MATHEMATICA, 66 (89), N° 2, 2024, pp. 210–215

HELMHOLTZ EQUATIONS FOR THE LAPLACE OPERATOR AND ITS POWERS

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Abstract. We show that a tempered distribution u on $\mathbb{R}^d$ is a solution of $(-\Delta)^s u = u$ if and only if $-\Delta u = u$. This result holds for any $s \in (0, \infty)$ and any dimension $d \geq 1$. Our proof uses Fourier analysis.

MSC 2020. 35J05.
Key words. Helmholtz equations.

1. INTRODUCTION AND MAIN RESULTS

The solutions in the sense of distributions of the classical Helmholtz equation $-u_{xx} = u$ in $\mathbb{R}$ are of the form $u(x) = a\cos x + b\sin x$. In higher dimensions, solutions of

$$-\Delta u = u \quad \text{in} \quad \mathbb{R}^d \tag{1}$$

are functions which are bounded, infinitely differentiable and vanishing at ∞. They are explicitly expressed in terms of Bessel functions and spherical harmonics, see [1]. Now consider the fractional Helmholtz equation

$$(-\Delta)^s u = u \quad \text{in} \quad \mathbb{R}^d, \tag{2}$$

where $(-\Delta)^s$, $0 < s < 1$, is the fractional power of the Laplace operator which is defined in Section 2. In dimension one, Fall and Weth [3] proved that bounded solutions of the fractional equation (2) are in the form $u(x) = a\cos x + b\sin x$. In higher dimensions, Guan, Murugan and Wei [5] proved that functions which are bounded and vanishing at ∞ are solutions of the fractional Helmholtz equation (2) if and only if they are solutions of the classical Helmholtz equation (1). The authors extend this result for $1 < s \leq 2$ and $s \in \mathbb{N}^*$ when solutions are assumed to be bounded an infinitely differentiable functions on $\mathbb{R}^d$.

The following result complement the ones of [5] and [3]. Let $S'(\mathbb{R}^d)$ denote the space of tempered distributions on $\mathbb{R}^d$.

THEOREM 1.1. *Let $d \geq 1$ and $s > 0$. For $u \in S'(\mathbb{R}^d)$, we have $(-\Delta)^s u = u$ if and only if $-\Delta u = u$.*

The author thanks the referee for his helpful comments and suggestions.

DOI: 10.24193/mathcluj.2024.2.05

Taking into account that solutions of equation (1) are necessarily bounded functions on $\mathbb{R}^d$ and that functions of polynomial growth at infinity are in $S'(\mathbb{R}^d)$, we immediately conclude from Theorem 1.1 that:

Corollary 1.2. *The fractional Helmholtz equation* (2) *has no unbounded solutions of polynomial growth at infinity.*

Since solutions of equation (1) in higher dimensions are functions which are bounded, infinitely differentiable and vanishing at ∞, it follows from Theorem 1.1 that:

Corollary 1.3. *Let $d \geq 2$ and $s \in (0,\infty)$. All solutions in $S'(\mathbb{R}^d)$ of $(-\Delta)^s u = u$ are functions which are bounded, infinitely differentiable and vanishing at ∞.*

Next, we investigate solutions of the modified Helmholtz equation

$$-(-\Delta)^s u = u \quad \text{in} \quad \mathbb{R}^d. \tag{3}$$

Theorem 1.4. *Let $d \geq 1$ and $s \in (0,\infty)$. If $u \in S'(\mathbb{R}^d)$ is a solution of equation* (3) *then $u = 0$.*

Since continuous functions on $\mathbb{R}^d$ with slow growth are tempered distributions, it follows from Theorem 1.4 that

Corollary 1.5. *Let $u \in C(\mathbb{R}^d)$ be a solution of equation* (3)*. If*

$$|u(x)| \leq C\,(1+|x|)^m \quad \forall\, x \in \mathbb{R}^d$$

with some $C > 0$ and a nonnegative integer m, then $u = 0$.

