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ON KIRCHHOFF-DOUBLE PHASE PROBLEMS WITHOUT (AR)-CONDITION

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Abstract. In this article, via a variational approach, we consider the existence of weak solutions for a class of Kirchhoff-double phase type problems, namely,

$$\begin{cases} -M\left(D(u)\right)\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $1 < p < q < N$. The aim of this article is to establish the existence of at least one nontrivial weak solution of the above problem without **(AR)**-condition, by using the Mountain Pass Theorem for an energy functional satisfying the Cerami condition.

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Key words. Kirchhoff-double phase type problems, variational methods, Cerami condition, Mountain Pass Theorem, Musielak-Orlicz space.

1. INTRODUCTION AND MAIN RESULTS

The study of differential equations and variational problems with double-phase operators is a new and important topic. Since it sheds light on multiple range of applications in the field of mathematical physics such as elasticity theory, strongly anisotropic materials, Lavrentiev's phenomenon, etc. (see [15–17]).

This paper is concerned with the existence of solutions to the following problem

$$(1) \quad \begin{cases} -M\left(D(u)\right)\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a smooth bounded domain, $1 < p < q < N$ and $\frac{q}{p} < 1 + \frac{1}{N}$, $\lambda > 0$ is a real number, $a : \overline{\Omega} \mapsto [0, +\infty)$ is Lipschitz continuous,

$$D(u) := \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right) \mathrm{d}x,\ M : \mathbb{R}_+ \to \mathbb{R}_+$$

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is a continuous function (a Kirchhoff-type function) and the nonlinear term $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

Since the original work of A. Ambrosetti and P.H. Rabinowitz [1], critical point theory has become one of the most important tools for determining solutions to elliptic equations of variational type. In particular, our elliptic problem (1) generalizes many works, since the function M can be $\not\equiv 1$. The main ingredient to obtaining the existence of solutions for superlinear problems is the condition proposed by A. Ambrosetti and P.H. Rabinowitz (the (**AR**)-condition for short).

Many authors have recently studied problem (1) in the case when $M \equiv 1$, and a plethora of results have been obtained, see for instance B. Ge et al. [8], W. Liu and G. Dai [12], K. Perera and M. Squassina [14] and the references therein.

On the other hand, there are much fewer results for the case $M \not\equiv 1$. For example, via a variational approach, A. Fiscella and A. Pinamonti [7] obtained a nontrivial weak solution of problem (1) with $\lambda = 1$ under the following conditions:

(M_1) There exists $\theta \in \left[1, \frac{p^*}{q}\right[$ such that for all $t \in \mathbb{R}_+$,

$$tM(t) \leq \theta \widehat{M}(t),$$

where $\widehat{M}(t) = \int_0^t M(\tau) \mathrm{d}\tau$ and $p^* = \frac{Np}{N-p}$.

(M') For all $\tau > 0$, there exists $\kappa = \kappa(\tau) > 0$ such that $M(t) \geq \kappa$, for all $t \geq \tau$.

(H_1') There exists an exponent $r \in \,]q\theta, p^*[$ such that for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ and

$$|g(x,t)| \leq q\theta\varepsilon |t|^{q\theta-1} + r\delta_\varepsilon |t|^{r-1} \quad \text{for a.e. } x \in \Omega \text{ and any } t \in \mathbb{R}.$$

(AR) There exist $\sigma \in \,]q\theta, p^*[$, $c > 0$ and $t_0 > 0$ such that

$$c < \sigma G(x,t) \leq tg(x,t), \quad \text{for a.e. } x \in \Omega \text{ and any } |t| \geq t_0,$$

where $G(x,t) = \int_0^t g(x,s) \mathrm{d}s$.

As we know, the main role of utilizing the famous Ambrosetti-Rabinowitz type condition is to ensure the boundedness of the Palais-Smale type sequences of the corresponding functional. This condition sometimes can be very restrictive and excludes many interesting nonlinearities. Indeed, there are several functions which are superlinear at infinity and at the origin but do not satisfy **(AR)**-condition. See Remark 1.1 below.

To state our main results, we first collect our assumptions on the function M and the nonlinearity g as follows:

(M_2) $M \in C(\mathbb{R}_+)$ satisfies $\inf\limits_{t \in \mathbb{R}_+} M(t) \geq m_0 > 0$, where m_0 is a constant.

(H_1) There exist $q < s < p^*$ and $C_0 > 0$ such that
$$|g(x,t)| \leq C_0 \left(1 + |t|^{s-1}\right).$$

(H_2) $\liminf_{|t|\to\infty} \frac{G(x,t)}{|t|^{\theta q}} = +\infty$ uniformly a.e. $x \in \Omega$, where $G(x,t) = \int_0^t g(x,s)\mathrm{d}s$ and θ comes from (M_1) above.

