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TCHEBYSHEV-TYPE INEQUALITIES WITH THREE FUNCTIONS

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Abstract. In this paper, Tchebyshev-type fractional integral inequalities with three functions are generalized by involving the k-weighted fractional integral of a function with respect to another function ψ.

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Key words. Tchebyshev inequality, Hilfer operator, weighted function.

1. INTRODUCTION

In [2], for the Tchebyshev functional

$$T(f,\, g)(x) = \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\,\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x,$$

Tchebyshev's inequality is given as follows:

$$T(f,\, g)(x) \geq 0,$$

where f and g are two integrable functions synchronous on $[a, b]$.

Belarbi and Dahmani [1] developed the following result about Tchebyshev's inequality using Riemann-Liouville fractional integral operators. Let f and g be two synchronous functions on $[0, +\infty[$. Then, for all $x > 0$, $\alpha > 0$,

$$\mathrm{I}^{\alpha}(f\,g)(x) \geq \frac{\Gamma(\alpha+1)}{x^{\alpha}}\mathrm{I}^{\alpha} f(x)\,\mathrm{I}^{\alpha} g(x).$$

In [10], Sulaiman proved the following result:

$$\begin{aligned}&\frac{x^{\beta}}{\Gamma(\alpha+1)}\,\mathrm{I}^{\alpha}(f\,g\,h)(x) + \frac{x^{\alpha}}{\Gamma(\alpha+1)}\,\mathrm{I}^{\beta}(f\,g\,f)(x)\\ &\geq \mathrm{I}^{\alpha} f(x)\,\mathrm{I}^{\beta}(g\,h)(x) + \mathrm{I}^{\beta} f(x)\,\mathrm{I}^{\alpha}(g\,h)(x) + \mathrm{I}^{\alpha} g(x)\,\mathrm{I}^{\beta}(f\,h)(x)\\ &\quad + \mathrm{I}^{\beta} g(x)\,\mathrm{I}^{\alpha}(f\,h)(x) - \mathrm{I}^{\alpha} h(x)\,\mathrm{I}^{\beta}(f\,g)(x) - \mathrm{I}^{\beta} h(x)\,\mathrm{I}^{\alpha}(f\,g)(x).\end{aligned}$$

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On the other hand, the weighted fractional integrals are defined for an integrable function f on the interval $[a,b]$ and for a differentiable function μ such that $\mu'(t) \neq 0$ for all $t \in [a,b]$, as follows:

$$ {}_{a^+}I_w^{\beta} f(x) = \frac{1}{w(x)\Gamma(\beta)} \int_a^x \mu'(s)(\mu(x)-\mu(s))^{\beta-1} w(s) f(s)\mathrm{d}s, \quad x > a, $$

where w is a weighted function (a positive measurable function)[4].

2. k-WEIGHTED FRACTIONAL OPERATOR

In this section, we present a definition of the k-weighted fractional integrals of a function f with respect to the function ψ and we prove that they are bounded in a specified space. Let $[a,b] \subseteq [0,+\infty)$, where $a < b$.

DEFINITION 2.1. Let $\alpha > 0$, $k > 0$ and ψ be a positive, strictly increasing differentiable function such that $\psi'(s) \neq 0$ for all $s \in [a,b]$. The left and right-sided k-weighted fractional integrals of a function f with respect to the function ψ on $[a,b]$ are defined respectively as follows:

$$ (1) \qquad {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) = \frac{1}{w(x)k\,\Gamma_k(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1} w(t) f(t)\mathrm{d}t, $$

where $a < x \leq b$.

$$ (2) \qquad {}_{b^-}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) = \frac{1}{w(x)k\,\Gamma_k(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\frac{\alpha}{k}-1} w(t) f(t)\mathrm{d}t, $$

where $a \leq x < b$, w is a weighted non-decreasing function and the k-gamma function is defined by

$$ \Gamma_k(\beta) = \int_0^{\infty} t^{\beta-1} e^{-\frac{t^k}{k}} \mathrm{d}t. $$

The space $L_p^W[a,b]$ of all real-valued Lebesgue measurable functions f on $[a,b]$ with norm conditions:

$$ \| f \|_p^W = \left(\int_a^b | f(x) |^p \, W(x)\mathrm{d}x \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < +\infty. $$

is known as weighted Lebesgue space, where W is a weight function (measurable and positive).

