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BLOW-UP OF SOLUTIONS FOR A VISCOELASTIC KIRCHHOFF EQUATION WITH A SOURCE, DELAY AND BALAKRISHNAN-TAYLOR DAMPING TERMS

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Abstract. A nonlinear viscoelastic Kirchhoff-type equation with a source, Balakrishnan-Taylor damping, dispersion and delay terms is studied. We prove the blow-up of solutions under suitable hypotheses.

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Key words. Kirchhoff equation, blow-up, delay term, viscoelastic term, source term.

1. INTRODUCTION

Let $\mathcal{H} = \Omega \times (0, \tau) \times (0, \infty)$, in the present work, we consider the following Kirchhoff equation

$$(1)\quad \begin{cases} |u_t|^\gamma u_{tt} - M(t)\Delta u(t) + \int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho - \Delta u_{tt}(t) \\ \qquad +\beta_1 |u_t(t)|^{m-2}u_t(t) + \beta_2 |u_t(t-\tau)|^{m-2}u_t(t-\tau) = ku|u|^{p-2}. \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \text{ in } \Omega \\ u_t(x, t-\tau) = f_0(x, t-\tau), \ \text{ in } \Omega \times (0,\tau) \\ u(x,t) = 0, \ \text{ in } \partial\Omega \times (0,\infty) \end{cases}$$

where

$$M(t) := \left(\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\right),$$

and $\Omega \in \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $p \geq 2$, ζ_0, ζ_1, σ, β_1, k are positive constants and β_2 is a real number. $\gamma \geq 0$ for $n = 1, 2$, and $0 \leq \gamma \leq \frac{4}{n-2}$ for $n \geq 3$. $m \geq 2$ for $n = 1, 2$, and $2 \leq m \leq \frac{n+2}{n-2}$ for $n \geq 3$. h is a positive function.

Physically, the relationship between the stress and strain history in the beam is inspired by Boltzmann theory called viscoelastic damping term, where the kernel of the term of memory is the function h. See [4, 7, 12–16, 20].

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From a mathematical point of view, the effect on the movement of vertically moving viscoelastic strings consisting of two different materials (such as electric wires) depends especially on the acceleration. This effect is represented by $|u_t|^\gamma u_{tt}$, where $|u_t|^\gamma$ is the material density, varying the velocity.

In [2], Balakrishnan and Taylor proposed a new model of damping called the Balakrishnan-Taylor damping, as it relates to the span problem and the plate equation. For more depth, here are some papers that are focused on the study of this damping [2, 3, 6, 7, 15, 16, 19, 21].

The effect of the delay often appears in many applications and practical problems and turns a lot of systems into different problems worth studying. Recently, the stability and the asymptotic behavior of evolution systems with time delay has been studied by many authors. See [5, 7, 8, 10, 12, 13, 16, 22].

The great importance of the source term in physics is that they appear in several issues and theories. It is also used in many applications such, e.g. optical applications. Many researchers also touched on this type of problem in several different issues, where the global existence of solutions, stability and explosion of solutions were studied. For more information, the reader is referred to [1, 5, 9, 11, 17, 23, 24].

We believe that based on all of the above, the combination of these terms of damping (memory term, Balakrishnan-Taylor damping, source, dispersion and the delay terms) in one particular problem with the addition of the delay term $(\beta_2||u_t(t-\tau)|^{m-2}u_t(t-\tau))$, constitutes a new problem worthy of study and research, a new problem different from the above, on which we will try to shed light on.

Our paper is divided into several sections. In the next section we lay down the hypotheses, concepts and lemmas we need. In Section 3, we state and prove the blow up of solutions.

