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PRIMAL-PROXIMITY SPACES

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Abstract. Proximity space is one of the common topics in mathematics, computer science, and pattern recognition. Recently, Acharjee et al. introduced a new structure named primal in mathematics. Thus, the main purpose of this paper is to introduce and study primal-proximity spaces. Also, we define two new operators via primal proximity spaces and investigate some of their fundamental properties. In addition, we obtain a new topology, which is weaker than old one, via these new operators. Moreover, we not only discuss some of their properties but also enrich with some examples.

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Key words. Primal, primal-proximity, primal topological space, Kuratowski closure operator.

1. INTRODUCTION

In many areas of mathematics and computer science, topology plays crucial roles. Applications of many topological ideas, to solve various problems of nature, have attracted researchers of different branches of science and social sciences. Many new notions have been introduced in topology, which have enriched topology with several new areas of research.

Some of the most important classical structures of topology are filters [34], ideals [20], and grills [7]. The definition of ideal was introduced by Kuratowski [20]. On the other hand, the notion of grill was introduced in [7]. It is important to observe that the notion of ideal is the dual of filter, but the one of ideal has helped researchers to introduce many new areas of topology, viz., ideal topological spaced [14], I-proximities [16], etc. But to the best of our knowledge, no literature was available on the dual structure of grill prior to [1].

Recently, Acharjee et al. [1] introduced a new structure named 'primal'. They obtained not only some fundamental properties related to primal but also some relationships between topological spaces and primal topological spaces.

A primal [1] is the dual of a grill while the dual of a filter is an ideal.

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Later, Al-Omari et al. [2, 3] introduced several new operators in primal topological spaces using the notion of a primal.

On the other hand, the notion of proximity [11] is also important in topology. Several forms of this notion such as I-proximity [16], μ-proximity [26,35], quasi proximity [30], and multiset proximity [19] have been studied by several researchers. Moreover, applications of proximity can be found in pattern recognition [28], region based theory of space [8,9], artificial intelligence [10], spatial analysis [6], etc. One may refer to [4, 5, 12, 13, 15, 17, 18, 21–25, 27, 29, 31–33] and many others for proximity.

In Section 3, we introduce a new type of proximity named primal-proximity. Also, we define a point-primal proximity operator and investigate some of its fundamental properties in Section 4. In addition, we prove that this operator is a Kuratowski closure operator under special conditions. Moreover, we define one more operator via the point-primal proximity operator. This operator comes across as a Kuratowski closure operator without any condition. Furthermore, we give not only some relationships but also several examples.

2. PRELIMINARIES

In this section, we discuss some preliminary definitions which will be used in the next sections.

Definition 2.1 ([1]). Let X be a non-empty set. A collection $\mathcal{P} \subseteq 2^X$ is called a *primal* on X if it satisfies the following conditions:

(a) $X \notin \mathcal{P}$,
(b) if $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$,
(c) if $A \cap B \in \mathcal{P}$, then $A \in \mathcal{P}$ or $B \in \mathcal{P}$.

Corollary 2.2 ([1]). *Let X be a non-empty set. A collection $\mathcal{P} \subseteq 2^X$ is a primal on X if and only if it satisfies the following conditions:*

(i) $X \notin \mathcal{P}$,
(ii) *if $B \notin \mathcal{P}$ and $B \subseteq A$, then $A \notin \mathcal{P}$,*
(iii) *if $A \notin \mathcal{P}$ and $B \notin \mathcal{P}$, then $A \cap B \notin \mathcal{P}$.*

Example 2.3 ([1]). Let X be a non-empty set. Then,

$$\mathcal{P} = \{A \subseteq X : |A^c| \geq \aleph_0\}$$

is a primal on X, where $\aleph_0$ is the smallest infinite cardinal number.

Definition 2.4 ([11]). A binary relation δ on 2^X is called an (Efremovich) *proximity* on X if δ satisfies the following five conditions:

(a) $A\delta B \Rightarrow B\delta A$,
(b) $A\delta(B \cup C) \Leftrightarrow A\delta B$ or $A\delta C$,
(c) $A\delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$,
(d) $A \cap B \neq \emptyset \Rightarrow A\delta B$,

(e) if $A \not\delta B$, then there exists $C, D \subseteq X$ such that $A \not\delta C^c$, $D^c \not\delta B$ and $C \cap D = \emptyset$.

A proximity space is a pair (X, δ) consisting of a set X and a proximity relation on X. We shall write $A\delta B$ if the sets $A, B \subseteq X$ are δ-related, otherwise we shall write $A \not\delta B$. Throughout this paper, the space $(X, \delta, \mathcal{P})$ means an Ef-proximity space (X, δ) with a primal $\mathcal{P}$ on X. Now, we give the following definition which will be used in Section 5.

Definition 2.5. In a space $(X, \delta, \mathcal{P})$, we say that a subset A of X is *locally* in $\mathcal{P}$ at $x \in X$ if there exists a δ-neighborhood U of x such that $U^c \cup A^c \notin \mathcal{P}$. Also for a subset A of X, the primal local function of A with respect to δ and $\mathcal{P}$, denoted by $A^\diamond(\delta, \mathcal{P})$, simply $A^\diamond(\mathcal{P})$ or $A^\diamond$, is the set

$$\bigcup\{x \in X : A \text{ is not primal locally in } \mathcal{P} \text{ at } x\}$$

i.e.,

$$A^\diamond(\delta, \mathcal{P}) = \bigcup\{x \in X : U^c \cup A^c \notin \mathcal{P} \text{ for every } \delta\text{-neighborhood } U \text{ of } x\}.$$

3. PRIMAL-PROXIMITY SPACES

In this section, we introduce the notion of primal-proximity on X and investigate some of its fundamental properties.

Definition 3.1. A binary relation $\hookrightarrow$ on 2^X with a primal $\mathcal{P}$ on a non-empty set X is called a *primal-proximity* on X if $\hookrightarrow$ satisfies the following conditions:

(a) if $A \hookrightarrow B$, then $B \hookrightarrow A$;
(b) $A \hookrightarrow (B \cup C)$ if and only if $A \hookrightarrow B$ or $A \hookrightarrow C$;
(c) if $A^c \notin \mathcal{P}$, then $A \not\hookrightarrow B$ for all $B \subseteq X$;
(d) if $(A \cap B)^c \in \mathcal{P}$, then $A \hookrightarrow B$;
(e) if $A \not\hookrightarrow B$, then there exist $C, D \subseteq X$ such that $A \not\hookrightarrow C^c$, $D^c \not\hookrightarrow B$ and $(C \cap D)^c \notin \mathcal{P}$.

Definition 3.2. A *primal-proximity space* is a pair $(X, \hookrightarrow)$ consisting of a set X and primal-proximity relation on a non-empty set X. We write $A \hookrightarrow B$ if the sets $A, B \subseteq X$ are $\hookrightarrow$-related, otherwise we write $A \not\hookrightarrow B$.

Remark 3.3. Let X be a non-empty set and $A \subseteq X$ such that $\mathcal{P} = 2^X \setminus \{X\}$.