(a) Taking $W \equiv 1$, the space $L_p^W[a,b]$ reduces to the classical space $L_p[a,b]$.

(b) If we choose $W(x) = w^p(x)\,\psi'(x)$ and $p = 1$, we get

$$ L_{X_w}[a,b] = \left\{ f : \| f \|_{X_w} = \int_a^b | w(x) f(x) | \, \psi'(x)\mathrm{d}x < \infty \right\}. $$

In the next theorem, we show that the k-weighted fractional operators are bounded.

Theorem 2.2. *The fractional integrals* (1), (2) *are defined for functions* $f \in L_{X_{\psi'}}[a,b]$, *existing almost everywhere and*

$$(3) \qquad {}_{a^+}\mathrm{I}^{\psi}_{w} f(x) \in L_{X_w}[a,b], \qquad {}_{b^-}\mathrm{I}^{\psi}_{w} f(x) \in L_{X_w}[a,b].$$

Moreover

$$(4) \quad \left\| {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w} f(x) \right\|_{X_w} \leq C \, \| f(x) \|_{X_w}, \qquad \left\| {}_{b^-}\mathrm{J}^{\alpha,\psi}_{k,w} f(x) \right\|_{X_w} \leq C \, \| f(x) \|_{X_w},$$

where

$$C = \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)}.$$

Proof. Let $f \in L_{X_w}[a,b]$, By applying Fubini's Theorem, we get

$$\begin{aligned}
&\left\| {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w} f \right\|_{X_w} = \int_a^b \mid w(x) \, {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w} f(x) \mid \psi'(x) \, \mathrm{d}x \\
&\leq \frac{1}{k \, \Gamma_k(\alpha)} \int_a^b \int_a^x |w(s) f(s)| \, \psi'(s) (\psi(x) - \psi(s))^{\frac{\alpha}{k} - 1} \psi'(x) \, \mathrm{d}s \, \mathrm{d}x \\
&= \frac{1}{k \, \Gamma_k(\alpha)} \int_a^b |w(s) f(s)| \left(\int_s^b (\psi(x) - \psi(s))^{\frac{\alpha}{k} - 1} \psi'(x) \, \mathrm{d}x \right) \psi'(s) \, \mathrm{d}s \\
&= \frac{1}{\alpha \, \Gamma_k(\alpha)} \int_a^b |w(s) f(s)| \, (\psi(b) - \psi(s))^{\frac{\alpha}{k}} \, \psi'(s) \, \mathrm{d}s \\
&\leq \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \int_a^b |w(s) f(s)| \, \psi'(s) \, \mathrm{d}s \\
&= C \, \|f\|_{X_w}.
\end{aligned}$$

Similarly

$$\int_a^b \mid w(x) \, {}_{b^-}\mathrm{J}^{\alpha,\psi}_{k,w} f(x) \mid \psi'(x) \mathrm{d}x \leq \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \, \|f\|_{X_w}.$$

This gives us our desired formulas (4) and (3). □

The k-weighted fractional operators depend on the functions w and ψ and produce particular types of k-weighted fractional integrals.

(a) Taking $w(\tau) = 1$, the k-weighted fractional reduces to the k-Hilfer operator of order $\alpha > 0$ or generalized k-fractional integrals.

$${}_{a^+}\mathcal{J}^{\alpha}_{k} f(x) = \frac{1}{k \, \Gamma_k(\alpha)} \int_a^x (\psi(x) - \psi(s))^{\frac{\alpha}{k} - 1} \psi'(s) f(s) \mathrm{d}s, \quad x > a,$$

$${}_{b^-}\mathcal{J}^{\alpha}_{k} f(x) = \frac{1}{k \, \Gamma_k(\alpha)} \int_x^b (\psi(s) - \psi(x))^{\frac{\alpha}{k} - 1} \psi'(s) f(s) \mathrm{d}s, \quad x < b.$$

(b) Taking $\psi(\tau) = \tau$, the k-weighted fractional is simplified to the k-weighted Riemann-Liouville fractional operator of order $\alpha > 0$

$$_{a^+}\mathcal{RL}^{\alpha}_{k,w}f(x) = \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x (x-s)^{\frac{\alpha}{k}-1}w(s)f(s)\mathrm{d}s, \quad x > a,$$

$$_{b^-}\mathcal{RL}^{\alpha}_{k,w}f(x) = \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_x^b (s-x)^{\frac{\alpha}{k}-1}w(s)f(s)\mathrm{d}s, \quad x < b.$$