2. PRELIMINARIES

For studying our problem, in this section we will need some materials. Firstly, we introduce the following hypothesis for β_2 and h:

(A1) $h : \mathbb{R}_+ \to \mathbb{R}_+$ are non-increasing C^1 functions satisfying

$$h(t) > 0, \;\; \zeta_0 - \int_0^\infty h(\varrho)\mathrm{d}\varrho = l > 0.$$

(A2)

$$|\beta_2| < \beta_1.$$

Let us introduce

$$(h \circ \psi)(t) := \int_\Omega \int_0^t h(t-\varrho)|\psi(t) - \psi(\varrho)|^2 \mathrm{d}\varrho \mathrm{d}x.$$

As in [22], we take the following new variables

$$y(x,\rho,t) = u_t(x,t-\tau\rho), \ \ (x,\rho,t) \in \Omega \times (0,1) \times \mathbb{R}_+,$$

which satisfy

$$\begin{cases} \tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0, \\ y(x,0,t) = u_t(x,t). \end{cases} \tag{2}$$

So, problem (1) can be written as

$$\begin{cases} |u_t|^\gamma u_{tt} - M(t)\Delta u(t) + \int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho - \Delta u_{tt}(t) \\ \quad +\beta_1|u_t(t)|^{m-2}u_t(t) + \beta_2|y(x,1,t)|^{m-2}y(x,1,t) = ku|u|^{p-2}. \\ \tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0. \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \text{ in } \ \Omega \\ y(x,\rho,0) = f_0(x,-\tau\rho), \ \text{ in } \ \Omega \times (0,1) \\ u(x,t) = 0, \ \text{ in } \ \partial\Omega \times (0,\infty), \end{cases} \tag{3}$$

where

$$(x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

Now, we give the energy functional.

Lemma 2.1. *The energy functional E, defined by*

$$\begin{aligned} E(t) =& \frac{1}{\gamma+2}\|u_t\|_{\gamma+2}^{\gamma+2} + \frac{1}{2}\left(\zeta_0 - \int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u(t)\|_2^2 \\ &+ \frac{1}{2}\|\nabla u_t(t)\|_2^2 + \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4 + \frac{1}{2}(h \circ \nabla u)(t) - \frac{k}{p}\|u(t)\|_p^p \\ &+ \frac{\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m \mathrm{d}\rho, \end{aligned} \tag{4}$$

satisfies

$$\begin{aligned} E'(t) \leq & - C_0\left(\|u_t(t)\|_m^m + \|y(x,1,t)\|_m^m\right) + \frac{1}{2}(h' \circ \nabla u)(t) \\ & - \frac{1}{2}h(t)\|\nabla u(t)\|_2^2 - \frac{\sigma}{4}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\nabla u(t)\|_2^2\right\}\right)^2 \leq 0, \end{aligned} \tag{5}$$

where $\xi > 0$ satisfies

$$\tau(m-1)|\beta_2| \leq \xi \leq \tau(m\beta_1 - |\beta_2|).$$

Proof. Taking the inner product of $(3)_1$ with u_t and then integrating over Ω, we find

$$
\begin{aligned}
&(|u_t|^{\gamma}u_{tt}(t),u_t(t))_{L^2(\Omega)}-(M(t)\Delta u(t),u_t(t))_{L^2(\Omega)}-(\Delta u_{tt}(t),u_t(t))_{L^2(\Omega)}\\
&+(\int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho,u_t(t))_{L^2(\Omega)}+\beta_1(|u_t|^{m-2}u_t,u_t)_{L^2(\Omega)}\\
&+\beta_2(|y(x,1,t)|^{m-2}y(x,1,t),u_t(t))_{L^2(\Omega)}-(ku|u|^{p-2},u_t(t))_{L^2(\Omega)}=0.
\end{aligned}
\tag{6}
$$

A direct calculation, gives

$$
(|u_t|^{\gamma}u_{tt}(t),u_t(t))_{L^2(\Omega)} = \frac{1}{\gamma+2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t(t)\|_{\gamma+2}^{\gamma+2}\right),
\tag{7}
$$

and

$$
-(\Delta u_{tt}(t),u_t(t))_{L^2(\Omega)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u_t(t)\|_2^2\right).
$$