(a) If $x \in A$, then $\{x\} \hookrightarrow A$.
(b) If $A \not\hookrightarrow B$, then $A \cap B = \emptyset$.

Suppose $A \cap B \neq \emptyset$. Then, there exists at least one point in X such that $a \in A \cap B$. Therefore, $(A \cap B)^c \neq X$. Hence, $(A \cap B)^c \in \mathcal{P}$ since $\mathcal{P} = 2^X \setminus \{X\}$. It follows that $A \hookrightarrow B$, which is impossible. Therefore, $A \cap B = \emptyset$.

Corollary 3.4. *Let $\hookrightarrow$ be a primal-proximity on a non-empty set X. Then, the following hold:*

(i) *if $B \not\hookrightarrow A$, then $A \not\hookrightarrow B$,*
(ii) *$A \not\hookrightarrow (B \cup C)$ if and only if $A \not\hookrightarrow B$ and $A \not\hookrightarrow C$,*
(iii) *if there exists $B \subseteq X$ such that $A \hookrightarrow B$, then $A^c \in \mathcal{P}$,*
(iv) *if $A \not\hookrightarrow B$, then $(A \cap B)^c \notin \mathcal{P}$,*
(v) *if $A \not\hookrightarrow B$, then there exist $C, D \subseteq X$ such that $A \not\hookrightarrow C^c$, $D^c \not\hookrightarrow B$ and $(C \cap D)^c \notin \mathcal{P}$.*

Example 3.5. Let $\mathcal{P}$ be a primal on a non-empty set X and $A, B \subseteq X$. We define a binary relation $\hookrightarrow$ on 2^X as:

$$A \hookrightarrow B \Leftrightarrow A^c, B^c \in \mathcal{P}.$$

Then, $\hookrightarrow$ is a primal-proximity relation. Indeed, one easily finds that $\hookrightarrow$ satisfies conditions, (i) to (iv). We are going to check that $\hookrightarrow$ also satisfies condition (v). Let $A \not\hookrightarrow B$. It follows that $A^c \notin \mathcal{P}$ or $B^c \notin \mathcal{P}$. If $A^c \notin \mathcal{P}$, by taking $C = A^c$ and $D = A$ we have the required properties. If $B^c \notin \mathcal{P}$, by taking $C = B$ and $D = B^c$, we again have the required properties.

Example 3.6. Let $\mathcal{P}$ be a primal on a non-empty set X and $A, B \subseteq X$. We define a binary relation $\hookrightarrow$ on 2^X as:

$$A \hookrightarrow B \Leftrightarrow (A \cap B)^c \in \mathcal{P}.$$

Then, $\hookrightarrow$ is a primal-proximity on X. It follows directly from the definition that $\hookrightarrow$ satisfies conditions (i) to (iv). To prove that $\hookrightarrow$ satisfies condition (v), let $A \not\hookrightarrow B$. It follows that $(A \cap B)^c \notin \mathcal{P}$. If we take $C = B^c$ and $D = B$, then the result can be proven.

Example 3.7. Let $(X, \tau, \mathcal{P})$ be a primal topological space such that $\mathcal{P} = 2^X \setminus \{X\}$. Let (X, τ) be a normal space and $A, B \subseteq X$. Define a binary relation $\hookrightarrow$ on 2^X as:

$$A \hookrightarrow B \Leftrightarrow (\mathrm{cl}(A) \cap \mathrm{cl}(B))^c \in \mathcal{P},$$

where the closure is taken with respect to τ. Then, the binary relation $\hookrightarrow$ is a primal-proximity on X.

Proof. (a) Let $A, B \subseteq X$.

$$\begin{aligned} A \hookrightarrow B &\Leftrightarrow (\mathrm{cl}(A) \cap \mathrm{cl}(B))^c \in \mathcal{P} \\ &\Leftrightarrow (\mathrm{cl}(B) \cap \mathrm{cl}(A))^c \in \mathcal{P} \\ &\Leftrightarrow B \hookrightarrow A. \end{aligned}$$

(b) Let $A, B, C \subseteq X$.

$$\begin{aligned} A \hookrightarrow (B\cup C) &\Leftrightarrow (\mathrm{cl}(A)\cap \mathrm{cl}(B\cup C))^c \in \mathcal{P} \\ &\Leftrightarrow (\mathrm{cl}(A)\cap (\mathrm{cl}(B)\cup \mathrm{cl}(C)))^c \in \mathcal{P} \\ &\Leftrightarrow ((\mathrm{cl}(A)\cap \mathrm{cl}(B))\cup(\mathrm{cl}(A)\cap \mathrm{cl}(C)))^c \in \mathcal{P} \\ &\Leftrightarrow (\mathrm{cl}(A)\cap \mathrm{cl}(B))^c \cap (\mathrm{cl}(A)\cap \mathrm{cl}(C))^c \in \mathcal{P} \\ &\Leftrightarrow (\mathrm{cl}(A)\cap \mathrm{cl}(B))^c \in \mathcal{P} \text{ or } (\mathrm{cl}(A)\cap \mathrm{cl}(C))^c \in \mathcal{P} \\ &\Leftrightarrow A \hookrightarrow B \text{ or } A \hookrightarrow C. \end{aligned}$$

(c) Let $A \hookrightarrow B$ with $\mathcal{P} = 2^X \setminus \{X\}$.

$$\left.\begin{array}{r} A \hookrightarrow B \Leftrightarrow (\mathrm{cl}(A)\cap \mathrm{cl}(B))^c \in \mathcal{P} \Rightarrow (\mathrm{cl}(A))^c \in \mathcal{P} \\ \mathcal{P} = 2^X \setminus \{X\} \end{array}\right\} \Rightarrow (\mathrm{cl}(A))^c \neq X$$

$$\left.\begin{array}{r} \Rightarrow \mathrm{cl}(A) \neq \emptyset \Rightarrow A \neq \emptyset \Rightarrow A^c \neq X \\ \mathcal{P} = 2^X \setminus \{X\} \end{array}\right\} \Rightarrow A^c \in \mathcal{P}.$$

(d) Let $A \not\hookrightarrow B$ with $\mathcal{P} = 2^X \setminus \{X\}$.

$$\left.\begin{array}{r} A \not\hookrightarrow B \Rightarrow (\mathrm{cl}(A)\cap \mathrm{cl}(B))^c \notin \mathcal{P} \\ \mathcal{P} = 2^X \setminus \{X\} \end{array}\right\} \Rightarrow (\mathrm{cl}(A)\cap \mathrm{cl}(B))^c = X$$

$$\Rightarrow \mathrm{cl}(A\cap B) \subseteq \mathrm{cl}(A)\cap \mathrm{cl}(B) = \emptyset \Rightarrow \mathrm{cl}(A\cap B) = \emptyset \Rightarrow A\cap B = \emptyset$$

$$\left.\begin{array}{r} \Rightarrow (A\cap B)^c = X \\ \mathcal{P} = 2^X \setminus \{X\} \end{array}\right\} \Rightarrow (A\cap B)^c \notin \mathcal{P}.$$