(c) Using $\psi(\tau) = \ln\tau$, the k-weighted fractional reduces to the k-weighted Hadamard fractional operator of order $\alpha > 0$

$$_{a^+}\mathcal{H}^{\alpha}_{k,w}f(x) = \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x \left(\ln\frac{x}{s}\right)^{\frac{\alpha}{k}-1}w(s)f(s)\frac{\mathrm{d}s}{s}, \quad x > a > 1,$$

$$_{b^-}\mathcal{H}^{\alpha}_{k,w}f(x) = \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_x^b \left(\ln\frac{s}{x}\right)^{\frac{\alpha}{k}-1}w(s)f(s)\frac{\mathrm{d}s}{s}, \quad 1 < x < b.$$

(d) Putting $\psi(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$ where $\rho > 0$, the k-weighted fractional makes it similar to the k-weighted Katugampola fractional operators of order $\alpha > 0$, or (k,s) weighted fractional [7].

$$_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)}\int_a^x \left(x^{\rho+1}-s^{\rho+1}\right)^{\frac{\alpha}{k}-1}w(s)f(s)s^{\rho}\mathrm{d}s,\ x > a,$$

$$_{b^-}\mathcal{K}^{\alpha}_{k,w}f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)}\int_x^b \left(s^{\rho+1}-x^{\rho+1}\right)^{\frac{\alpha}{k}-1}w(s)f(s)s^{\rho}\mathrm{d}s,\ x < b.$$

(e) Setting $\psi(\tau) = \frac{(\tau-a)^{\theta}}{\theta}$ ($\psi(\tau) = -\frac{(b-\tau)^{\theta}}{\theta}$) respectively where $\theta > 0$, the left sided (right sided) k-weighted fractional respectively is reduced to the k-weighted fractional conformable operator of order $\alpha > 0$ [6]:

$$_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x) = \frac{\theta^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)}\int_a^x \left((x-a)^{\theta}-(s-a)^{\theta}\right)^{\frac{\alpha}{k}-1}\frac{w(s)\,f(s)}{(s-a)^{1-\theta}}\mathrm{d}s,$$

for $x > a$; and

$$_{b^-}\mathcal{C}^{\alpha}_{k,w}f(x) = \frac{\theta^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)}\int_x^b \left((b-x)^{\theta}-(b-s)^{\theta}\right)^{\frac{\alpha}{k}-1}\frac{w(s)\,f(s)}{(b-s)^{1-\theta}}\mathrm{d}s,$$

for $x < b$.

3. TCHEBYSHEV-TYPE INEQUALITIES VIA k-WEIGHTED FRACTIONAL OPERATOR

We present some basic notations that we utilize in this study as well as a significant remark.

- A non-decreasing function on $[a, b]$ is an increasing or constant function on $[a, b]$.
- Let $a, b \in \mathbb{R}$ where $b > a$. Two functions f and g are said to be synchronous on $[a, b]$, if for all $x, y \in [a, b]$

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

REMARK 3.1. Since w is non-decreasing on $[a, b]$, by applying the notion of the k-weighted fractional integral (1), we get

$$\begin{aligned} {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(1)(x) &= \frac{1}{w(x)k\,\Gamma_k(\alpha)} \int_a^x \psi'(s)(\psi(x) - \psi(s))^{\frac{\alpha}{k}-1} w(s)\,\mathrm{d}s \\ &\leq \frac{1}{w(a)k\,\Gamma_k(\alpha)} \int_a^x \psi'(s)(\psi(x) - \psi(s))^{\frac{\alpha}{k}-1} w(b)\,\mathrm{d}s \\ &= \frac{w(b)}{w(a)\,k\,\Gamma_k(\alpha)} \int_a^x \psi'(s)(\psi(x) - \psi(s))^{\frac{\alpha}{k}-1}\,\mathrm{d}s, \end{aligned}$$

therefore

$$\frac{w(b)\,(\psi(x) - \psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha + k)} \geq {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(1)(x). \tag{5}$$