Integrating by parts, we find

$$
\begin{aligned}
&-(M(t)\Delta u(t),u_t(t))_{L^2(\Omega)}\\
&=-(\left(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\right)\Delta u(t),u_t(t))_{L^2(\Omega)}\\
&=\left(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\right)\int_\Omega \nabla u(t).\nabla u_t(t)\mathrm{d}x\\
&=\left(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega |\nabla u(t)|^2\mathrm{d}x\right\}\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\left(\zeta_0+\frac{\zeta_1}{2}\|\nabla u\|_2^2\right)\|\nabla u(t)\|_2^2\right\}+\frac{\sigma}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\nabla u(t)\|_2^2\right\}^2,
\end{aligned}
\tag{8}
$$

and we have

$$
\begin{aligned}
&\left(\int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho,u_t(t)\right)_{L^2(\Omega)}\\
&=\int_0^t h(t-\varrho)(\Delta u(\varrho),u_t(t))_{L^2(\Omega)}\mathrm{d}\varrho\\
&=-\int_0^t h(t-\varrho)\left[\int_\Omega \nabla u(x,\varrho)\nabla u(x,t)\mathrm{d}x\right]\mathrm{d}\varrho,
\end{aligned}
$$

and

$$
-\nabla u(x,\varrho).\nabla u(x,t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{|\nabla u(x,\varrho)-\nabla u(x,t)(t)|^2\right\}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{|\nabla u(x,t)|^2\right\},
$$

then

$$
\begin{aligned}
&-\int_0^t h(t-\varrho)(\nabla u(\varrho),\nabla u_t(t))_{L^2(\Omega)}\mathrm{d}\varrho\\
&=-\int_0^t h(t-\varrho)\int_\Omega\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{|\nabla u(x,\varrho)-\nabla u(x,t)|^2\right\}\right]\mathrm{d}x\mathrm{d}s.\\
&\quad-\int_0^t h(t-\varrho)\int_\Omega\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{|\nabla u(x,t)|^2\right\}\right]\mathrm{d}x\mathrm{d}\varrho\\
&=\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2\mathrm{d}x\right\}\right]\mathrm{d}\varrho\\
&\quad-\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\nabla u(x,t)\|_2^2\right\}\right]\mathrm{d}x\mathrm{d}\varrho.
\end{aligned}
\tag{9}
$$

We use **(A1)** and we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2\mathrm{d}x\right\}\right]\mathrm{d}\varrho\\
&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^t h(t-\varrho)\left[\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2\mathrm{d}x\right]\right\}\mathrm{d}\varrho\\
&\quad-\frac{1}{2}\int_0^t h'(t-\varrho)\left[\int_\Omega|\nabla u(x,t)-\nabla u(x,\varrho)|^2\mathrm{d}x\right]\mathrm{d}\varrho\\
&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(h\circ\nabla u)(t)-\frac{1}{2}(h'\circ\nabla u)(t),
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\nabla u(t)\|_2^2\right\}\right]\mathrm{d}x\mathrm{d}\varrho\\
&=-\frac{1}{2}\left(\int_0^t h(t-\varrho)\mathrm{d}\varrho\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\nabla u(t)\|_2^2\right\}\right)\mathrm{d}x\\
&=-\frac{1}{2}\left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\nabla u(t)\|_2^2\right\}\right)\mathrm{d}x\\
&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u(t)\|_2^2\right\}+\frac{1}{2}h(t)\|\nabla u(t)\|_2^2.
\end{aligned}
\tag{11}
$$

By substituting (10) and (11) into (9), we get

$$
\begin{aligned}
&\left(\int_0^t h(t-\varrho)\Delta u(\varrho)\mathrm{d}\varrho,u_t(t)\right)_{L^2(\Omega)}\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}(h\circ\nabla u)(t)-\frac{1}{2}\left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u(t)\|_2^2\right\}\\
&\quad-\frac{1}{2}(h'\circ\nabla u)(t)+\frac{1}{2}h(t)\|\nabla u(t)\|_2^2,
\end{aligned}
\tag{12}
$$

and we have

$$-(ku|u|^{p-2},u_t(t))_{L^2(\Omega)} = -\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{k}{p}\|u(t)\|_p^p\right\}. \tag{13}$$