(e) Let $A \not\hookrightarrow B$. Then, $(\mathrm{cl}(A)\cap \mathrm{cl}(B))^c \notin \mathcal{P}$. So, $\mathrm{cl}(A)\cap \mathrm{cl}(B) = \emptyset$. Since (X, τ) is normal space, there exist two disjoint open sets in τ, C and D such that $\mathrm{cl}(A) \subseteq C$ and $\mathrm{cl}(B) \subseteq D$. Hence, C^c is closed and $\mathrm{cl}(A)\cap C^c = \emptyset$. This implies $\mathrm{cl}(A) \not\hookrightarrow C^c$. Since $C\cap D = \emptyset$, we have $C \subseteq D^c$. It follows that $\mathrm{cl}(C) \subseteq D^c$ since D^c is closed. Therefore, $\mathrm{cl}(C)\cap \mathrm{cl}(B) = \emptyset$ and $(\mathrm{cl}(C)\cap \mathrm{cl}(B))^c \notin \mathcal{P}$. Hence, $C \not\hookrightarrow B$. Let $E = C^c$. Then, $A \not\hookrightarrow B$ implies that there exists a subset E such that $A \not\hookrightarrow E$, $E^c \not\hookrightarrow B$, and $(E\cap E^c)^c \notin \mathcal{P}$. □

4. POINT-PRIMAL PROXIMITY OPERATOR

This section introduces the point-primal proximity operator. Here, we study several properties of a primal-proximity space using this operator.

Definition 4.1. Let $(X, \hookrightarrow)$ be a primal-proximity space. Then, the operator $\overset{\hookrightarrow}{(\cdot)} : 2^X \to 2^X$ defined by

$$\overset{\hookrightarrow}{A} = \{x \in X | \{x\} \hookrightarrow A\}$$

is said to be *point-primal proximity operator*. Moreover, $\overset{\hookrightarrow}{A}$ is said to be the *point-primal proximity* of A.

We now provide the following lemma without the proof.

Lemma 4.2. *Let $\mathcal{P}$ be a primal on a non-empty set X. If $A \hookrightarrow B$, $A \subseteq C$, and $B \subseteq D$, then $C \hookrightarrow D$.*

Lemma 4.3. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B \subseteq X$. If $B \not\hookrightarrow A$, then $\overset{\hookrightarrow}{A} \subseteq B^c$.*

Proof. Suppose $\overset{\hookrightarrow}{A} \cap B \neq \emptyset$. Then, there exists at least a point $x \in \overset{\hookrightarrow}{A} \cap B$. So, $x \in \overset{\hookrightarrow}{A}$ and $x \in B$, i.e. $\{x\} \hookrightarrow A$ and $\{x\} \subseteq B$. By Lemma 4.2, it implies that $A \hookrightarrow B$, which is a contradiction. Hence, $\overset{\hookrightarrow}{A} \subseteq B^c$. □

Theorem 4.4. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B \subseteq X$. If $B \not\hookrightarrow A$, then $B \not\hookrightarrow \overset{\hookrightarrow}{A}$.*

Proof. Let $B \not\hookrightarrow A$. Then by (e) of Definition 3.1, there exist $C, D \subseteq X$ such that $B \not\hookrightarrow C^c$, $D^c \not\hookrightarrow A$ and $(C \cap D)^c \notin \mathcal{P}$. This result, combined with Lemma 4.3, implies that $\overset{\hookrightarrow}{A} \subseteq D$. Now, we want to prove that $\overset{\hookrightarrow}{A} \subseteq C^c$. Let $x \in \overset{\hookrightarrow}{A}$, then $\{x\} \hookrightarrow A$. Suppose $x \in C$. Then $x \in C \cap D$ and $(C \cap D)^c \subseteq X \setminus \{x\}$, so $X \setminus \{x\} \notin \mathcal{P}$. Then, by Definition 3.1 (c), we have $\{x\} \not\hookrightarrow A$, which is a contradiction. Hence, $x \in C^c$. So, $\overset{\hookrightarrow}{A} \subseteq C^c$. Now, we have by Lemma 4.2, $B \not\hookrightarrow \overset{\hookrightarrow}{A}$. Hence, the theorem is proven. □

Due to Theorem 4.4 and (a) of Definition 3.1, we have the following corollary.

Corollary 4.5. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B \subseteq X$. If $B \not\hookrightarrow A$, then $\overset{\hookrightarrow}{B} \not\hookrightarrow \overset{\hookrightarrow}{A}$.*

Theorem 4.6. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B \subseteq X$. Then, the following properties hold:*

(i) *if $A \subseteq B$, then $\overset{\hookrightarrow}{A} \subseteq \overset{\hookrightarrow}{B}$;*
(ii) $\overset{\hookrightarrow}{(A \cap B)} \subseteq \overset{\hookrightarrow}{A} \cap \overset{\hookrightarrow}{B}$;
(iii) $\overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{B} = \overset{\hookrightarrow}{(A \cup B)}$;
(iv) $\overset{\hookrightarrow}{\overset{\hookrightarrow}{A}} \subseteq \overset{\hookrightarrow}{A}$;
(v) *if $A^c \notin \mathcal{P}$, then $\overset{\hookrightarrow}{A} = \emptyset$;*
(vi) $\overset{\hookrightarrow}{\emptyset} = \emptyset$;
(vii) $\overset{\hookrightarrow}{A} \setminus \overset{\hookrightarrow}{B} \subseteq \overset{\hookrightarrow}{(A \setminus B)}$;
(viii) *if $B^c \notin \mathcal{P}$, then $\overset{\hookrightarrow}{(A \cup B)} = \overset{\hookrightarrow}{A} = \overset{\hookrightarrow}{(A \setminus B)}$;*
(ix) *if $[(A \setminus B) \cup (B \setminus A)]^c \notin \mathcal{P}$, then $\overset{\hookrightarrow}{A} = \overset{\hookrightarrow}{B}$.*

Proof. (i) Let $A \subseteq B$ and $x \in \overset{\hookrightarrow}{A}$. Then, $\{x\} \hookrightarrow A$. Since $A \subseteq B$, by Lemma 4.2, we have $\{x\} \hookrightarrow B$. Hence, $x \in \overset{\hookrightarrow}{B}$.

(ii) Let $A, B \subseteq X$. From (i), it is not difficult to see that $\overset{\hookrightarrow}{(A \cap B)} \subseteq \overset{\hookrightarrow}{A}$ and $\overset{\hookrightarrow}{(A \cap B)} \subseteq \overset{\hookrightarrow}{B}$. Thus, we get $\overset{\hookrightarrow}{(A \cap B)} \subseteq \overset{\hookrightarrow}{A} \cap \overset{\hookrightarrow}{B}$.

(iii) Let $A, B \subseteq X$. From (i), we can easily find that $\overset{\hookrightarrow}{A} \subseteq \overset{\hookrightarrow}{(A \cup B)}$ and $\overset{\hookrightarrow}{B} \subseteq \overset{\hookrightarrow}{(A \cup B)}$. Thus, obviously $\overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{B} \subseteq \overset{\hookrightarrow}{(A \cup B)}$.

