2) by $r$, where $r\in\mathcal{N}$, and using Lemma 2.1(i), we obtain

$$d(a)r[x,y]=0 \text{ for all } x,y,r\in\mathcal{N},$$

which reduces to

$$d(a)\mathcal{N}[x,y]=\{0\} \text{ for all } x,y\in\mathcal{N}.$$

In the light of the 3-primeness of $\mathcal{N}$, the latter expression shows that

$$[x,y]=0 \text{ or } d(a)=0 \text{ for all } x,y\in\mathcal{N},$$

and therefore, $\mathcal{N}$ is a commutative ring or $d(a)=0$.

(ii) Let $a\in Z(\mathcal{N})$ and assume that

$$d(x\circ y)-H(x\circ y)=0 \text{ for all } x,y\in\mathcal{N}. \tag{3}$$

Putting $ax$ instead of $x$ in (3) and using the same techniques as above, we arrive at

$$d(a)\mathcal{N}(x\circ y)=\{0\} \text{ for all } x,y\in\mathcal{N}.$$

By the 3-primeness of $\mathcal{N}$, we conclude that

$$x\circ y=0 \text{ or } d(a)=0 \text{ for all } x,y\in\mathcal{N}.$$

Clearly, the first condition implies that $xy=-yx$ for all $x,y\in\mathcal{N}$. Replacing $y$ by $yt$, where $t\in\mathcal{N}$, we obtain $xyt=y(-x)t=yt(-x)$ which means that $y[t,-x]=0$ for all $x,y,t\in\mathcal{N}$. It follows that $[t,-x]y[t,-x]=0$ and hence $[t,-x]\mathcal{N}[t,-x]=\{0\}$ for all $t,x\in\mathcal{N}$. In view of the 3-primeness of $\mathcal{N}$, the last result shows that $\mathcal{N}$ is a commutative ring, which forces that $2\mathcal{N}=\{0\}$. □

## 3. MAIN RESULT

Since [8, Theorem 1], [1, Theorem 2.2], and [5, Theorem 4.1] are actually results obtained using the derivations, if we consider them here in terms of derivation combined with a left multiplier, we get the following result:

THEOREM 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $d$ be a derivation of $\mathcal{N}$, and $H$ be a left multiplier of $\mathcal{N}$. Then the following assertions are equivalent:*

(i) $d([x,y])-H([x,y])=0$ *for all* $x,y\in\mathcal{N}$,
(ii) $d([x,y])-[H(x),y])=0$ *for all* $x,y\in\mathcal{N}$,
(iii) $d([x,y])-[x,H(y)])=0$ *for all* $x,y\in\mathcal{N}$,
(iv) $\mathcal{N}$ *is a commutative ring.*

*Proof.* (i)⇒(iv) Assume that

$$d([x,y]) - H([x,y]) = 0 \text{ for all } x,y \in \mathcal{N}. \tag{4}$$

It follows that

$$d([x,y]) = H([x,y]) \text{ for all } x,y \in \mathcal{N}. \tag{5}$$

Taking $yx$ instead of $y$ in (5), we get

$$yd([x,y]) + d(y)[x,y] - H(y)[x,y] = 0 \text{ for all } x,y \in \mathcal{N}. \tag{6}$$

Replacing $y$ by $[u,v]$ in (6), we find that

$$[u,v]d([x,[u,v]]) + d([u,v])[x,[u,v]] - H([u,v])[x,[u,v]] = 0 \text{ for all } u,v,x \in \mathcal{N}.$$

From (4), the last equation can be written as

$$[u,v]H([x,[u,v]]) = 0 \text{ for all } u,v,x \in \mathcal{N}, \tag{7}$$

which leads to

$$[u,v]H(x)[u,v] - [u,v]H([u,v])x = 0 \text{ for all } u,v,x \in \mathcal{N}. \tag{8}$$

For $x = xt$, equation (8) gives

$$[u,v]H(x)[t,[u,v]] = 0 \text{ for all } u,v,x,t \in \mathcal{N}. \tag{9}$$

Taking $x = xz$ in (9), we obtain

$$[u,v]H(x)z[t,[u,v]] = 0 \text{ for all } u,v,x,t,z \in \mathcal{N}.$$

That is,

$$[u,v]H(x)\mathcal{N}[t,[u,v]] = \{0\} \text{ for all } u,v,x,t \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, the above relation leads to

$$[u,v]H(x) = 0 \text{ or } [t,[u,v]] = 0 \text{ for all } u,v,x,t \in \mathcal{N}. \tag{10}$$

Suppose there exist two elements $u_0, v_0 \in \mathcal{N}$ such that $[u_0,v_0]H(x) = 0$ for all $x \in \mathcal{N}$. This implies, in view of (4), that

$$[u_0,v_0]d([r,t]) = 0 \text{ for all } r,t \in \mathcal{N}. \tag{11}$$

According to (7), we have

$$[u,v]d([x,[u,v]]) = 0 \text{ for all } u,v,x \in \mathcal{N}.$$

Equivalently,

$$[u,v]d(x[u,v]) = [u,v]d([u,v]x) \text{ for all } u,v,x \in \mathcal{N},$$

which implies that

$$[u,v]d(x)[u,v] + [u,v]xd([u,v]) = [u,v]^2d(x) + [u,v]d([u,v])x, \tag{12}$$

for all $u,v,x \in \mathcal{N}$.