Now, multiplying the equation $(3)_2$ by $-y\xi$, integrating over $\Omega\times(0,1)$, and by using $(2)_2$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\xi}{m}\int_\Omega\int_0^1 |y(x,\rho,t)|^m \mathrm{d}\rho\mathrm{d}x\\
&= -(\frac{\xi}{\tau})\int_\Omega\int_0^1 |y|^{m-1}y_\rho \mathrm{d}\rho\mathrm{d}x\\
&= -\frac{\xi}{m\tau}\int_\Omega\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\rho}|y(x,\rho,s,t)|^m \mathrm{d}\rho\mathrm{d}x\\
&= \frac{\xi}{m\tau}\int_\Omega\Big(|y(x,0,t)|^m - |y(x,1,t)|^m\Big)\mathrm{d}x\\
&= \frac{\xi}{m\tau}\Big(\int_\Omega |u_t(t)|^m\mathrm{d}x - \int_\Omega |y\,(x,1,t)\,|^m\mathrm{d}x\Big)\\
&= \frac{\xi}{m\tau}\Big(\|u_t(t)\|_m^m - \|y\,(x,1,t)\,\|_m^m\Big),
\end{aligned} \tag{14}$$

By Young's inequality, we have

$$\begin{aligned}
&\beta_2\Big(|y(x,1,t)|^{m-2}y(x,1,t),u_t(t)\Big)_{L^2(\Omega)}\\
&\le \frac{|\beta_2|}{m}\|u_t(t)\|_m^m + \frac{(m-1)|\beta_2|}{m}\|y\,(x,1,t)\,\|_m^m.
\end{aligned} \tag{15}$$

By replacing (7)-(8) and (12)-(15) into (6), we find (4) and (5). where $C_0 = \min\left\{\beta_1 - \frac{\xi}{m\tau} - \frac{|\beta_2|}{m},\ \frac{\xi}{m\tau} - \frac{(m-1)|\beta_2|}{m}\right\}$. This completes of the proof. □

THEOREM 2.2. *Suppose that* **(A1)-(A2)** *are satisfied. Let*

$$\begin{cases} 2 < p < \frac{2n-2}{n-2}, & n \ge 3,\\ p \ge 2, & n = 1,2. \end{cases}$$

Then, for any $u_0, u_1 \in H_0^1(\Omega)\cap L^2(\Omega)$, *and* $f_0 \in L^2(\Omega,(0,1))$, *there exists a weak solution* u *of problem* (3) *such that*

$$\begin{aligned}
&u \in C(]0,T[,H_0^1(\Omega))\cap C^1(]0,T[,L^2(\Omega)),\\
&u_t \in C(]0,T[,H_0^1(\Omega))\cap L^2(]0,T[,L^2(\Omega,(0,1))).
\end{aligned}$$

LEMMA 2.3 ([18]). *There exists a positive constant* $c(\Omega) > 0$, *such that*

$$\Big(\int_\Omega |u|^p\mathrm{d}x\Big)^{\frac{s}{p}} \le c\Big(\int_\Omega |u|^p\mathrm{d}x + \|\nabla u\|_2^2\Big),$$

for any $2 \le s \le p$.

COROLLARY 2.4 ([18]). *There exists a positive constant* $c(\Omega) > 0$*, such that*

$$\|u\|_2^2 \;\leq\; c\Bigg[\bigg(\int_\Omega |u|^p \mathrm{d}x\bigg)^{\frac{2}{p}} + \|\nabla u\|_2^{\frac{4}{p}}\Bigg].$$

3. BLOW UP RESULT

In this section, we prove the blow up result of the solution of problem (3). First, we define the functional

$$\begin{aligned}\mathbb{H}(t) = -E(t) \;&=\; -\frac{1}{\gamma+2}\|u_t\|_{\gamma+2}^{\gamma+2} - \frac{1}{2}\bigg(\zeta_0 - \int_0^t h(\varrho)\mathrm{d}\varrho\bigg)\|\nabla u(t)\|_2^2 \\ &\quad -\frac{1}{2}\|\nabla u_t(t)\|_2^2 - \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4 - \frac{1}{2}(h\circ\nabla u)(t) \\ &\quad +\frac{k}{p}\|u(t)\|_p^p - \frac{\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m \mathrm{d}\rho. \end{aligned} \tag{16}$$