(H_3) $\lim_{t\to 0} \frac{g(x,t)}{|t|^{p-1}} = 0$ uniformly in x .

(H_4) There exist $c_1, r_1 \geq 0$ and $\eta > \frac{N}{p}$ such as
$$|G(x,t)|^\eta \leq c_1 |t|^{\eta p} F(x,t),$$
for all $(x,t) \in \Omega \times \mathbb{R}$, $|t| \geq r_1$ and $F(x,t) := \frac{1}{\theta q} g(x,t)t - G(x,t) \geq 0$.

(H_5) There exist $\mu > q$ and $\beta > 0$ such that
$$\mu G(x,t) \leq t g(x,t) + \beta |t|^p, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

REMARK 1.1. (a) Hypotheses (H_4) and (H_5), which are important in obtaining a compactness condition of Palais-Smale type, can be found in [10, 11] with Ω is replaced by the entire space $\mathbb{R}^N$.
(b) Let $g(x,t) = a|t|^{p-2} t \ln(1 + |t|)$ with $a > 0$. It is easy to see that the function g does not satisfy **(AR)**-condition, but it satisfies $(H_1)-(H_5)$.

Now we can state our main results.

THEOREM 1.2. *Assume that* $(M_1), (M_2), (H_1), (H_2)$, (H_3) *and* (H_4) *hold. Then, for all* $\lambda > 0$, *problem* (1) *has at least one nontrivial solution in* $W_0^{1,\mathcal{H}}$.

THEOREM 1.3. *Assume that* $(M_1), (M_2), (H_1), (H_2)$,$(H_3)$ *and* (H_5) *hold. Then, for all* $\lambda > 0$, *problem* (1) *has at least one nontrivial solution in* $W_0^{1,\mathcal{H}}(\Omega)$.

2. PRELIMINARIES

To study problem (1), we need some definitions and basic properties of $W_0^{1,\mathcal{H}}$ which form the so-called Musielak–Orlicz–Sobolev space. For more details, see [3, 5, 6, 13] and the references therein.

Denote by $N(\Omega)$ the set of all generalized N-functions (N stands for nice). Let us denote by $\mathcal{H} : \Omega \times [0, +\infty[\to [0, +\infty[$ the functional defined as
$$\mathcal{H}(x,t) = t^p + a(x)t^q, \quad \text{for a.e. } x \in \Omega \text{ and for any } t \in [0, +\infty[\,,$$
with $1 < p < q$ and $0 \leq a(.) \in L^1(\Omega)$. It is clear that $\mathcal{H}$ is a generalized N-function, locally integrable and
$$\mathcal{H}(x, 2t) \leq 2^q \mathcal{H}(x,t), \quad \text{for a.e. } x \in \Omega \text{ and for any } t \in [0, +\infty[\,,$$
which is called condition (Δ_2).

We designate the Musielak–Orlicz space by

$$L^{\mathcal{H}}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mathcal{H}(x, |u|) \mathrm{d}x < +\infty \Big\},$$

equipped with the so-called Luxembourg norm

$$|u|_{\mathcal{H}} = \inf \Big\{ \lambda > 0 : \int_\Omega \mathcal{H}(x, |\frac{u}{\lambda}|) \mathrm{d}x \leq 1 \Big\}.$$

The Musielak–Orlicz–Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined as

$$W^{1,\mathcal{H}}(\Omega) = \Big\{ u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega) \Big\},$$

with the norm

$$\| u \| = |u|_{\mathcal{H}} + |\nabla u|_{\mathcal{H}}.$$

We denote by $W_0^{1,\mathcal{H}}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. With these norms, the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are separable reflexive Banach spaces [5, 9].

On $L^{\mathcal{H}}(\Omega)$, we consider the function $\rho_{\mathcal{H}} : L^{\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \left(|u|^p + a(x)|u|^q \right) \mathrm{d}x.$$

The relationship between $\rho_{\mathcal{H}}$ and $|\cdot|_{\mathcal{H}}$ is established by the next result.

Proposition 2.1 ([12]). *For $u \in L^{\mathcal{H}}(\Omega)$, $(u_n) \subset L^{\mathcal{H}}(\Omega)$ and $\lambda > 0$, we have*

(i) *For $u \neq 0$, $|u|_{\mathcal{H}} = \lambda \Longleftrightarrow \rho_{\mathcal{H}}(\frac{u}{\lambda}) = 1$;*

(ii) $|u|_{\mathcal{H}} < 1 \, (= 1, > 1) \Longleftrightarrow \rho_{\mathcal{H}}(u) < 1 \, (= 1, > 1)$;

(iii) $|u|_{\mathcal{H}} > 1 \Longrightarrow |u|_{\mathcal{H}}^p \leq \rho_{\mathcal{H}}(u) \leq |u|_{\mathcal{H}}^q$;

(iv) $|u|_{\mathcal{H}} < 1 \Longrightarrow |u|_{\mathcal{H}}^q \leq \rho_{\mathcal{H}}(u) \leq |u|_{\mathcal{H}}^p$;

(v) $\lim_{n\to+\infty} |u_n|_{\mathcal{H}} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho_{\mathcal{H}}(u_n) = 0$ *and* $\lim_{n\to+\infty} |u_n|_{\mathcal{H}} = +\infty \Leftrightarrow \lim_{n\to+\infty} \rho_{\mathcal{H}}(u_n) = +\infty$.