THEOREM 3.2. *Let f, g and h be three monotonic functions on $[a, b]$ and ψ be an increasing and positive function on $[a, b]$, having a continuous derivative ψ' on $[a, b]$ and also $x > a$, α, β, $k > 0$. If for all $x, y \in [a, b]$*

$$(f(x) - f(y))(g(x) - g(y))(h(x) - h(y)) \geq 0. \tag{6}$$

Then the following inequalities hold:

$$\begin{aligned} &\frac{w(b)\,(\psi(x) - \psi(a))^{\frac{\beta}{k}}}{w(a)\,\Gamma_k(\alpha + k)}\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g\,h)(x) \\ &\quad - \frac{w(b)\,(\psi(x) - \psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha + k)}\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,g\,h)(x) \\ &\geq {}_{a^+}\mathrm{J}_{k}^{\beta,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(g\,h)(x) - {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(g\,h)(x) \\ &\quad + {}_{a^+}\mathrm{J}_{k}^{\beta,\psi} g(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,h)(x) - {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} g(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,h)(x) \\ &\quad + {}_{a^+}\mathrm{J}_{k}^{\beta,\psi} h(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x) - {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} h(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,g)(x). \end{aligned}$$

Proof. Given the hypothesis (6), for all $t, s \in [a,b]$, we have

$$(f(t)-f(s))(g(t)-g(s))(h(t)-h(s)) \geq 0,$$

then

$$\begin{aligned} f\,g\,h(t) - f\,g\,h(s) \geq & f(s)g(t)h(t) - f(t)g(s)h(s) + g(s)f(t)h(t) \\ & - g(t)f(s)h(s) + h(s)f(t)g(t) - h(t)f(s)g(s). \end{aligned} \tag{7}$$

Multiplying the inequality (7) by

$$\frac{\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}w(t)}{w(x)\,k\,\Gamma_k(\alpha)}$$

and integrating with respect to t over (a,x), we get

$$\begin{aligned} & {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g\,h)(x) - (f\,g\,h)(s)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(1)(x) \\ & \geq f(s)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(g\,h)(x) - g(s)h(s)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) \\ & \quad + g(s)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,h)(x) - f(s)h(s)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}g(x) \\ & \quad + h(s)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x) - f(s)g(s)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}h(x). \end{aligned} \tag{8}$$

Now, multiplying the above inequality (8) by

$$\frac{\psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1}w(s)}{w(x)k\,\Gamma_k(\beta)}$$

and integrating with respect to s over (a,x), we get

$$\begin{aligned} & {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(1)(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(f\,g\,h)(x) - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(1)(x)\,{}_{a^+}\mathrm{J}_{k}^{\beta,\psi}(f\,g\,h)(x) \\ & \geq {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(g\,h)(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(g\,h)(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) \\ & \quad + {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}g(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,h)(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,h)(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}g(x) \\ & \quad + {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}h(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,g)(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}h(x). \end{aligned} \tag{9}$$

Combining inequality (9) and inequality (5) yields inequality (6). □

REMARK 3.3. We present some special cases of the above Theorem 3.2.

(a) Taking $w = 1$, $\psi(x) = 1$ and $k = 1$, we get Theorem 2.2 in [10].

(b) If we choose $w = 1$ and $\psi(x) = \ln x$, we obtain Theorem 5 in [3].

THEOREM 3.4. *Let f and g be two synchronous functions on $[a,b]$, h be a positive function and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$ and also $x > a$, $\alpha, \beta, k > 0$, then the following inequalities hold:*

$$
\begin{aligned}
&\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\beta}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g\,h)(x) \\
&\quad+\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,g\,h)(x) \\
&\geq {}_{a^+}\mathrm{J}_{k}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(g\,h)(x)+{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(g\,h)(x) \\
&\quad+{}_{a^+}\mathrm{J}_{k}^{\beta,\psi}g(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,h)(x)+{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}g(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,h)(x) \\
&\quad-{}_{a^+}\mathrm{J}_{k}^{\beta,\psi}h(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x)-{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}h(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,g)(x).
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g\,h)(x) \\
&\geq \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\Big\{{}_{a^+}\mathrm{J}_{k}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(g\,h)(x) \\
&\quad+{}_{a^+}\mathrm{J}_{k}^{\beta,\psi}g(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,h)(x)-{}_{a^+}\mathrm{J}_{k}^{\beta,\psi}h(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x)\Big\}.
\end{aligned} \tag{11}
$$

Proof. Since f and g are synchronous functions on $[a,b]$, then for all $t, s \in [a,b]$, we have

$$(f(t)-f(s))(g(t)-g(s))(h(t)+h(s)) \geq 0,$$

then

$$
\begin{aligned}
f\,g\,h(t)+f\,g\,h(s) \geq\;& f(s)g(t)h(t)+f(t)g(s)h(s)+g(s)f(t)h(t) \\
&+g(t)f(s)h(s)-h(s)f(t)g(t)-h(t)f(s)g(s),
\end{aligned}
$$

The rest of the proof of inequality (10) is similar to that in Theorem 3.2.