THEOREM 3.1. *Assume* **(A1)-(A2)** *hold, and suppose that* $E(0) < 0$*, Then, the solution of problem* (3) *blows up in finite time.*

Proof. From (5), we have

$$E(t) \leq E(0) \leq 0.$$

Therefore

$$\mathbb{H}'(t) = -E'(t) \;\geq\; C_0\bigg(\|u_t(t)\|_m^m + \|y(x,1,t)\|_m^m\bigg),$$

hence

$$\begin{aligned}&\mathbb{H}'(t) \geq C_0\|u_t(t)\|_m^m \geq 0\\ &\mathbb{H}'(t) \geq C_0\|y(x,1,t)\|_m^m \geq 0.\end{aligned} \tag{17}$$

By (16), we have

$$0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) \;\leq\; \frac{k}{p}\|u|_p^p. \tag{18}$$

We set

$$\mathcal{K}(t) \;=\; \mathbb{H}^{1-\alpha} + \frac{\varepsilon}{\gamma+1}\int_\Omega u|u_t|^\gamma u_t \mathrm{d}x + \varepsilon\int_\Omega \nabla u \nabla u_t \mathrm{d}x + \varepsilon\frac{\sigma}{4}\|\nabla u\|_2^4, \tag{19}$$

where $\varepsilon > 0$ will be assigned later and

$$\frac{2(p-1)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{20}$$

By multiplying $(3)_1$ by u and with a derivative of (19), we get

$$\begin{aligned}\mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\gamma+1}\|u_t\|_{\gamma+2}^{\gamma+2} + \varepsilon\|\nabla u_t\|_2^2 + \varepsilon k\int_\Omega \|u\|_p^p \\ &- \varepsilon\zeta_0\|\nabla u\|_2^2 - \varepsilon\zeta_1\|\nabla u\|_2^4 + \underbrace{\varepsilon\int_\Omega \nabla u\int_0^t h(t-\varrho)\nabla u(\varrho)\mathrm{d}\varrho\mathrm{d}x}_{J_1} \\ &- \underbrace{\varepsilon\beta_1\int_\Omega u.u_t.|u_t|^{m-2}\mathrm{d}x}_{J_2} - \underbrace{\varepsilon\beta_2\int_\Omega u.y(x,1,t).|y(x,1,t)|^{m-2}\mathrm{d}x}_{J_3}.\end{aligned} \tag{21}$$

We have

$$\begin{aligned}J_1 &= \varepsilon\int_0^t h(t-\varrho)\mathrm{d}\varrho\int_\Omega \nabla u.(\nabla u(\varrho)-\nabla u(t))\mathrm{d}x\mathrm{d}\varrho + \varepsilon\int_0^t h(\varrho)\mathrm{d}\varrho\|\nabla u\|_2^2 \\ &\geq \frac{\varepsilon}{2}\left(\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u\|_2^2 - \frac{\varepsilon}{2}(h\circ\nabla u),\end{aligned}$$

and, for $\delta_1, \delta_2 > 0$

$$\begin{aligned}J_2 &\geq -\varepsilon\delta_1\|u\|_2^2 - \varepsilon\frac{c_1}{4\delta_1}\|u\|_m^m, \\ J_3 &\geq -\varepsilon\delta_2\|u\|_2^2 - \varepsilon\frac{c_2}{4\delta_2}\|y(x,1,t)\|_m^m.\end{aligned}$$

From (21), we find

$$\begin{aligned}\mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\gamma+1}\|u_t\|_{\gamma+2}^{\gamma+2} + \varepsilon\|\nabla u_t\|_2^2 + \varepsilon k\|u\|_p^p \\ &- \varepsilon\zeta_1\|\nabla u\|_2^4 - \varepsilon\left[\left(\zeta_0 - \frac{1}{2}\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u\|_2^2 - \frac{\varepsilon}{2}(h\circ\nabla u)\right. \\ &- \varepsilon(\delta_1+\delta_2)\|u\|_2^2 - \varepsilon\frac{c_1}{4\delta_1}\|u\|_m^m - \varepsilon\frac{c_2}{4\delta_2}\|y(x,1,t)\|_m^m.\end{aligned} \tag{22}$$