Proposition 2.2 ([5]). (i) *If $1 \leq r \leq p^*$, then there is a continuous embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$. In particular, if $1 \leq r < p^*$, then the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.*

(ii) *In $W_0^{1,\mathcal{H}}(\Omega)$, the following Poincaré-type inequality holds, that is, there is a constant $C_0 > 0$ independent of u such that*

$$|u|_{\mathcal{H}} \leq C_0 |\nabla u|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

By the above Proposition, there exists $c_r > 0$ such that

$$|u|_r \leq c_r \|u\| \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $|\cdot|_r$ denotes the usual norm in $L^r(\Omega)$. It follows from (ii) of Proposition 2.2 that $|\nabla u|_{\mathcal{H}}$ and $\|u\|$ are equivalent norms on $W_0^{1,\mathcal{H}}(\Omega)$. In the following discussion, we equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm $|\nabla u|_{\mathcal{H}}$ and write $\|u\| = |\nabla u|_{\mathcal{H}}$ for simplicity.

Let $J : W_0^{1,\mathcal{H}}(\Omega) \to \left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$ be defined by

$$\langle J(u), v\rangle = \int_\Omega \left(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \nabla u.\nabla v \mathrm{d}x, \tag{2}$$

for all $u, v \in W_0^{1,\mathcal{H}}(\Omega)$, where $\left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$ denotes the dual space of $W_0^{1,\mathcal{H}}(\Omega)$ and $\langle \cdot, \cdot\rangle$ stands for the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and $\left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$.

Proposition 2.3 ([12]).

(i) *J is a continuous, bounded and strictly monotone operator.*

(ii) *J is a mapping of type (S_+), i.e, if $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and*

$$\overline{\lim_{n\to+\infty}} \langle J(u_n) - J(u), u_n - u\rangle \leq 0,$$

then $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

(iii) *J is a homeomorphism.*

In this paper, we denote by $Y = W_0^{1,\mathcal{H}}$, $Y^* = \left(W_0^{1,\mathcal{H}}\right)^*$ the dual space. We notice that problem (1) has a variational structure, in fact, its solutions can be searched as critical points of the energy functional $I_\lambda : Y \to \mathbb{R}$ given by

$$I_\lambda(u) = \phi(u) - \lambda\psi(u),$$

where

$$\phi(u) = \widehat{M}\left(\int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right) \mathrm{d}x\right) \quad \text{and} \quad \psi(u) = \int_\Omega G(x, u)\mathrm{d}x.$$

Then, it follows from assumption (H_1) that $I_\lambda \in C^1(Y, \mathbb{R})$, and its Fréchet derivative is

$$\begin{aligned}
&\langle I'_\lambda(u), v\rangle \\
&= M\left(\int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right) \mathrm{d}x\right) \int_\Omega \left(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \nabla u\nabla v\mathrm{d}x \\
&\quad - \lambda \int_\Omega g(x, u)v\mathrm{d}x,
\end{aligned}$$

for any $u, v \in Y$.

Let $u \in Y$. We say that u is a **weak solution** of the problem (1) if

$$\begin{aligned}
&M\left(\int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right) \mathrm{d}x\right) \quad \int_\Omega \left(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \nabla u\nabla v\mathrm{d}x \\
&= \lambda \int_\Omega g(x, u)v\mathrm{d}x,
\end{aligned}$$

for all $v \in Y$.

Next we give the definition of the Cerami condition which was introduced by G. Cerami in [4].

Definition 2.4. Let $(X, \|\cdot\|)$ be a real Banach space and let $J \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that J satisfies the $(C)_c$-Cerami condition if any sequence $(u_n) \subset X$ such that

$$J(u_n) \to c \quad \text{and} \quad \|J'(u_n)\|_{X^*}(1 + \|u_n\|) \to 0$$

has a convergent subsequence. If this condition is satisfied at every level $c \in \mathbb{R}$, then we say that J satisfies the (C)-condition.

Remark 2.5. It is clear from the above definition that if J satisfies the (PS)-condition, then it satisfies the (C)-condition. However, there are functionals that satisfy the (C)-condition but do not satisfy the condition (PS)-condition (see [4]). Consequently, the (C)-condition is weaker than the (PS)-condition.

Now, we present the following theorem which will play a fundamental role in the proof of main theorems.