Replacing $x$ with $[r,t]$ and $[u,v] = [u_0,v_0]$ in (12), by (11) we infer that

$$\big([u_0,v_0]rt - [u_0,v_0]tr\big)d([u_0,v_0]) = 0 \text{ for all } r,t \in \mathcal{N}.$$

For $t = H(yz)$ and using the fact that $[u_0, v_0]H(y) = 0$ for all $y, z \in \mathcal{N}$, we can see that

$$[u_0, v_0]rH(y)zd([u_0, v_0]) = 0 \text{ for all } r, y, z \in \mathcal{N}.$$

By the 3-primeness of $\mathcal{N}$, we obtain

$$[u_0, v_0] = 0 \text{ or } d([u_0, v_0]) = 0 \text{ or } H = 0.$$

From the last expression, equation (10) forces

$$(13) \qquad [u, v] = 0 \text{ or } d([u, v]) = 0 \text{ or } [u, v] \in Z(\mathcal{N}) \text{ for all } u, v \in \mathcal{N}.$$

If there exist $u_0, v_0 \in \mathcal{N}$ such that $[u_0, v_0] \in Z(\mathcal{N})$, then $d([u_0, v_0]) = 0$ or $\mathcal{N}$ is a commutative ring by Lemma 2.3(i). In this case, (13) becomes $d([u, v]) = 0$ for all $u, v \in \mathcal{N}$ or $\mathcal{N}$ is a commutative ring. In view of [10, Theorem 2.3], the first condition yields that $\mathcal{N}$ is a commutative ring. Consequently, in both cases, $\mathcal{N}$ is a commutative ring.

Let's show (ii)⇒(iv). For this, assume that

$$(14) \qquad d([x, y]) - [H(x), y] = 0 \text{ for all } x, y \in \mathcal{N}.$$

Substituting $xy$ for $y$ in the (14) and using it again, we obtain

$$d(x)[x, y] + x[H(x), y] = [H(x), xy] \text{ for all } x, y \in \mathcal{N}$$

which leads to

$$(15) \quad d(x)[x, y] + xH(x)y - xyH(x) = H(x)xy - xyH(x) \text{ for all } x, y \in \mathcal{N}$$

On the other hand, from (14) we can show that $H(x)x = xH(x)$ for all $x \in \mathcal{N}$; and therefore (15) yields $d(x)[x, y] = 0$ for all $x, y \in \mathcal{N}$. In such a way that,

$$(16) \qquad d(x)xy = d(x)yx \text{ for all } x, y \in \mathcal{N}.$$

Now, putting $yt$ instead of $y$ in (16) and using it again, we arrive at $d(x)y[x, t] = 0$ for all $x, y, t \in \mathcal{N}$. So that, $d(x)\mathcal{N}[x, t] = \{0\}$ for all $x, t \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, the last result proves that for each $x \in \mathcal{N}$, we have either $d(x) = 0$ or $d(x) \in Z(\mathcal{N})$. Since the first condition assures also $d(x) \in Z(\mathcal{N})$, we conclude that $d(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. Accordingly, $\mathcal{N}$ is a commutative ring by Lemma 2.1(ii).

Using the same proof as above, we can prove the implication (iii)⇒(iv).

Implications (iv)⇒(i), (iv)⇒(ii) and (iv)⇒(iii) are obvious. □

We now state some consequences of the previous theorem, just to take $H = \mathrm{id}_{\mathcal{N}}$ or $H = 0$.

Corollary 3.2 ([8, Theorem 1]). *If $\mathcal{R}$ is a prime ring admitting a derivation $d$ satisfying $d([x, y]) = [x, y]$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Corollary 3.3 ([5, Theorem 2.2]). *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 3.4 ([1, Theorem 4.1]). *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation $d$ such that $d([x,y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

The conclusion of Theorem 3.1 remains valid if we replace the product $[x, y]$ by $x \circ y$. Indeed, we obtain the following result:

Theorem 3.5. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a derivation $d$ and a left multiplier $H$ such that $d \neq 0$ or $H \neq 0$, then the following assertions are equivalent:*

(i) $d(x \circ y) - H(x \circ y) = 0$ *for all* $x, y \in \mathcal{N}$,
(ii) $\mathcal{N}$ *is a commutative ring of characteristics* 2 .