At this point, by setting δ_1, δ_1 so that, for large κ (which will be specified later)

$$\frac{c_1}{4C_0\delta_1} = \frac{\kappa\mathbb{H}^{-\alpha}(t)}{2}, \quad \frac{c_2}{4C_0\delta_2} = \frac{\kappa\mathbb{H}^{-\alpha}(t)}{2},$$

by (17) and substituting in (22), we get

$$\begin{aligned}\mathcal{K}'(t) &\geq [(1-\alpha) - \varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\gamma+1}\|u_t\|_{\gamma+2}^{\gamma+2} + \varepsilon\|\nabla u_t\|_2^2 \\ &- \frac{\varepsilon}{2}(h\circ\nabla u) - \varepsilon\zeta_1\|\nabla u\|_2^4 - \varepsilon\left(\zeta_0 - \frac{1}{2}\int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u\|_2^2 \\ &- \varepsilon\left(\frac{c_3\mathbb{H}^\alpha(t)}{2C_0\kappa}\right)\|u\|_2^2 + \varepsilon k\|u\|_p^p,\end{aligned} \tag{23}$$

where $c_3 = c_1 + c_2$. Now, for $0 < a < 1$, from (16)

$$\begin{aligned}
\varepsilon k\|u\|_p^p &= \varepsilon ak\|u\|_p^p + \frac{\varepsilon p(1-a)}{\gamma+2}\|u_t\|_{\gamma+2}^{\gamma+2} + \varepsilon p(1-a)\mathbb{H}(t) \\
&+\varepsilon\frac{p(1-a)}{2}\left(\zeta_0 - \int_0^t h(\varrho)\mathrm{d}\varrho\right)\|\nabla u\|_2^2 + \varepsilon\frac{p(1-a)}{2}\|\nabla u_t\|_2^2 \\
&+\varepsilon\frac{\zeta_1 p(1-a)}{2}\|\nabla u\|_2^4 - \varepsilon\frac{p(1-a)}{2}(h\circ\nabla u) \\
&+\frac{\varepsilon p(1-a)\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m \mathrm{d}\rho.
\end{aligned}$$

Substituting in (23), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq& \left\{(1-\alpha) - \varepsilon\kappa\right\}\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon ak\|u\|_p^p \\
&+\varepsilon\left\{\frac{p(1-a)}{\gamma+2} + \frac{1}{\gamma+1}\right\}\|u_t\|_{\gamma+2}^{\gamma+2} + \varepsilon\left\{1 + \frac{p(1-a)}{2}\right\}\|\nabla u_t\|_2^2 \\
&+\varepsilon\left\{\frac{p(1-a)}{2}\left(\zeta_0 - \int_0^t h(\varrho)\mathrm{d}\varrho\right) - \left(\zeta_0 - \frac{1}{2}\int_0^t h(\varrho)\mathrm{d}\varrho\right)\right\}\|\nabla u\|_2^2 \\
&+\varepsilon\zeta_1\left\{\frac{p(1-a)}{2} - 1\right\}\|\nabla u\|_2^4 + \varepsilon\left\{\frac{p(1-a)}{2} - \frac{1}{2}\right\}(h\circ\nabla u) \\
&-\varepsilon\left(\frac{c_3\mathbb{H}^\alpha(t)}{2C_0\kappa}\right)\|u\|_2^2 + \varepsilon p(1-a)\mathbb{H}(t) \\
&+\frac{\varepsilon p(1-a)\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m \mathrm{d}\rho.
\end{aligned} \tag{24}$$