Conversely, let $y \in \overset{\hookrightarrow}{(A \cup B)}$. Then, $\{y\} \hookrightarrow A \cup B$. Due to Definition 3.1, either $\{y\} \hookrightarrow A$ or $\{y\} \hookrightarrow B$. It indicates that either $y \in \overset{\hookrightarrow}{A}$ or $y \in \overset{\hookrightarrow}{B}$. So, we can conclude that $\overset{\hookrightarrow}{(A \cup B)} \subseteq \overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{B}$.

(iv) Let $A \subseteq X$ and let $x \notin \overset{\hookrightarrow}{A}$. Then, $\{x\} \not\hookrightarrow A$. So, by Theorem 4.4, we have $\{x\} \not\hookrightarrow \overset{\hookrightarrow}{A}$. Hence, $x \notin \overset{\hookrightarrow}{\overset{\hookrightarrow}{A}}$. Thus, we get $\overset{\hookrightarrow}{\overset{\hookrightarrow}{A}} \subseteq \overset{\hookrightarrow}{A}$.

(v) Let $A^c \notin \mathcal{P}$. Then, by (c) of Definition 3.1, $A \not\hookrightarrow B$ for all subsets B of X. Therefore, we have $A \not\hookrightarrow \{x\}$ for all $x \in X$. Again, by (a) of Definition 3.1, $\{x\} \not\hookrightarrow A$ for all $x \in X$. This means $x \notin \overset{\hookrightarrow}{A}$ for all $x \in X$. Hence, $\overset{\hookrightarrow}{A} = \emptyset$.

(vi) Let $\overset{\hookrightarrow}{\emptyset} \neq \emptyset$. Thus, we assume that $x \in \overset{\hookrightarrow}{\emptyset}$. Hence, $\{x\} \hookrightarrow \emptyset$. By (a) of Definition 3.1, $\emptyset \hookrightarrow \{x\}$. Again, by (c) of Definition 3.1, $\emptyset^c = X \in \mathcal{P}$; which is a contradiction to the definition of primal $\mathcal{P}$. Hence, $\overset{\hookrightarrow}{\emptyset} = \emptyset$.

(vii) For all $A, B \subseteq X$, $A = (A \setminus B) \cup (A \cap B)$. By (iii), we have

$$\overset{\hookrightarrow}{A} = \overset{\hookrightarrow}{(A \setminus B)} \cup \overset{\hookrightarrow}{(A \cap B)} \subseteq \overset{\hookrightarrow}{(A \setminus B)} \cup \overset{\hookrightarrow}{B}.$$

Hence, $\overset{\hookrightarrow}{A} \setminus \overset{\hookrightarrow}{B} \subseteq \overset{\hookrightarrow}{(A \setminus B)}$.

(viii) If $B^c \notin \mathcal{P}$, then $\overset{\hookrightarrow}{(A \cup B)} = \overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{B} = \overset{\hookrightarrow}{A} \cup \emptyset = \overset{\hookrightarrow}{A}$. Also, $\overset{\hookrightarrow}{A} \setminus \overset{\hookrightarrow}{B} \subseteq \overset{\hookrightarrow}{(A \setminus B)}$, thus $\overset{\hookrightarrow}{A} \subseteq \overset{\hookrightarrow}{(A \setminus B)}$. Moreover, $\overset{\hookrightarrow}{(A \setminus B)} = \overset{\hookrightarrow}{(A \cap B^c)} \subseteq \overset{\hookrightarrow}{A} \cap \overset{\hookrightarrow}{B^c} \subseteq \overset{\hookrightarrow}{A}$. Hence,

$$\overset{\hookrightarrow}{(A \cup B)} = \overset{\hookrightarrow}{A} = \overset{\hookrightarrow}{(A \setminus B)}.$$

(ix) If $[(A \setminus B) \cup (B \setminus A)]^c \notin \mathcal{P}$, then $(A \setminus B)^c \notin \mathcal{P}$ and $(B \setminus A)^c \notin \mathcal{P}$. Since $\overset{\hookrightarrow}{A} = \overset{\hookrightarrow}{[(A \setminus B) \cup (A \cap B)]}$ and $(A \setminus B)^c \notin \mathcal{P}$, by using (viii), we get $\overset{\hookrightarrow}{A} = \overset{\hookrightarrow}{(A \cap B)} \subseteq \overset{\hookrightarrow}{B}$. It follows that $\overset{\hookrightarrow}{A} \subseteq \overset{\hookrightarrow}{B}$. Similarly, since $\overset{\hookrightarrow}{B} = \overset{\hookrightarrow}{[(B \setminus A) \cup (B \cap A)]}$ and $(B \setminus A)^c \notin \mathcal{P}$, by using (viii), we get $\overset{\hookrightarrow}{B} = \overset{\hookrightarrow}{(B \cap A)} \subseteq \overset{\hookrightarrow}{A}$. It follows that $\overset{\hookrightarrow}{B} \subseteq \overset{\hookrightarrow}{A}$. Hence, $\overset{\hookrightarrow}{A} = \overset{\hookrightarrow}{B}$. □

Remark 4.7. Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A \subseteq X$. The inclusion $A \subseteq \overset{\hookrightarrow}{A}$ need not be true in general as shown by the following example:

Example 4.8. Let $X = \{a, b, c\}$, $\mathcal{P} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$, and the binary relation $\hookrightarrow$ on 2^X defined as in Example 3.6. For the subset $A = \{b\}$, we have $A = \{b\} \nsubseteq \emptyset = \overset{\hookrightarrow}{A}$.

Theorem 4.9. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B \subseteq X$. Then, the following statements hold:*

(i) $A \cap \overset{\hookrightarrow}{B} = \emptyset$ *for all* $A^c \notin \mathcal{P}$ *and* $B \subseteq X$,

(ii) $\{x\} \hookrightarrow X$ *for all* $x \in X$ *if and only if* $\mathcal{P} = 2^X \setminus \{X\}$,

(iii) *if* $\mathcal{P} = 2^X \setminus \{X\}$, *then* $\overset{\hookrightarrow}{X} = X$.

Proof. (i) Let $A^c \notin \mathcal{P}$ and suppose $A \cap \overset{\hookrightarrow}{B} \neq \emptyset$. It follows that $A \not\hookrightarrow B$ since $A^c \notin \mathcal{P}$ and also $\overset{\hookrightarrow}{B} \nsubseteq A^c$. Hence, by Lemma 4.3, we have $A \hookrightarrow B$, which is a contradiction. Thus, $A \cap \overset{\hookrightarrow}{B} = \emptyset$.