*Proof.* (i)⇒(ii). We will discuss according to the values of $d$ and $H$. Firstly, suppose that $H \neq 0$ and

$$d(x \circ y) - H(x \circ y) = 0 \text{ for all } x, y \in \mathcal{N}. \tag{17}$$

Replacing $y$ by $xy$ in (17), we get

$$d(x(x \circ y)) - H(x(x \circ y)) = 0 \text{ for all } x, y \in \mathcal{N}. \tag{18}$$

Substituting $u \circ v$ for $x$ in (18) and using (17), we obtain

$$(u \circ v)H((u \circ v) \circ y) = 0 \text{ for all } u, v, y \in \mathcal{N}. \tag{19}$$

It follows that

$$(u \circ v)H(u \circ v)y = -(u \circ v)H(y)(u \circ v) \text{ for all } u, v, y \in \mathcal{N}. \tag{20}$$

Putting $yrt$ instead of $y$ in (20) and using it again, we infer that

$$\big(-(u \circ v)H(yr)(u \circ v)\big)t = -(u \circ v)H(yr)t(u \circ v) \text{ for all } u, v, y, r, t \in \mathcal{N},$$

which can be written as

$$(u \circ v)H(y)r[-(u \circ v), t] = 0 \text{ for all } u, v, y, r, t \in \mathcal{N}.$$

Hence,

$$(u \circ v)H(y)\mathcal{N}[-(u \circ v), t] = \{0\} \text{ for all } u, v, y, t \in \mathcal{N}.$$

In view of the 3-primeness of $\mathcal{N}$, it follows that

$$(u \circ v)H(y) = 0 \text{ or } -(u \circ v) \in Z(\mathcal{N}) \text{ for all } u, v, y \in \mathcal{N}. \tag{21}$$

Suppose that there are two elements $u_0, v_0 \in \mathcal{N}$ which satisfy

$$(u_0 \circ v_0)H(y) = 0 \text{ for all } y \in \mathcal{N}, \tag{22}$$

then from (20) and (17), we obtain

$$(u_0 \circ v_0)d((u_0 \circ v_0) \circ y) = 0 \text{ for all } y \in \mathcal{N}.$$

Using the property defining of $d$, we get

$$(u_0 \circ v_0)d((u_0 \circ v_0)y + (u_0 \circ v_0)d(y(u_0 \circ v_0)) = 0 \text{ for all } y \in \mathcal{N}.$$

Thereby, for all $y \in \mathcal{N}$, we have

$$(u_0 \circ v_0)d(u_0 \circ v_0)y + (u_0 \circ v_0)^2 d(y) + (u_0 \circ v_0)d(y)(u_0 \circ v_0) + (u_0 \circ v_0)yd(u_0 \circ v_0) = 0.$$

Replacing $y$ by $r \circ s$ in the preceding expression and applying (17) together (22), we find that

$$(u_0 \circ v_0)(r \circ s)d(u_0 \circ v_0) = 0 \text{ for all } r, s \in \mathcal{N}.$$

It follows that

$$\big((u_0 \circ v_0)rs + (u_0 \circ v_0)sr\big)d(u_0 \circ v_0) = 0 \text{ for all } r, s \in \mathcal{N}.$$

Now, taking $r = H(rt)$ and invoking (22), we obtain

$$(u_0 \circ v_0)sH(r)td(u_0 \circ v_0) = 0 \text{ for all } r, s, t \in \mathcal{N}$$

which implies that

$$(u_0 \circ v_0)\mathcal{N}H(r)\mathcal{N}d(u_0 \circ v_0) = \{0\} \text{ for all } r \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, the above relation yields

$$(u_0 \circ v_0) = 0 \text{ or } d(u_0 \circ v_0) = 0 \text{ or } H = 0.$$

Consequently, (21) shows that

$$d(u \circ v) = 0 \text{ or } -(u \circ v) \in Z(\mathcal{N}) \text{ for all } u, v \in \mathcal{N}. \tag{23}$$

that is,

$$d(-(u \circ v)) \in Z(\mathcal{N}) \text{ for all } u, v \in \mathcal{N}.$$

Now, to complete this proof, we discuss according to the nature of the set $d(Z(\mathcal{N}))$.

**Case 1.** If $d(Z(\mathcal{N})) = \{0\}$. In this case, from (23) and the additivity of $d$, it is easily to show that $d(u \circ v) = 0$ for all $u, v \in \mathcal{N}$.