Theorem 2.6 ([2]). *Let X be a real Banach space, let $J : X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$ that satisfies (C)-condition, $J(0) = 0$ and the following conditions hold:*

(i) *There exist positive constants ρ and α such that $J(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.*

(ii) *There exists a function $e \in X$ such that $\|e\| > \rho$ and $J(e) \leq 0$.*

Then, the functional J has a critical value $c \geq \alpha$, that is, there exists $u \in X$ such that $J(u) = c$ and $J'(u) = 0$ in X^*.

3. PROOFS OF MAIN RESULTS

First of all, we begin by showing that the (C)-condition holds.

Lemma 3.1. *Assume that $(M_1), (M_2), (H_1), (H_2)$ and (H_4) hold. Then, for all $\lambda > 0$, I_λ satisfies the (C)-condition.*

Proof. Let $(u_n) \subset Y$ be a Cerami sequence for I_λ, namely,

$$I_\lambda(u_n) \to c \quad \text{and} \quad \|I'_\lambda(u_n)\|_{Y^*}(1 + \|u_n\|) \to 0. \tag{3}$$

We need to prove the boundedness of the sequence (u_n) in Y. To this end, by contradiction, it is assumed that $\|u_n\| \to +\infty$ as $n \to +\infty$. For n large enough, by (M_1), we obtain

$$c + 1 \geq I_\lambda(u_n) - \frac{1}{\theta q}\langle I'_\lambda(u_n), u_n\rangle$$

$$
\begin{aligned}
&= \widehat{M}\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right) - \lambda\int_\Omega G(x,u_n)\mathrm{d}x\\
&- \frac{1}{\theta q}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (|\nabla u_n|^p + a(x)|\nabla u_n|^q)\mathrm{d}x\right)\\
&+ \frac{\lambda}{\theta q}\int_\Omega g(x,u_n)u_n\mathrm{d}x\\
&\geq \frac{1}{\theta}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\\
&- \frac{1}{\theta q}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (|\nabla u_n|^p + a(x)|\nabla u_n|^q)\mathrm{d}x\right)\\
&- \lambda\int_\Omega G(x,u_n)\mathrm{d}x + \frac{\lambda}{\theta q}\int_\Omega g(x,u_n)u_n\mathrm{d}x\\
&\geq \frac{1}{\theta q}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (|\nabla u_n|^p + a(x)|\nabla u_n|^q)\mathrm{d}x\right)\\
&- \frac{1}{\theta q}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (|\nabla u_n|^p + a(x)|\nabla u_n|^q)\mathrm{d}x\right)\\
&- \lambda\int_\Omega G(x,u_n)\mathrm{d}x + \frac{\lambda}{\theta q}\int_\Omega g(x,u_n)u_n\mathrm{d}x\\
&\geq \lambda\int_\Omega F(x,u_n)\mathrm{d}x.
\end{aligned}
$$

Therefore

$$
c+1 \geq \lambda\int_\Omega F(x,u_n)\mathrm{d}x. \tag{4}
$$

Because $\|u_n\| > 1$ for n large enough, using (M_1) and (M_2), we obtain

$$
\begin{aligned}
c &= I_\lambda(u_n) + \circ(1)\\
&= \widehat{M}\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right) - \lambda\int_\Omega G(x,u_n)\mathrm{d}x + \circ(1)\\
&\geq \frac{1}{\theta}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\\
&\quad - \lambda\int_\Omega G(x,u_n)\mathrm{d}x + \circ(1)\\
&\geq \frac{m_0}{\theta q}\|u_n\|^p - \lambda\int_\Omega G(x,u_n)\mathrm{d}x + \circ(1),
\end{aligned}
$$

which implies that

$$\frac{m_0}{\lambda\theta q} \leq \limsup_{n\to+\infty} \int_\Omega \frac{|G(x,u_n)|}{\|u_n\|^p} \mathrm{d}x. \tag{5}$$

For $0 \leq a' < b'$, put

$$\Lambda_n(a',b') := \Big\{ x \in \Omega : a' \leq |u_n(x)| < b' \Big\}.$$

Let a sequence (ω_n) be defined by $\omega_n = \frac{u_n}{\|u_n\|}$. Then,

$$\|\omega_n\| = 1 \quad \text{and} \quad |\omega_n|_r \leq c_r \|\omega_n\| = c_r \quad \text{for} \quad r \in [1, p^*[\,.$$

Up to a subsequence, for $\omega \in Y$, we may assume that

$$\begin{aligned} \omega_n &\rightharpoonup \omega && \text{in } Y; \\ \omega_n &\to \omega && \text{in } L^r(\Omega), 1 \leq r < p^*; \\ \omega_n(x) &\to \omega(x) && \text{a.e. } x \in \Omega. \end{aligned} \tag{6}$$

Next, we need to distinguish two cases.