(ii) If $\{x\} \hookrightarrow X$ for all $x \in X$, then by (iii) of Corollary 3.4, we have $\{x\}^c \in \mathcal{P}$ for all $x \in X$. Hence, $\mathcal{P} = 2^X \setminus \{X\}$. Conversely, if $\mathcal{P} = 2^X \setminus \{X\}$, then $(\{x\} \cap X)^c = (\{x\})^c \in \mathcal{P}$ and by (d) of Definition 3.1, we have $\{x\} \hookrightarrow X$ for all $x \in X$.

(iii) Let $x \in X$. Since $\mathcal{P} = 2^X \setminus \{X\}$, then $(\{x\})^c = (\{x\} \cap X)^c \in \mathcal{P}$ and by (d) of Definition 3.1, we get $\{x\} \hookrightarrow X$ for all $x \in X$. Hence, $\overset{\hookrightarrow}{X} = X$. □

Theorem 4.10. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space. If $A, B, C \subseteq X$ and $B \subsetneqq C$ such that $A \not\hookrightarrow B$ but $A \hookrightarrow C$, then $A \hookrightarrow (C \setminus B)$.*

Proof. Let $A, B, C \subseteq X$ and $B \subsetneqq C$. We consider $D = C \setminus B$. Since $A \hookrightarrow C$, then $A \hookrightarrow B \cup (C \setminus B) = B \cup D$. Then by Definition 3.1, $A \hookrightarrow B$ or $A \hookrightarrow D$. Now, $A \hookrightarrow B$ is not possible since we consider $A \not\hookrightarrow B$. Then obviously, $A \hookrightarrow (C \setminus B)$. □

Theorem 4.11. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B \subseteq X$. If $A \not\hookrightarrow B$, then there exists $C \subseteq X$ such that $A \not\hookrightarrow C$ and $B \not\hookrightarrow C^c$.*

Proof. Since $A \not\hookrightarrow B$, thus by (e) of Definition 3.1, there exist $M, N \subseteq X$ such that $A \not\hookrightarrow M^c$, $N^c \not\hookrightarrow B$ and $(M \cap N)^c \notin \mathcal{P}$. Let $M = X \setminus C$ and $N = C$. Then, $(M \cap N)^c = X \notin \mathcal{P}$. Also, $A \not\hookrightarrow (X \setminus C)^c$, $C^c \not\hookrightarrow B$. It yields $A \not\hookrightarrow C$, $B \not\hookrightarrow C^c$. Hence, the proof is completed. □

Corollary 4.12. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B, C \subseteq X$. If $A \not\hookrightarrow B$ and $B \hookrightarrow C$, then $A \not\hookrightarrow C$.*

5. PROXIMAL CLOSED SETS AND TOPOLOGY

In this section, proximal closed sets are defined. Moreover, various results between a primal-proximity space and a primal topological space are obtained using proximal closed sets and related notions.

Definition 5.1. Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space. Then, a subset F of X is called *proximity-closed* if and only if $\{x\} \hookrightarrow F$ implies $x \in F$.

Lemma 5.2. *If there is a point $x \in X$ such that $A \hookrightarrow \{x\}$ and $\{x\} \hookrightarrow B$, then $A \hookrightarrow B$.*

Proof. Suppose $A \not\hookrightarrow B$, by Theorem 4.11, there exists a subset C such that $A \not\hookrightarrow C$ and $C^c \not\hookrightarrow B$. Now, either $x \in C$ or $x \in C^c$.

Case 1. If $x \in C$, then $A \not\hookrightarrow \{x\}$. For if $A \hookrightarrow \{x\}$, then by Lemma 4.2, we get $A \hookrightarrow C$ which is a contradiction.

Case 2. If $x \in C^c$, then $\{x\} \not\hookrightarrow B$. Therefore, if $A \hookrightarrow \{x\}$ and $\{x\} \hookrightarrow B$, then $A \hookrightarrow B$. □

Theorem 5.3. *The collection of complements of all proximity-closed sets of $(X, \hookrightarrow, \mathcal{P})$ forms a topology on X. This topology is denoted by $\overset{\hookrightarrow}{\tau}$.*

Proof. Since X and $\emptyset$ are proximity-closed in $(X, \hookrightarrow, \mathcal{P})$, their complements $\emptyset$ and X are in $\overset{\hookrightarrow}{\tau}$.

Let $\{F_i : i \in I\}$ be a collection of proximity-closed sets. If

$$\{x\} \hookrightarrow \bigcap\{F_i : i \in I\},$$

then $\{x\} \hookrightarrow F_i$ for every $i \in I$, by Lemma 4.2. Since F_i is proximity-closed, $x \in F_i$ for every $i \in I$. Hence,

$$x \in \bigcap\{F_i : i \in I\} \text{ and } \bigcap\{F_i : i \in I\} \text{ is proximity-closed.}$$

Therefore, if $(X \setminus F_i) \in \overset{\hookrightarrow}{\tau}$ for every $i \in I$, then

$$\bigcup\{X \setminus F_i : i \in I\} \text{ is the complement of } \bigcap\{F_i : i \in I\},$$

which belongs to $\overset{\hookrightarrow}{\tau}$.

Finally, let F_1 and F_2 be two proximity-closed sets. If $\{x\} \hookrightarrow F_1 \cup F_2$, then $\{x\} \hookrightarrow F_1$ or $\{x\} \hookrightarrow F_2$. Thus, $x \in F_1$ or $x \in F_2$ since F_1 and F_2 are proximity-closed. This implies $x \in F_1 \cup F_2$. Thus, $F_1 \cup F_2$ is proximity-closed. Therefore, if $X \setminus F_1 \in \overset{\hookrightarrow}{\tau}$ and $X \setminus F_2 \in \overset{\hookrightarrow}{\tau}$, then

$$(X \setminus F_1) \cap (X \setminus F_2) = X \setminus (F_1 \cup F_2) \in \overset{\hookrightarrow}{\tau}.$$

Hence, $\overset{\hookrightarrow}{\tau}$ is a topology on X. □

Theorem 5.4. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space. The set $\overset{\hookrightarrow}{A}$ is the closure of A where the closure is taken with respect to the topology $\overset{\hookrightarrow}{\tau}$ and denoted by* $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$.

Proof. Let $x \in \overset{\hookrightarrow}{A}$. Then, $\{x\} \hookrightarrow A$. By Lemma 4.2, $\{x\} \hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$ since $A \subseteq \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$ and $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$ is proximity-closed in $\overset{\hookrightarrow}{\tau}$. Thus, $x \in \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$. Hence, $\overset{\hookrightarrow}{A} \subseteq \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$.

Conversely, let $x \notin \overset{\hookrightarrow}{A}$. Then, $\{x\} \not\hookrightarrow A$. By Theorem 4.11, there exists a subset C such that $\{x\} \not\hookrightarrow C$ and $C^c \not\hookrightarrow A$. Since there is no point of C^c which is related to A, then $\overset{\hookrightarrow}{A} \subseteq C$. By Lemma 4.2, $\{x\} \not\hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$. Thus, $\overset{\hookrightarrow}{A}$ is proximity-closed in $\overset{\hookrightarrow}{\tau}$. Therefore, $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A) \subseteq \overset{\hookrightarrow}{A}$. Hence, $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A) = \overset{\hookrightarrow}{A}$. □

Definition 5.5 ([20]). The operator $\Phi : 2^X \to 2^X$ is a *Kuratowski closure operator* provided:

(a) $\Phi(\emptyset) = \emptyset$;
(b) $A \subseteq \Phi(A)$ for every $A \in 2^X$;
(c) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ for any $A, B \in 2^X$;
(d) $\Phi(\Phi(A)) = \Phi(A)$ for every $A \in 2^X$.