• If $d \neq 0$, based on the proof of [10, Theorem 3.5], more precisely up to relation (3.15), we have $d(u) \in Z(\mathcal{N})$ or $u \in Z(\mathcal{N})$ for all $u \in \mathcal{N}$. But, since in this case $d$ is a derivation and by virtue of Lemma 2.1(i), we get $d(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$ and hence $\mathcal{N}$ is a commutative ring by Lemma 2.2.

So, for all $u, v, t \in \mathcal{N}$, we have

$$d(u \circ tv) = 0 = d(u(tv + tv)) = d(u)t(v + v) + ud(tv + tv). \tag{24}$$

Also, $d(u \circ uv) = 0 = d(u)(u \circ v) + ud(u \circ v) = d(u)v(u + u)$ which implies that $d(u + u) = 0$ for all $u \in \mathcal{N}$. Thereby, (24) shows that $d(u)\mathcal{N}(v + v) = \{0\}$ for all $u, v \in \mathcal{N}$. Since $d \neq 0$, we conclude that $2\mathcal{N} = \{0\}$.

• If $d = 0$, in this case our assumptions forced $H$ to be nonzero. On the other hand, (17) yields $H(u \circ v) = 0$ for all $u, v \in \mathcal{N}$. In such a way that $H(u)v + H(v)u = 0$ for all $u, v \in \mathcal{N}$. Replacing $v$ by $v \circ t$, we get $H(u)vt = -H(u)tv$ for all $u, v, t, r \in \mathcal{N}$. Again, taking $v = -v$ and $t = tr$ in the preceding equation and using it, we arrive at $H(u)t[v, r] = 0$ for all $u, v, t, r \in \mathcal{N}$ which yields $H(u)\mathcal{N}[v, r] = \{0\}$ for all $u, v, r \in \mathcal{N}$.

In view of the 3-primeness of $\mathcal{N}$ and $H \neq 0$, we conclude that $\mathcal{N} \subseteq Z(\mathcal{N})$ and hence $\mathcal{N}$ is a commutative ring by Lemma 2.1(iii). Thus, the expression $H(u \circ v) = 0$ for all $u, v \in \mathcal{N}$ shows that $H(u)\mathcal{N}(2v) = \{0\}$ for all $u, v \in \mathcal{N}$ which implies, because of $H \neq 0$, that $2\mathcal{N} = \{0\}$.

**Case 2.** If $d(Z(\mathcal{N})) \neq \{0\}$, then there exists an element $z_0 \in Z(\mathcal{N})$ such that $d(z_0) \neq 0$ and therefore $\mathcal{N}$ is a commutative ring of characteristic equal 2 by Lemma 2.3(ii).

Consequently, in both cases we find that $\mathcal{N}$ is a commutative ring and $2\mathcal{N} = \{0\}$.

It remains to discuss the case where $H = 0$ and $d \neq 0$. In this case, our hypotheses give $d(x \circ y) = 0$ for all $x, y \in \mathcal{N}$. Using the same reasoning as in **Case 1** above, we find that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$ and from the Lemma 2.1(ii) we get that $\mathcal{N}$ is a commutative ring, which implies that

$$0 = d(2xy) = d(2xyz) = d(2xy)z + (2x)yd(z) = (2x)yd(z),$$

for all $x, y, z \in \mathcal{N}$.

The 3-primeness of $\mathcal{N}$ with $d \neq 0$ force that $2\mathcal{N} = \{0\}$, so $\mathcal{N}$ is a commutative ring of characteristic 2.

Conversely, the proof is obvious. □

As an immediate consequence of the previous theorem, we find the following result:

Corollary 3.6. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation $d$ and a nonzero left multiplier $H$, then the following assertions are equivalent:*

(i) $d(x \circ y) = 0$*, for all* $x, y \in \mathcal{N}$*,*
(ii) $H(x \circ y) = 0$*, for all* $x, y \in \mathcal{N}$*,*
(iii) $d(x \circ y) - x \circ y = 0$*, for all* $x, y \in \mathcal{N}$*,*
(iv) $\mathcal{N}$ *is a commutative ring of characteristics* 2 *.*

The following example shows that the 3-primeness of $\mathcal{N}$ cannot be omitted in all our theorems and corollaries.

Example 3.7. Let $S$ be an any noncommutative left near-ring and define $\mathcal{N}$, $d$ and $H$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y \in \mathcal{S} \right\}, \ d \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$H \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we can see that $\mathcal{N}$ is not a 3-prime near-ring, $d$ is a nonzero derivation of $\mathcal{N}$, and $H$ is a nonzero left multiplier of $\mathcal{N}$ which satisfies all the identities of our theorems and corollaries. But, $\mathcal{N}$ is a noncommutative ring.

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