Finally, we establish mean value properties of solutions of the fractional Helmholtz equation (2). Denote by

$$\eta_r(x) = \begin{cases} \frac{C(d,s) r^{2s}}{|x|^d (|x|^2 - r^2)^s}, & \text{if } |x| > r \\ 0, & \text{if } |x| \leq r. \end{cases}$$

The constant $C(d,s)$ is chosen so that

$$\int_{\mathbb{R}^d} \eta_r(x)\,\mathrm{d}x = 1,$$

and therefore

$$C(d,s) = \frac{\Gamma(d/2)\sin(\pi s)}{\pi^{1+d/2}}.$$

For a continuous function u on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \frac{|u(x)|}{(1+|x|)^{d+2s}}\mathrm{d}x < \infty, \tag{4}$$

we define

$$M(u,x,r) := \int_{\mathbb{R}^d} u(x+y)\eta_r(y)\mathrm{d}y.$$

A continuous function u on $\mathbb{R}^d$ satisfying (4) is a solution of $(-\Delta)^s u = 0$ in the sense of distributions if and only if u satisfies the mean value property $M(u, x, r) = u(x)$ for every $x \in \mathbb{R}^d$ and every $r > 0$, see for instance [7].

THEOREM 1.6. *Let $s \in (0, 1)$ and $d \geq 1$. If u is a solution of the fractional Helmholtz equation* (2) *then*

$$M(u, x, r) = \frac{\Gamma(d/2)}{\Gamma(s)} G^{20}_{13}\left(\frac{r^2}{4} \,\middle|\, \begin{matrix} 1 \\ s, \quad 0, \quad 1 - d/2 \end{matrix}\right) u(x)$$

for every $x \in \mathbb{R}^d$ and every $r > 0$, where G denotes Meijer's G-function.

2. PROOFS OF THEOREMS

Let $C_c^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support. For $s \in (0, 1)$, the fractional Laplacian $(-\Delta)^s$ is defined, for $\varphi \in C_c^\infty(\mathbb{R}^d)$, by

$$\begin{aligned}(-\Delta)^s \varphi(x) &= \mathcal{A}_{d,s} P.V \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} \mathrm{d}y \\ &= \mathcal{A}_{d,s} \lim_{\varepsilon \to 0} \int_{\{|x-y| \geq \varepsilon\}} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} \mathrm{d}y,\end{aligned}$$

where

$$\mathcal{A}_{d,s} = 2^{2s} \Gamma(\frac{d}{2} + s)/(\pi^{d/2} |\Gamma(-s)|).$$

Let D be a bounded domain of $\mathbb{R}^d$. Using the Taylor's expansion of $\varphi \in C_c^\infty(D)$, we obtain

$$|(-\Delta)^s \varphi(x)| \leq \frac{C(D, s)}{(1 + |x|)^{d+2s}} \max_{|\alpha| \leq 2} \left(\sup_{x \in D} |\partial^\alpha \varphi(x)| \right), \tag{5}$$

see for instance [2, Lemma 3.5]. Therefore, $(-\Delta)^s u$ defines a distribution for $u \in L_s(\mathbb{R}^d)$ by the formula

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^d} u(x) (-\Delta)^s \varphi(x) \mathrm{d}x,$$

where

$$L_s(\mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R};\ \int_{\mathbb{R}^d} \frac{|u(x)|}{(1 + |x|)^{d+2s}} \mathrm{d}x < \infty \right\}.$$

However, it is not clear whether $(-\Delta)^s u \in S'(\mathbb{R}^d)$ the space of tempered distribution since an analogue estimation of (5) for $\varphi \in S(\mathbb{R}^d)$ seems to be not valid.

Let $[s]$ denote the integer part of $s > 0$. For $s \in (0, \infty)$, a solution of $(-\Delta)^s u = u$ is understood to be a tempered distribution u such that

$$(-\Delta)^{s-[s]} (-\Delta)^{[s]} u = u.$$

LEMMA 2.1. *Let $s > 0$ and $u \in S'(\mathbb{R}^d)$. If $(-\Delta)^s u \in S'(\mathbb{R}^d)$ then*

$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\widehat{u} \tag{6}$$

in $S'(\mathbb{R}^d)$, where $\widehat{u}$ denotes the Fourier transform of u.