Theorem 5.6. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space such that*

$$\mathcal{P} = 2^X \setminus \{X\}.$$

Then, the operator

$$\overset{\hookrightarrow}{A} = \{x \in X | \{x\} \hookrightarrow A\}$$

on a primal-proximity space $(X, \hookrightarrow, \mathcal{P})$ is a Kuratowski closure operator.

Proof. (a) By (vi) of Theorem 4.6, $\overset{\hookrightarrow}{\emptyset} = \emptyset$.

(b) If $x \in A$, then $\{x\} \hookrightarrow A$. Hence, $x \in \overset{\hookrightarrow}{A}$. This shows that $A \subseteq \overset{\hookrightarrow}{A}$.

(c) By (iii) of Theorem 4.6, $\overset{\hookrightarrow}{(A \cup B)} = \overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{B}$.

(d) By (iv) of Theorem 4.6, we have always $\overset{\hookrightarrow}{\overset{\hookrightarrow}{A}} \subseteq \overset{\hookrightarrow}{A}$. Now, let $x \notin \overset{\hookrightarrow}{\overset{\hookrightarrow}{A}}$. Then, $\{x\} \not\hookrightarrow \overset{\hookrightarrow}{A}$. By (iv) of Corollary 3.4, we have

$$\left(\{x\} \cap \overset{\hookrightarrow}{A}\right)^c \notin \mathcal{P}.$$

Since $\mathcal{P} = 2^X \setminus \{X\}$, we get

$$\left(\{x\} \cap \overset{\hookrightarrow}{A}\right)^c = X,$$

which means that $\{x\} \cap \overset{\hookrightarrow}{A} = \emptyset$. Thus, we have $x \notin \overset{\hookrightarrow}{A}$. Hence,

$$\overset{\hookrightarrow}{A} \subseteq \overset{\hookrightarrow}{\overset{\hookrightarrow}{A}} \text{ and } \overset{\hookrightarrow}{\overset{\hookrightarrow}{A}} = \overset{\hookrightarrow}{A},$$

which completes the proof and this topology is denoted by $\overset{\hookrightarrow}{\tau}$. □

THEOREM 5.7. *Let* $(X, \hookrightarrow, \mathcal{P})$ *be a primal-proximity space. Then, the operator* $\mathrm{cl}^* : 2^X \to 2^X$ *defined by*

$$\mathrm{cl}^*(A) = A \cup \overset{\hookrightarrow}{A}$$

satisfies the Kuratowski closure axioms and induces a topology on X *called* τ^**, which is given by*

$$\tau^* = \{A \subseteq X \mid \mathrm{cl}^*(A^c) = A^c\}.$$

Proof. (a) By (vi) of Theorem 4.6, we have $\mathrm{cl}^*(\emptyset) = \emptyset \cup \overset{\hookrightarrow}{\emptyset} = \emptyset$.

(b) Let $A \subseteq X$. Since $\mathrm{cl}^*(A) = A \cup \overset{\hookrightarrow}{A}$, we have $A \subseteq \mathrm{cl}^*(A)$.

(c) Let $A, B \subseteq X$. By (iii) of Theorem 4.6, we have

$$\begin{aligned}
\mathrm{cl}^*(A \cup B) &= (A \cup B) \cup (\overset{\hookrightarrow}{A \cup B}) \\
&= (A \cup B) \cup \left(\overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{B}\right) \\
&= \left(A \cup \overset{\hookrightarrow}{A}\right) \cup \left(B \cup \overset{\hookrightarrow}{B}\right) \\
&= \mathrm{cl}^*(A) \cup \mathrm{cl}^*(B).
\end{aligned}$$

(d) Let $A \subseteq X$. By (iv) of Theorem 4.6, we have

$$\begin{aligned}
\mathrm{cl}^*(\mathrm{cl}^*(A)) &= \mathrm{cl}^*(A) \cup \overset{\hookrightarrow}{\mathrm{cl}^*(A)} \\
&= \left(A \cup \overset{\hookrightarrow}{A}\right) \cup \overset{\hookrightarrow}{\left(A \cup \overset{\hookrightarrow}{A}\right)} \\
&= \left(A \cup \overset{\hookrightarrow}{A}\right) \cup \left(\overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{\overset{\hookrightarrow}{A}}\right) \\
&= \left(A \cup \overset{\hookrightarrow}{A}\right) \cup \overset{\hookrightarrow}{A} \\
&= A \cup \overset{\hookrightarrow}{A} \\
&= \mathrm{cl}^*(A).
\end{aligned}$$

□

Theorem 5.8. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space. Then, the following properties hold:*

(i) *$B \not\hookrightarrow A$ if and only if $B \not\hookrightarrow \mathrm{cl}^*(A)$,*

(ii)

$$\mathrm{cl}^*\left(\overset{\hookrightarrow}{A}\right) = \overset{\hookrightarrow}{A},$$

(iii)

$$\mathrm{cl}^*\left(\overset{\hookrightarrow}{A}\right) = \overset{\hookrightarrow}{\mathrm{cl}^*(A)}.$$

Proof. (i) Let $B \not\hookrightarrow A$. Then, by Theorem 4.4, we have $B \not\hookrightarrow \overset{\hookrightarrow}{A}$. Hence, by (b) of Definition 3.1, $B \not\hookrightarrow (A \cup \overset{\hookrightarrow}{A}) = \mathrm{cl}^*(A)$ if and only if $B \not\hookrightarrow A$ and $B \not\hookrightarrow \overset{\hookrightarrow}{A}$.

(ii) Let $A \subseteq X$. By (iv) of Theorem 4.6, we have

$$\mathrm{cl}^*\left(\overset{\hookrightarrow}{A}\right) = \overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{\overset{\hookrightarrow}{A}} = \overset{\hookrightarrow}{A}.$$

(iii) Let $A \subseteq X$. By (iii) of Theorem 4.6, we have

$$\mathrm{cl}^*\left(\overset{\hookrightarrow}{A}\right) = \overset{\hookrightarrow}{A} \cup \overset{\hookrightarrow}{\overset{\hookrightarrow}{A}} = \overset{\hookrightarrow}{\left(A \cup \overset{\hookrightarrow}{A}\right)} = \overset{\hookrightarrow}{\mathrm{cl}^*(A)}.$$

□

Theorem 5.9. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A, B, H \subseteq X$ such that $A \subseteq B$. If $A \hookrightarrow B$ and $\{b\} \hookrightarrow H$ for all $b \in B$, then $A \hookrightarrow H$.*

Proof. Suppose $A \not\hookrightarrow H$. Then, there exist $C, D \subseteq X$ such that $A \not\hookrightarrow C^c$, $D^c \not\hookrightarrow B$, and $(C \cap D)^c \notin \mathcal{P}$. This result, combined with $A \hookrightarrow B$ and (b) of Definition 3.1, implies that $B \nsubseteq C^c$, that is $B \cap C \neq \emptyset$. It follows that there is a point $x \in X$ such that $\{x\} \hookrightarrow H$ and $x \in C$. Then, there are two cases either $x \in D$ or $x \notin D$.