Case 1. If $\omega = 0$, then, we have $\omega_n \to 0$ in $L^r(\Omega)$ for any $r \in [1, p^*[$. Then, by (H_1), we have

$$\int_{\Lambda_n(0,r_1)} \frac{|G(x,u_n)|}{\|u_n\|^p} \mathrm{d}x \leq \frac{C_0(r_1 + r_1^s)\mathrm{meas}(\Omega)}{\|u_n\|^p} \to 0 \quad \text{as } n \to +\infty. \tag{7}$$

Let $\eta' = \frac{\eta}{\eta-1}$. Because $\eta > \frac{N}{p}$, then, $1 < p\eta' < p^*$. Hence, using (H_4), (4) and (6), we deduce that

$$\begin{aligned} &\int_{\Lambda_n(r_1,+\infty)} \frac{|G(x,u_n)|}{\|u_n\|^p} \mathrm{d}x = \int_{\Lambda_n(r_1,+\infty)} \frac{|G(x,u_n)|}{|u_n|^p} |\omega_n|^p \mathrm{d}x \\ &\leq \left(\int_{\Lambda_n(r_1,+\infty)} \frac{|G(x,u_n)|^\eta}{|u_n|^{p\eta}} \mathrm{d}x \right)^{\frac{1}{\eta}} \left(\int_{\Lambda_n(r_1,+\infty)} |\omega_n|^{p\eta'} \mathrm{d}x \right)^{\frac{1}{\eta'}} \\ &\leq c_1^{\frac{1}{\eta}} \left(\int_{\Lambda_n(r_1,+\infty)} F(x,u_n) \mathrm{d}x \right)^{\frac{1}{\eta}} \left(\int_\Omega |\omega_n|^{p\eta'} \mathrm{d}x \right)^{\frac{1}{\eta'}} \\ &\leq \left(\frac{c_1}{\lambda}(c+1) \right)^{\frac{1}{\eta}} \left(\int_\Omega |\omega_n|^{p\eta'} \mathrm{d}x \right)^{\frac{1}{\eta'}} \to 0, \quad \text{as } n \to +\infty. \end{aligned} \tag{8}$$

From (7) and (8), we get

$$\begin{aligned} \int_\Omega \frac{|G(x,u_n)|}{\|u_n\|^p} \mathrm{d}x &= \int_{\Lambda_n(0,r_1)} \frac{|G(x,u_n)|}{\|u_n\|^p} \mathrm{d}x + \int_{\Lambda_n(r_1,+\infty)} \frac{|G(x,u_n)|}{\|u_n\|^p} \mathrm{d}x \\ &\to 0, \quad \text{as } n \to +\infty, \end{aligned}$$

which contradicts (5).

Case 2. If $\omega \neq 0$. Let $\Lambda_0 = \Big\{x \in \Omega : \omega(x) \neq 0\Big\}$. Then, $\mathrm{meas}(\Lambda_0) > 0$. For all $x \in \Lambda_0$, by (6), we have

$$|u_n(x)| = |\omega_n(x)|\ \|u_n\| \to +\infty.$$

Thus

$$\Lambda_0 \subset \Lambda_n(r_1, +\infty), \qquad \text{for } n \text{ large enough }.$$

As the proof of (7), we obtain that

$$\int_{\Lambda_n(0,r_1)} \frac{|G(x,u_n)|}{\|u_n\|^{\theta q}}\mathrm{d}x \leq \frac{C_0(r_1 + r_1^s)\mathrm{meas}(\Omega)}{\|u_n\|^{\theta q}} \to 0 \quad \text{ as } n \to +\infty. \tag{9}$$