Proof. We first assume that $s \in (0,1)$. For $\varphi \in S(\mathbb{R}^d)$, we have

$$\widehat{(-\Delta)^s \varphi}(\xi) = |\xi|^{2s}\widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^d.$$

The proof of this identity can be found in many papers, see for instance [8]. Thus

$$\langle \widehat{(-\Delta)^s u}, \varphi\rangle = \langle u, \widehat{(-\Delta)^s \varphi}\rangle = \langle u, |\xi|^{2s}\widehat{\varphi}\rangle = \langle |\xi|^{2s}\widehat{u}, \varphi\rangle.$$

Hence (6) holds for $s \in (0,1)$. Now, for any $s > 0$, we obtain

$$\widehat{(-\Delta)^s u} = |\xi|^{2(s-[s])}\widehat{(-\Delta)^{[s]} u} = |\xi|^{2(s-[s])}|\xi|^{2[s]}\widehat{u} = |\xi|^{2s}\widehat{u}.$$

This completes the proof. □

The following simple result is the key of our proofs.

LEMMA 2.2. *For $s \in (0,\infty)$, let ϕ_s be the function defined for $\xi \in \mathbb{R}^d$ by*

$$\phi_s(\xi) = \frac{1-|\xi|^{2s}}{1-|\xi|^2}.$$

Then

(i) *The functions ϕ_s and $1/\phi_s$ are C^∞ on $\mathbb{R}^d$.*
(ii) *The functions ϕ_s, $1/\phi_s$ and their derivatives of arbitrary order have polynomial growth at infinity.*
(iii) *The multiplication by ϕ_s is an automorphism on $S(\mathbb{R}^d)$.*

Proof. It is clear that ϕ_s is C^∞ on $\mathbb{R}^d\backslash\{\xi;\ |\xi| = 1\}$. Let $\varepsilon > 0$ small enough. For

$$\xi \in \Omega_\varepsilon := \{\xi;\ 1-\varepsilon < |\xi|^2 < 1+\varepsilon\},$$

we write

$$\phi_s(\xi) = f_s(1-|\xi|^2) \quad \text{with} \quad f_s(r) = \frac{1-(1-r)^s}{r},\ r \in (-\varepsilon,\varepsilon).$$

The functions $r \to 1-(1-r)^s$ and $r \to r$ are analytic on $(-\varepsilon,\varepsilon)$. Since f_s is continuous at 0, this implies that f_s is C^∞ on $(-\varepsilon,\varepsilon)$ as quotient of two analytic functions. Thus ϕ_s is C^∞ on Ω_ε. Consequently, ϕ_s is C^∞ on the whole of $\mathbb{R}^d$. It is easily seen that the functions $\xi \to 1-|\xi|^{2s}$ and $\xi \to 1-|\xi|^2$ and their derivatives have polynomial growth at infinity, and hence so does ϕ_s. Similar arguments show that $1/\phi_s$ is C^∞ on $\mathbb{R}^d$ and $1/\phi_s$ and its derivatives have polynomial growth at infinity. We omit the proof to avoid repetition. The multiplication operator $\varphi \to \phi_s\varphi$ is an automorphism on $S(\mathbb{R}^d)$ follows immediately from the first and the second statements. □

Proof of Theorem 1.1. Let $d \geq 1$ and $s > 0$. Let $u \in S'(\mathbb{R}^d)$ be a solution of equation (2). By applying the Fourier transform on both sides of (2), we obtain using (6) that

$$(1 - |\xi|^{2s})\widehat{u} = 0 \quad \text{in} \quad S'(\mathbb{R}^d).$$

By Lemma 2.2, the multiplication by ϕ_s is an automorphism on $S(\mathbb{R}^d)$. This implies that

$$0 = \langle \widehat{u}, (1 - |\xi|^{2s})\varphi\rangle = \langle \widehat{u}, (1 - |\xi|^2)\psi\rangle, \quad \psi \in S(\mathbb{R}^d),$$

which means that $|\xi|^2\widehat{u} = \widehat{u}$ in $S'(\mathbb{R}^d)$. Applying the inverse of the Fourier transform, we conclude that u is a solution of equation (1). The same steps show that any solution $u \in S'(\mathbb{R}^d)$ of equation (1) is a solution of equation (2). This completes the proof of the theorem. □