Case 1. Let $x \in D$. Hence $X \setminus \{x\} \notin \mathcal{P}$, by (c) of Definition 3.1, implies $\{x\} \not\hookrightarrow H$ for any subset H of X, which is a contradiction.

Case 2. Let $x \in D^c$. Then, $\{x\} \not\hookrightarrow B$. This result, combined with (c) and (d) of Definition 3.1, implies $\{x\} \not\hookrightarrow H$ which is a contradiction. Hence, $A \hookrightarrow H$. □

Example 5.10. Let $(X, \tau, \mathcal{P})$ be a primal topological space and '$\hookrightarrow$' be a binary relation on 2^X defined as $A \hookrightarrow B$ if and only if $(A \cap \mathrm{cl}(B))^c \in \mathcal{P}$. Then, '$\hookrightarrow$' is not a primal-proximity relation on 2^X but satisfies (b), (c), (d) and (e) of Definition 3.1. Hence, in this case, $\tau \subseteq \tau^*$.

Proof. We want to show that $\mathrm{cl}^*(A) \subseteq \mathrm{cl}(A)$ for all $A \subseteq X$. Let $x \in \mathrm{cl}^*(A) = A \cup \overset{\hookrightarrow}{A}$. Then, $x \in A$ or $x \in \overset{\hookrightarrow}{A}$. If $x \in A$, then $x \in \mathrm{cl}(A)$.

Now, if $x \in \overset{\hookrightarrow}{A}$, then $\{x\} \hookrightarrow A$. Hence, $(\{x\} \cap \mathrm{cl}(A))^c \in \mathcal{P}$ and so

$$(\{x\} \cap \mathrm{cl}(A))^c \neq X.$$

Thus, $\{x\} \cap \mathrm{cl}(A) \neq \emptyset$ which means $x \in \mathrm{cl}(A)$. Therefore, $\tau \subseteq \tau^*$. □

EXAMPLE 5.11. Let $(X, \tau, \mathcal{P})$ be a primal topological space and '$\hookrightarrow$' be a binary relation on 2^X defined as $A \hookrightarrow B$ if and only if $(A \cap \mathrm{cl}^\diamond(B))^c \in \mathcal{P}$. Then '$\hookrightarrow$' is not a primal-proximity relation on 2^X but satisfies (b), (c), (d) and (e) of Definition 3.1. Hence, in this case, $\tau^\diamond \subseteq \tau^*$.

Proof. We want to show that $\mathrm{cl}^*(A) \subseteq \mathrm{cl}^\diamond(A)$ for all $A \subseteq X$.

Let $x \in \mathrm{cl}^*(A) = A \cup \overset{\hookrightarrow}{A}$. Then, $x \in A$ or $x \in \overset{\hookrightarrow}{A}$. If $x \in A$, then

$$x \in A \subseteq A \cup A^\diamond = \mathrm{cl}^\diamond(A).$$

Now, if $x \in \overset{\hookrightarrow}{A}$, then $\{x\} \hookrightarrow A$. Hence, $(\{x\} \cap \mathrm{cl}^\diamond(A))^c \in \mathcal{P}$ and so,

$$(\{x\} \cap \mathrm{cl}^\diamond(A))^c \neq X.$$

Thus, $\{x\} \cap \mathrm{cl}^\diamond(A) \neq \emptyset$ which means $x \in \mathrm{cl}^\diamond(A)$. Therefore, $\tau^\diamond \subseteq \tau^*$. □

DEFINITION 5.12. Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, X is said to be a *primal-regular space* if for all $x \in X$ and $\tau^\diamond$-closed set F such that $(\{x\} \cap F)^c \notin \mathcal{P}$, there exist two open sets H, G such that $x \in H$ and $F \subseteq G$ and $(H \cap G)^c \notin \mathcal{P}$.

THEOREM 5.13. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Let X be a primal-regular space and '$\hookrightarrow$' be a binary relation on 2^X as defined in Example* 5.11*, then $\tau^\diamond = \tau^*$.*

Proof. In order to prove the theorem, it suffices to show $\mathrm{cl}^\diamond(A) = \mathrm{cl}^*(A)$ for all subsets A of X.

Let $x \in \mathrm{cl}^*(A)$. Then, $x \in A$ or $x \in \overset{\hookrightarrow}{A}$. If $x \in A$, then $x \in \mathrm{cl}^\diamond(A)$.

Now, if $x \in \overset{\hookrightarrow}{A}$, then $\{x\} \hookrightarrow A$. Hence, $(\{x\} \cap \mathrm{cl}^\diamond(A))^c \in \mathcal{P}$ which means $\{x\} \cap \mathrm{cl}^\diamond(A) \neq \emptyset$. Consequently, we have $x \in \mathrm{cl}^\diamond(A)$. Thus, $\mathrm{cl}^*(A) \subseteq \mathrm{cl}^\diamond(A)$.

Now, let $x \notin \mathrm{cl}^*(A)$. Then, $x \notin A$ and $x \notin \overset{\hookrightarrow}{A}$. It follows that $\{x\} \not\hookrightarrow A$ and hence by Example 5.11, it implies that $(\{x\} \cap \mathrm{cl}^\diamond(A))^c \notin \mathcal{P}$.

Since X is primal-regular space and $\tau^c \subseteq \tau^{\diamond c}$, there exist two open sets H and G such that $x \in H$ and $A \subseteq \mathrm{cl}^\diamond(A) \subseteq G$ and $(H \cap G)^c \notin \mathcal{P}$. Hence, $(H \cap A)^c \notin \mathcal{P}$ and since $x \in H \in \tau$ and $(H \cap A)^c \notin \mathcal{P}$, then $x \notin A^\diamond$. So, $x \notin \mathrm{cl}^\diamond(A)$. It follows that $\mathrm{cl}^\diamond(A) \subseteq \mathrm{cl}^*(A)$. Hence, $\mathrm{cl}^\diamond(A) = \mathrm{cl}^*(A)$. □

EXAMPLE 5.14. Let $(X, \tau, \mathcal{P})$ be a primal topological space and '$\hookrightarrow$' be a binary relation on 2^X defined as $A \hookrightarrow B$ if and only if

$$(\mathrm{cl}^\diamond(A) \cap \mathrm{cl}^\diamond(B))^c \in \mathcal{P}.$$

Then, '$\hookrightarrow$' is not a primal-proximity relation on 2^X, but satisfies (a)-(d) of Definition 3.1.