Furthermore, using (H_2), (9), the fact that $\widehat{M}(t) \leq \widehat{M}(1)(1+t^\theta)$ for all $t \in \mathbb{R}_+$ and Fatou's Lemma, we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{I_\lambda(u_n)}{\|u_n\|^{\theta q}} \\
&= \lim_{n\to+\infty} \left(\frac{\widehat{M}\left(\int_\Omega(\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)}{\|u_n\|^{\theta q}} - \lambda \int_\Omega \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\mathrm{d}x \right) \\
&\leq \lim_{n\to+\infty} \left(\frac{\widehat{M}(1)\left(1 + \left(\int_\Omega(\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)^\theta\right)}{\|u_n\|^{\theta q}} \right. \\
&\qquad \left. - \lambda \int_{\Lambda_n(0,r_1)} \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\mathrm{d}x - \lambda \int_{\Lambda_n(r_1,+\infty)} \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\mathrm{d}x \right) \\
&\leq \lim_{n\to+\infty} \left(\frac{\widehat{M}(1)\left(\|u_n\|^{\theta q} + \frac{1}{p^\theta}\|u_n\|^{\theta q}\right)}{\|u_n\|^{\theta q}} - \lambda \int_{\Lambda_n(r_1,+\infty)} \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\mathrm{d}x \right) \\
&= \lim_{n\to+\infty} \left(\widehat{M}(1)\left(1 + \frac{1}{p^\theta}\right) - \lambda \int_{\Lambda(r_1,+\infty)} \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\mathrm{d}x \right) \\
&\leq \limsup_{n\to+\infty} \left(\widehat{M}(1)\left(1 + \frac{1}{p^\theta}\right) - \lambda \int_{\Lambda_n(r_1,+\infty)} \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\mathrm{d}x \right) \\
&= \widehat{M}(1)\left(1 + \frac{1}{p^\theta}\right) - \lambda \liminf_{n\to+\infty} \int_{\Lambda_n(r_1,+\infty)} \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\mathrm{d}x \\
&= \widehat{M}(1)\left(1 + \frac{1}{p^\theta}\right) - \lambda \liminf_{n\to+\infty} \int_\Omega \frac{G(x,u_n)}{\|u_n\|^{\theta q}}\ \chi_{\Lambda_n(r_1,+\infty)}(x)\mathrm{d}x \\
&\leq \widehat{M}(1)\left(1 + \frac{1}{p^\theta}\right) - \lambda \int_\Omega \liminf_{n\to+\infty} \frac{G(x,u_n)}{|u_n|^{\theta q}}\ \chi_{\Lambda_n(r_1,+\infty)}(x)|\omega_n|^{\theta q}\mathrm{d}x,
\end{aligned}$$

which implies that

$$\lim_{n\to+\infty}\frac{I_\lambda(u_n)}{\|u_n\|^{\theta q}}=-\infty. \tag{10}$$

On the other hand, we have

$$\lim_{n\to+\infty}\frac{I_\lambda(u_n)}{\|u_n\|^{\theta q}}=\lim_{n\to+\infty}\frac{c+\circ(1)}{\|u_n\|^{\theta q}}=0. \tag{11}$$

Combining (10) and (11), we get a contradiction. Therefore, (u_n) is bounded in Y.

Finally, we need to prove that any (C)-sequence has a convergent subsequence. Let $(u_n)\subset Y$ be a (C)-sequence. Then, (u_n) is bounded in Y. Passing to the limit, if necessary, to a subsequence, from Proposition 2.2, we have

$$\begin{aligned} &u_n\rightharpoonup u \text{ in } Y; \qquad u_n\to u \text{ in } L^r(\Omega); \qquad u_n(x)\to u(x) \text{ a.e. } x\in\Omega;\\ &\nabla u_n\rightharpoonup\nabla u \text{ in } \left(L^{\mathcal{H}}(\Omega)\right)^N; \qquad \varphi_{\mathcal{H}}(\nabla u_n)\to k \text{ in } \mathbb{R}, \end{aligned} \tag{12}$$

where $1\le r<p^*$ and $\varphi_{\mathcal{H}}(u)=\int_\Omega\left(\frac{|u|^p}{p}+a(x)\frac{|u|^q}{q}\right)\mathrm{d}x$.

If $k=0$, because $\varphi_{\mathcal{H}}(v)\ge\frac{\rho_{\mathcal{H}}(v)}{q}\ge 0$ for all $v\in Y$, then it follows from Proposition 2.1 that $u_n\to 0$ in Y. Hence, let us suppose $k>0$.

By (H_1), It is easy to compute directly that

$$\begin{aligned} &\int_\Omega|g(x,u_n)-g(x,u)||u_n-u|\,\mathrm{d}x\le\int_\Omega(|g(x,u_n)|+|g(x,u)|)\,|u_n-u|\,\mathrm{d}x\\ &\le\int_\Omega\left[C_0\left(1+|u_n|^{s-1}\right)+C_0\left(1+|u|^{s-1}\right)\right]|u_n-u|\,\mathrm{d}x\\ &\le 2C_0\int_\Omega|u_n-u|\,\mathrm{d}x+C_0\int_\Omega|u_n|^{s-1}|u_n-u|\,\mathrm{d}x+C_0\int_\Omega|u|^{s-1}|u_n-u|\,\mathrm{d}x\\ &\le 2C_0\int_\Omega|u_n-u|\,\mathrm{d}x+C_0\left(\int_\Omega|u_n|^{(s-1)s'}\,\mathrm{d}x\right)^{\frac{1}{s'}}\left(\int_\Omega|u_n-u|^s\,\mathrm{d}x\right)^{\frac{1}{s}}\\ &\quad+C_0\left(\int_\Omega|u|^{(s-1)s'}\mathrm{d}x\right)^{\frac{1}{s'}}\left(\int_\Omega|u_n-u|^s\,\mathrm{d}x\right)^{\frac{1}{s}}\\ &=2C_0\int_\Omega|u_n-u|\,\mathrm{d}x+C_0\left(\int_\Omega|u_n|^s\,\mathrm{d}x\right)^{\frac{s-1}{s}}\left(\int_\Omega|u_n-u|^s\,\mathrm{d}x\right)^{\frac{1}{s}}\\ &\quad+C_0\left(\int_\Omega|u|^s\,\mathrm{d}x\right)^{\frac{s-1}{s}}\left(\int_\Omega|u_n-u|^s\,\mathrm{d}x\right)^{\frac{1}{s}}\\ &=2C_0\,|u_n-u|_1+C_0\,|u_n|_s^{s-1}\,|u_n-u|_s+C_0|u|_s^{s-1}\,|u_n-u|_s\\ &\to 0,\qquad\text{as } n\to\infty, \end{aligned}$$