We get the acquired inequality (11) by putting $\beta=\alpha$ through the inequality (10). □

REMARK 3.5. We present some special cases of the above Theorem 3.4.

(a) Taking $w=1$, $\psi(x)=1$ and $k=1$, we get Theorem 2.1 in [10].

(b) Choose $w=1$, $\psi(x)=\ln x$, we obtain Theorem 4 in [3].

Taking $h(x)=\frac{1}{2}$ in the above Theorem 3.4, we get the following corollary.

Corollary 3.6. *Let f and g be two synchronous functions on $[a,b]$ and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$ and also $x>a$, $\alpha,\beta,\ k>0$, then the following inequalities hold:*

$$\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\beta}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x)+\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(f\,g)(x)$$
$$\geq\ {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}g(x)+\ {}_{a^+}\mathrm{J}_{k}^{\beta,\psi}f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}g(x).$$

and

$${}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x)\geq\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}g(x).$$

Remark 3.7. We present some special cases of the above Corollary 3.6.

(a) Taking $w=1$, $k=1$, $b=+\infty$ and $a=0$, we obtain Theorem 7 and Theorem 6 in [8].

(b) Choose $w=1$ and $\psi(x)=\ln x$, we obtain Theorem 3 in [3].

(c) Taking $w=1$, $k=1$, $a=0$ and $\psi(\tau)=\frac{\tau^{\xi+\eta}}{\xi+\eta}$, gives Theorem 2.2 and Theorem 2.1 in [5].

(d) Taking $k=1$, $a=0$ and $\psi(\tau)=\frac{\tau^{\theta}}{\theta}$, yields Theorem 6 and Theorem 5 in [9].

Theorem 3.8. *Let f and g be two integrable functions on $[a,b]$, h be a positive function and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$ and also $x>a$, $\alpha,\beta,\ k>0$, then the following inequalities hold:*

$$\begin{aligned}&{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}h^2(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f^2(x)+\ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}g^2(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}h^2(x)\\&\geq 2\ {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(f\,h)(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}(g\,h)(x).\end{aligned}\tag{12}$$

Proof. For all $t,s\in[a,b]$, we have

$$(f(t)h(s)-h(t)g(s))^2\geq 0,$$

then

$$f^2(t)h^2(s)+h^2(t)g^2(s)\geq 2\,f(t)f(t)g(s)h(s).$$

The rest of the proof of inequality (12) is similar to that in Theorem 3.2. □

Putting $h(x)=1$ through the Theorem 3.8, we get the following corollary.

Corollary 3.9. *Let f and g be two integrable functions on $[a, b]$ and ψ be an increasing and positive function on $[a, b]$, having a continuous derivative ψ' on $[a, b]$ and also $x > a$, α, β, $k > 0$, then the following inequalities hold:*

$$\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\beta}{k}}}{w(a)\,\Gamma_k(\beta+k)}\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f^2(x)+\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{J}^{\beta,\psi}_{k,w}g^2(x)$$
$$\geq 2\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k}f(x)\,{}_{a^+}\mathrm{J}^{\beta,\psi}_{k,w}g(x)$$

and for $\alpha = \beta$ and $f = g$, yields

$${}_{a^+}\mathrm{J}^{\beta,\psi}_{k,w}f^2(x)\geq 2\,\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\left({}_{a^+}\mathrm{J}^{\alpha,\psi}_{k}f(x)\right)^2.$$

Remark 3.10. We present some special cases of the above Corollary 3.9.

(a) Taking $w = 1$, $\psi(x) = 1$ and $k = 1$, we get Theorem 2.3 in [10].

(b) Taking $w = 1$ and $\psi(x) = \ln x$, we obtain Theorem 6 in [3].

4. CONCLUSION

This work gives a new result for Tchebyshev-type inequalities with three functions using k-weighted fractional operators, as well as additional related weight inequalities based on the functions w and ψ.

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