According to (18), Corollary 2.4 and Young's inequality, we get

$$\begin{aligned}
\mathbb{H}^\alpha(t)\|u\|_2^2 &\leq \left(\frac{k}{p}\int_\Omega |u|^p \mathrm{d}x\right)^\alpha \|u\|_2^2 \\
&\leq c\left[\left(\int_\Omega |u|^p\mathrm{d}x\right)^{\alpha+\frac{2}{p}} + \left(\int_\Omega |u|^p\mathrm{d}x\right)^\alpha \|\nabla u\|_2^{\frac{4}{p}}\right] \\
&\leq c\left[\left(\int_\Omega |u|^p\mathrm{d}x\right)^{\frac{(\alpha p+2)}{p}} + \left(\int_\Omega |u|^p\mathrm{d}x\right)^{\frac{\alpha p}{(p-2)}} + \|\nabla u\|_2^2\right].
\end{aligned}$$

By (20), yields

$$2 < \alpha p + 2 \leq p \text{ and } 2 < \frac{\alpha p^2}{p-2} \leq p.$$

Hence, Lemma 2.3 gives

$$\mathbb{H}^\alpha(t)\|u\|_2^2 \leq c\left(\|u\|_p^p + \|\nabla u\|_2^2\right). \tag{25}$$

Combining (24) and (25), we get

$$\begin{aligned}
&\mathcal{K}'(t)\\
&\geq \Big\{(1-\alpha)-\varepsilon\kappa\Big\}\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon\Big(ak-\frac{c_4}{2C_0\kappa}\Big)\|u\|_p^p\\
&\quad+\varepsilon\Big\{\frac{p(1-a)}{\gamma+2}+\frac{1}{\gamma+1}\Big\}\|u_t\|_{\gamma+2}^{\gamma+2}+\varepsilon\Big\{1+\frac{p(1-a)}{2}\Big\}\|\nabla u_t\|_2^2\\
&\quad+\varepsilon\Big\{\frac{p(1-a)}{2}\Big(\zeta_0-\int_0^t h(\varrho)\mathrm{d}\varrho\Big)-\Big(\zeta_0-\frac{1}{2}\int_0^t h(\varrho)\mathrm{d}\varrho\Big)-\frac{c_4}{2C_0\kappa}\Big\}\|\nabla u\|_2^2\\
&\quad+\varepsilon\zeta_1\Big\{\frac{p(1-a)}{2}-1\Big\}\|\nabla u\|_2^4+\varepsilon\Big\{\frac{p(1-a)}{2}-\frac{1}{2}\Big\}(h\circ\nabla u)\\
&\quad+\varepsilon p(1-a)\mathbb{H}(t)+\frac{\varepsilon p(1-a)\xi}{m}\int_0^1\|y(x,\rho,t)\|_m^m\mathrm{d}\rho.
\end{aligned}$$

In this stage, we take $a>0$ small enough so that

$$\lambda_1=\frac{p(1-a)}{2}-1>0,$$

and we assume

$$\int_0^\infty h(\varrho)\mathrm{d}\varrho<\frac{\frac{p(1-a)}{2}-1}{\frac{p(1-a)}{2}-\frac{1}{2}}=\frac{2\lambda_1}{2\lambda_1+1},$$

which gives

$$\lambda_2 \;=\; \Big\{\Big(\frac{p(1-a)}{2}-1\Big)-\Big(\int_0^t h(\varrho)\mathrm{d}\varrho\Big)\Big(\frac{p(1-a)}{2}-\frac{1}{2}\Big)\Big\}>0.$$

Then we choose κ so large that

$$\lambda_3=ak-\frac{c_4}{2C_0\kappa}>0,\qquad \lambda_4=\lambda_2-\frac{c_4}{2C_0\kappa}>0.$$

Finally, we fixed κ, a, and we appoint ε small enough so that

$$\lambda_5=(1-\alpha)-\varepsilon\kappa>0,\qquad \text{and}\qquad \mathcal{K}(0)>0.$$