where $\frac{1}{s}+\frac{1}{s'}=1$. Therefore

$$\int_\Omega |g\,(x,u_n)-g(x,u)||u_n-u|\,\mathrm{d}x \to 0, \qquad \text{as } n\to\infty.$$

On the other hand, using (M_2) and (12), we obtain

$$M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p+\frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\to M(k)\neq 0, \qquad \text{as } n\to+\infty.$$

Next, since $u_n \rightharpoonup u$, by (3), we have

$$\langle I'_\lambda(u_n), u_n-u\rangle \to 0, \qquad \text{as } n\to+\infty.$$

Then

$$\langle I'_\lambda(u_n), u_n-u\rangle = M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p+\frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\langle J(u_n), u_n-u\rangle \\ -\int_\Omega g(x,u_n)(u_n-u)\mathrm{d}x\to 0, \quad \text{as } n\to+\infty,$$

where J is given in (2).

Since J is a mapping of type (S_+) in view of Proposition 2.3, we conclude that $u_n\to u$ in Y. The proof is complete. □

LEMMA 3.2. *Assume that* $(M_1),(M_2),(H_1),(H_2)$ *and* (H_5) *hold. Then, for all* $\lambda>0$, I_λ *satisfies the* (C)*-condition.*

Proof. Let $(u_n)\subset Y$ be a Cerami sequence for I_λ satisfying (3). As in the proof of Lemma 3.1, we only prove that (u_n) is bounded in Y. By contradiction, suppose that $\|u_n\|\to+\infty$ as $n\to+\infty$. Let a sequence (ω_n) be defined by $\omega_n=\frac{u_n}{\|u_n\|}$. Then, $\|\omega_n\|=1$ and $|\omega_n|_r\le c_r\|\omega_n\|=c_r$ for $r\in[1,p^*[$. Up to a subsequence, for $\omega\in Y$, we may assume that

$$\begin{aligned} \omega_n &\rightharpoonup \omega \quad \text{in } Y;\\ \omega_n &\to \omega \quad \text{in } L^r(\Omega), 1\le r<p^*;\\ \omega_n(x) &\to \omega(x) \quad \text{a.e. } x\in\Omega. \end{aligned} \tag{13}$$

By virtue of $(M_1),(M_2)$ and (H_5), we have

$$\begin{aligned} &c+1\\ &\ge I_\lambda(u_n)-\frac{1}{\mu}\langle I'_\lambda(u_n),u_n\rangle\\ &\ge \frac{1}{\theta}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p+\frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p+\frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\\ &\quad -\frac{1}{\mu}M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p+\frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (|\nabla u_n|^p+a(x)|\nabla u_n|^q)\mathrm{d}x\right)\\ &\quad -\lambda\int_\Omega G(x,u_n)\mathrm{d}x+\frac{\lambda}{\mu}\int_\Omega g(x,u_n)u_n\mathrm{d}x \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{\theta q} M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (|\nabla u_n|^p + a(x)|\nabla u_n|^q)\mathrm{d}x\right)\\
&\quad - \frac{1}{\mu} M\left(\int_\Omega (\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q)\mathrm{d}x\right)\left(\int_\Omega (|\nabla u_n|^p + a(x)|\nabla u_n|^q)\mathrm{d}x\right)\\
&\quad - \lambda \int_\Omega G(x,u_n)\mathrm{d}x + \frac{\lambda}{\mu}\int_\Omega g(x,u_n)u_n \mathrm{d}x\\
&\geq m_0\left(\frac{1}{\theta q} - \frac{1}{\mu}\right)\|u_n\|^p - \frac{\lambda\beta}{\mu}\int_\Omega |u|^p \mathrm{d}x\\
&\geq m_0 \frac{\mu - \theta q}{\theta q \mu}\|u_n\|^p - \frac{\lambda\beta}{\mu}|u_n|_p^p.
\end{aligned}$$

Therefore,

$$1 \leq \frac{\lambda\beta\theta q}{m_0(\mu-\theta q)} \limsup_{n\to+\infty} |\omega_n|_p^p. \tag{14}$$

It follows from (13) and (14) that $\omega \neq 0$. Similar to the proof of Lemma 3.1, by (10) and (11), we can conclude a contradiction. The proof is complete. □

Proof of Theorem 1.2. Let $X = Y$ and $J = I_\lambda$. We know that I_λ satisfies (C)-condition in Y from Lemma 3.1 and $I_\lambda(0) = 0$. To apply Theorem 2.6, we will show that I_λ has a mountain pass geometry.