DEFINITION 5.15. Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, X is said to be a *primal-normal space* if for two $\tau^\diamond$-closed sets F_1, F_2 such that $(F_1 \cap F_2)^c \notin \mathcal{P}$, there exist two open sets H and G such that

$$F_1 \subseteq H, \qquad F_2 \subseteq G \quad \text{and} \quad (H \cap G)^c \notin \mathcal{P}.$$

THEOREM 5.16. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. If X is a primal-normal space and a binary relation defined as in Example* 5.14 *and (X, τ) is T_1-space, then $\tau^\diamond = \tau^*$.*

Proof. In order to prove the theorem, it suffices to show that $\mathrm{cl}^\diamond(A) = \mathrm{cl}^*(A)$ for all subsets A of X.

Let $x \in \mathrm{cl}^*(A)$. Then, $x \in A$ or $x \in \overset{\hookrightarrow}{A}$. If $x \in A$, then $x \in \mathrm{cl}^\diamond(A)$.

Now, if $x \in \overset{\hookrightarrow}{A}$, then $\{x\} \hookrightarrow A$. Hence,

$$(\mathrm{cl}^\diamond(\{x\}) \cap \mathrm{cl}^\diamond(A))^c \in \mathcal{P}.$$

Since (X, τ) is T_1-space and $\tau^c \subseteq \tau^{\diamond c}$, then $[\{x\} \cap \mathrm{cl}^\diamond(A)]^c \in \mathcal{P}$ and so, $\{x\} \cap \mathrm{cl}^\diamond(A) \neq \emptyset$. Consequently, we have $x \in \mathrm{cl}^\diamond(A)$. Hence, $\mathrm{cl}^*(A) \subseteq \mathrm{cl}^\diamond(A)$.

Now, let $x \notin \mathrm{cl}^*(A)$. Then, $x \notin A$ and $x \notin \overset{\hookrightarrow}{A}$. It follows that $\{x\} \not\hookrightarrow A$ and hence by Example 5.14, it implies that

$$(\mathrm{cl}^\diamond(\{x\}) \cap \mathrm{cl}^\diamond(A))^c \notin \mathcal{P}.$$

Since (X, τ) is primal-normal space, T_1-space and $\tau^c \subseteq \tau^{\diamond c}$, there exist two open sets H and G such that $\{x\} \subseteq H$, $A \subseteq \mathrm{cl}^\diamond(A) \subseteq G$ and $(H \cap G)^c \notin \mathcal{P}$.

Hence, $(H \cap A)^c \notin \mathcal{P}$. Since $x \in H \in \tau$ and $(H \cap A)^c \notin \mathcal{P}$, thus $x \notin A^\diamond$. So, $x \notin \mathrm{cl}^\diamond(A)$. It follows that $\mathrm{cl}^\diamond(A) \subseteq \mathrm{cl}^*(A)$ and hence, $\mathrm{cl}^\diamond(A) = \mathrm{cl}^*(A)$. □

THEOREM 5.17. *Let $(X, \hookrightarrow, \mathcal{P})$ be a primal-proximity space and $A \subseteq X$. Then, $A \in \overset{\hookrightarrow}{\tau}$ if and only if $\{x\} \not\hookrightarrow A^c$ for every $x \in A$.*

Proof. Let $A \in \overset{\hookrightarrow}{\tau}$ and $x \in A$. Then, A^c is proximity-closed in $\overset{\hookrightarrow}{\tau}$ and $x \notin A^c$. Hence, we get $\{x\} \not\hookrightarrow A^c$.

Conversely, if for every $x \in A$, we have $\{x\} \not\hookrightarrow A^c$. Then, $\{x\} \hookrightarrow A^c$ implies that $x \notin A$. This means that $\{x\} \hookrightarrow A^c$ implies $x \in A^c$. Hence, A^c is proximity-closed in $\overset{\hookrightarrow}{\tau}$. Thus, $A \in \overset{\hookrightarrow}{\tau}$. □

THEOREM 5.18. *Let* $(X, \hookrightarrow, \mathcal{P})$ *be a primal-proximity space and* $A, B \subseteq X$ *such that* $A \not\hookrightarrow B$. *Then, the following conditions hold:*

(i) $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B) \subseteq A^c$ *where* $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$ *means the closure of* B *with respect to* $\overset{\hookrightarrow}{\tau}$.

(ii) *if* $\mathcal{P} = 2^X \setminus \{X\}$, *then* $B \subseteq \mathrm{int}_{\overset{\hookrightarrow}{\tau}}(A^c)$ *where* $\mathrm{int}_{\overset{\hookrightarrow}{\tau}}(A^c)$ *means the interior of* A^c *with respect to* $\overset{\hookrightarrow}{\tau}$.

Proof. (i) Since the closure is taken with respect to $\overset{\hookrightarrow}{\tau}$ and $A \not\hookrightarrow B$, we have $\overset{\hookrightarrow}{B} = \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B) \subseteq A^c$,

(ii) If $x \in B$, then $\{x\} \hookrightarrow B$. This implies that $\{x\} \not\hookrightarrow A$. Because if $\{x\} \hookrightarrow A$, then by Lemma 5.2, we get $A \hookrightarrow B$. Hence, $x \notin \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$ which means $x \in (\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A))^c = \mathrm{int}_{\overset{\hookrightarrow}{\tau}}(A^c)$. Hence, we have $B \subseteq \mathrm{int}_{\overset{\hookrightarrow}{\tau}}(A^c)$. □

THEOREM 5.19. *Let* $(X, \hookrightarrow, \mathcal{P})$ *be a primal-proximity space and* $A, B \subseteq X$. *Then,* $A \hookrightarrow B$ *if and only if* $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A) \hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$, *where* $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$ *means the closure of* A *with respect to* $\overset{\hookrightarrow}{\tau}$.

Proof. If $A \hookrightarrow B$, then by Lemma 4.2, $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A) \hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$ since $A \subseteq \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$ and $B \subseteq \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$. If $A \not\hookrightarrow B$, then there exists a subset E of X such that $A \not\hookrightarrow E$ and $E^c \not\hookrightarrow B$ and $(E \cap E^c)^c \notin \mathcal{P}$. Hence, $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B) \subseteq E$ by (i) of Theorem 5.18. This implies that $A \not\hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$. Because if $A \hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$ then by Lemma 4.2, we have $A \hookrightarrow E$ since $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B) \subseteq E$. Now if $A \not\hookrightarrow B$, then $A \not\hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$. Also, $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B) \not\hookrightarrow A$ by a similar proof. Again it follows that $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B) \not\hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A)$. Hence, $A \hookrightarrow B$ if and only if $\mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(A) \hookrightarrow \mathrm{cl}_{\overset{\hookrightarrow}{\tau}}(B)$. □

6. CONCLUSION

In this paper, we introduced a new type of proximity space called primal-proximity space. Later, we defined point-primal proximity operator and investigated some of its fundamental properties. We also proved that this operator is a Kuratowski closure operator under special condition. Moreover, one more operator via point-primal proximity operator was defined. Furthermore, we gave not only some relationships but also several examples.

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