Thus, for some $\eta>0$, estimate (24) becomes

$$\begin{aligned}
\mathcal{K}'(t)\geq\eta\Big\{&\mathbb{H}(t)+\|u_t\|_{\gamma+2}^{\gamma+2}+\|\nabla u_t\|_2^2+\|\nabla u\|_2^2+(h\circ\nabla u)+\|u\|_p^p\\
&+\|\nabla u\|_2^4+\int_0^1\|y(x,\rho,t)\|_m^m\mathrm{d}\rho\Big\}.
\end{aligned}\tag{26}$$

Next, by using Holder's and Young's inequalities, we obtain

$$\Big|\int_\Omega u|u_t|^\gamma u_t\mathrm{d}x\Big|^{\frac{1}{1-\alpha}}\;\leq\; c\Big[\|u\|_\gamma^{\frac{\theta}{1-\alpha}}+\|u_t\|_{\gamma+2}^{\frac{\mu}{1-\alpha}}\Big],\tag{27}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$.

We take $\mu = (\gamma+2)(1-\alpha)$, to get

$$\frac{\theta}{1-\alpha} = \frac{\gamma+2}{(1-\alpha)(\gamma+2)-1} \leq p.$$

Further, for $s = \frac{\gamma+2}{(1-\alpha)(\gamma+2)-1}$, estimate (27) gives

$$\left|\int_\Omega u|u_t|^\gamma u_t \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq c\Big[\|u\|_p^s + \|u_t\|_{\gamma+2}^{\gamma+2}\Big].$$

Then, Lemma 2.3 yields

$$\left|\int_\Omega u|u_t|^\gamma u_t \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq c\Big[\|u\|_p^p + \|u_t\|_{\gamma+2}^{\gamma+2} + \|\nabla u\|_2^2\Big].$$

Similarly, we have

$$\left|\int_\Omega \nabla u \nabla u_t \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq c\Big[\|\nabla u\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla u_t\|_2^{\frac{\mu}{1-\alpha}}\Big],$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

We take $\theta = 4(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{4}{4(1-\alpha)-1} \leq 2$$

and

$$\left|\int_\Omega \nabla u \nabla u_t \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq c\Big\{\|\nabla u\|_2^4 + \|\nabla u_t\|_2^2\Big\}.$$

Hence,

$$\begin{aligned}
&\mathcal{K}^{\frac{1}{1-\alpha}}(t) \\
&= \left(\mathbb{H}^{1-\alpha} + \frac{\varepsilon}{\gamma+1}\int_\Omega u|u_t|^\gamma u_t \mathrm{d}x + \varepsilon\int_\Omega \nabla u \nabla u_t \mathrm{d}x + \varepsilon\frac{\sigma}{4}\|\nabla u\|_2^4\right)^{\frac{1}{1-\alpha}} \\
&\leq c\left(\mathbb{H}(t) + \left|\int_\Omega u|u_t|^\gamma u_t \mathrm{d}x\right|^{\frac{1}{1-\alpha}} + \left|\int_\Omega \nabla u \nabla u_t \mathrm{d}x\right|^{\frac{1}{1-\alpha}} + \|\nabla u\|_2^{\frac{4}{1-\alpha}}\right) \\
&\leq c\left(\mathbb{H}(t) + \|u\|_p^p + \|u_t\|_{\gamma+2}^{\gamma+2} + \|\nabla u\|_2^2 + \|\nabla u\|_2^4 + \|\nabla u_t\|_2^2\right) \\
&\leq c\bigg(\mathbb{H}(t) + \|u\|_p^p + \|u_t\|_{\gamma+2}^{\gamma+2} + \|\nabla u\|_2^2 + \|\nabla u\|_2^4 + \|\nabla u_t\|_2^2 \\
&\qquad + (h \circ \nabla u) + \int_0^1 \|y(x,\rho,t)\|_m^m \mathrm{d}\rho\bigg).
\end{aligned} \tag{28}$$

From (26) and (28), we obtain

$$\mathcal{K}'(t) \geq \Gamma\mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{29}$$

where $\Gamma > 0$, depends only on η and c. By integration of (29), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \Gamma\frac{\alpha}{(1-\alpha)}t}.$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\Gamma\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then, the proof is completed. □

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