Proof of Theorem 1.4. Let $s \in (0, \infty)$ and $u \in S'(\mathbb{R}^d)$ a solution of equation (3). As in the proof of Theorem 1.1, by applying the Fourier transform in both sides of (3), we obtain

$$-|\xi|^{2s}\widehat{u} = \widehat{u} \quad \text{in} \quad S'(\mathbb{R}^d).$$

This means that $(1 + |\xi|^{2s})\widehat{u} = 0$ which implies that $\widehat{u} = 0$, and hence $u = 0$ since the Fourier transform is injective on $S'(\mathbb{R}^d)$. This completes the proof of the theorem. □

Proof of Theorem 1.6. Let $u \in C(\mathbb{R}^d) \cap L_s(\mathbb{R}^d)$ be a solution of the Helmholtz equation (2). By Theorem 1.1, u is a solution of the classical Helmholtz equation (1). Kuznetsov [6] proved that solutions of (1) satisfy the mean value properties over spheres

$$M^\circ(u, x, r) = j_{d/2-1}(r)u(x) \tag{7}$$

for every $x \in \mathbb{R}^d$ and every $r > 0$, where

$$M^\circ(u, x, r) := \frac{1}{|\partial B_r|}\int_{\partial B_r} u(x + y)\sigma(\mathrm{d}y)$$

and $j_{d/2-1}$ is the normalized Bessel function defined on $\mathbb{R}$ by

$$j_{d/2-1}(r) := \Gamma(d/2)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n + d/2)}\left(\frac{r}{2}\right)^{2n}.$$

Here, B_r denotes the ball of radius r centered at the origin of $\mathbb{R}^d$ and $\sigma(\mathrm{d}y)$ is the surface area measure on the sphere ∂B_r. By spherical coordinates, we have

$$\begin{aligned} M(u, x, r) &= \int_{\mathbb{R}^d} u(x + y)\eta_r(y)\mathrm{d}y \\ &= \frac{2}{\Gamma(s)\Gamma(1 - s)}\int_r^{\infty}\frac{r^{2s}}{t(t^2 - r^2)^s}M^\circ(u, x, t)\,\mathrm{d}t. \end{aligned}$$

Thus, it follows from (7) that

$$\begin{aligned} M(u,x,r) &= \frac{2\,r^{2s}}{\Gamma(s)\Gamma(1-s)}\int_r^\infty \frac{j_{d/2-1}(t)}{t(t^2-r^2)^s}\mathrm{d}t\ u(x) \\ &= \frac{1}{\Gamma(s)\Gamma(1-s)}\int_1^\infty \frac{j_{d/2-1}(r\sqrt{t})}{t(t-1)^s}\mathrm{d}t\ u(x) \\ &= \frac{\Gamma(d/2)}{\Gamma(s)}\,\frac{r^2}{4}\,G_{13}^{20}\left(\frac{r^2}{4}\,\middle|\,\begin{matrix} 0 \\ s-1, \quad -1, \quad -d/2 \end{matrix}\right)u(x). \end{aligned}$$

The last equality follows from 6.592 (3) in [4]. The fact that

$$\frac{r^2}{4}\,G_{13}^{20}\left(\frac{r^2}{4}\,\middle|\,\begin{matrix} 0 \\ s-1, \quad -1, \quad -d/2 \end{matrix}\right) = G_{13}^{20}\left(\frac{r^2}{4}\,\middle|\,\begin{matrix} 1 \\ s, \quad 0, \quad 1-d/2 \end{matrix}\right)$$

completes the proof. □

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Received November 23, 2023
Accepted February 28, 2024

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