First, we affirm that there exists $\sigma, M > 0$ such that

$$I_\lambda(u) \geqslant M, \quad \forall u \in Y \quad \text{with} \quad \|u\| = \sigma.$$

In virtue of (H_1) and (H_3), we deduce that for any $\varepsilon > 0$, there is a $c_\varepsilon > 0$ such that

$$\begin{aligned}
|g(x,t)| &\leq \varepsilon|t|^{p-1} + c_\varepsilon|t|^{s-1}, &&\text{for all } (x,t) \in \Omega\times\mathbb{R},\\
|G(x,t)| &\leq \varepsilon|t|^p + c_\varepsilon|t|^s, &&\text{for all } (x,t) \in \Omega\times\mathbb{R},
\end{aligned} \tag{15}$$

where $s \in]q, p^*[$.

Therefore, in view of (15) and Proposition 2.2, for $u \in Y$ with $\|u\| < 1$ sufficiently small, we get

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q)\mathrm{d}x\right) - \lambda\int_\Omega G(x,u)\mathrm{d}x\\
&\geq \frac{m_0}{\theta q}\int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q)\mathrm{d}x - \lambda\int_\Omega (\varepsilon|u|^p + c_\varepsilon|u|^s)\mathrm{d}x\\
&\geq \frac{m_0}{\theta q}\|u\|^q - \lambda\varepsilon c_p^p\|u\|^p - \lambda c_\varepsilon c_s^s\|u\|^s.
\end{aligned}$$

Conclusively, there exists $\sigma, M > 0$ such that $I_\lambda(u) \geqslant M$ for any $u \in Y$ with $\|u\| = \sigma$.

Next, we affirm that there exists $e \in Y$ such that $\|e\| > \sigma$ and $I_\lambda(e) \leq 0$.

In fact, by (H_2), for all $T > 0$, there exists $\delta_T > 0$ such that

$$G(x,t) \geq T|t|^{\theta q}, \quad \text{for } |t| > \delta_T \text{ and for almost all } x \in \Omega.$$

Next, from (H_1), for any $x \in \Omega$ and $|t| \leq \delta_T$, we obtain

$$|G(x,t)| \leq C_0(1 + |\delta_T|^{s-1}).$$

The combination of the above two inequalities implies that there exists $C_T > 0$ such that

$$G(x,t) \geq T|t|^{\theta q} - C_T, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Hence, for $u_1 \in Y$ with $u_1 > 0$ on Ω and $t > 1$ large enough, we obtain

$$\begin{aligned} I_\lambda(tu_1) &= \widehat{M}\left(\int_\Omega (\frac{1}{p}|\nabla tu_1|^p + \frac{a(x)}{q}|\nabla tu_1|^q)\mathrm{d}x\right) - \lambda \int_\Omega G(x, tu_1)\mathrm{d}x \\ &\leq \widehat{M}(1)\left(1 + \left(\int_\Omega (\frac{t^p}{p}|\nabla u_1|^p + \frac{t^q}{q}a(x)|\nabla u_1|^q)\mathrm{d}x\right)^\theta\right) \\ &\quad - \lambda T|t|^{\theta q}\int_\Omega |u_1|^{\theta q}\mathrm{d}x + \lambda C_T \mathrm{meas}(\Omega) \\ &\leq \frac{\widehat{M}(1)}{p}t^{\theta q}\int_\Omega (|\nabla u_1|^p + a(x)|\nabla u_1|^q)\mathrm{d}x - \lambda T|t|^{\theta q}\int_\Omega |u_1|^{\theta q}\mathrm{d}x \\ &\quad + \widehat{M}(1) + \lambda C_T \mathrm{meas}(\Omega). \end{aligned}$$

As

$$\frac{\widehat{M}(1)}{p}\int_\Omega (|\nabla u_1|^p + a(x)|\nabla u_1|^q)\mathrm{d}x - \lambda T\int_\Omega |u_1|^{\theta q}\mathrm{d}x < 0,$$

for $T > 0$ large enough, we deduce

$$I_\lambda(tu_1) \to -\infty, \quad \text{as } t \to +\infty.$$

Hence, there exists $t_1 > 1$ and $e = t_1 u_1 \in Y$ with $\|e\| > \sigma$ such that $I_\lambda(e) \leq 0$.

Finally, all conditions of Theorem 2.6 are satisfied, so that, for all $\lambda > 0$, the problem (1) has a nontrivial weak solution in Y. □

Proof of Theorem 1.3. Taking into account Lemma 3.2, the rest of the proof is totally similar to the proof of Theorem 1.2. □

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