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NOVEL CRITERIA FOR STABILITY AND CONVERGENCE OF SOLUTIONS FOR NON-AUTONOMOUS NONLINEAR SYSTEMS

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Abstract. In this paper, we give a new integral inequality which is used to study the asymptotic behavior of solutions for a class of nonlinear dynamic systems with small perturbation using a numerical approach. We provide some new results on the stability of perturbed systems where necessary and sufficient condition is derived. We show that the perturbed nonlinear system can be globally uniformly practically asymptotically stable provided that the bound of perturbation is small enough. A numerical example is presented to illustrate the validity of the main result.
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1. INTRODUCTION

In studying the effect of perturbations of various types on the solutions of a nonlinear differential equation, one must assume some stability property for the unperturbed system. A useful kind of stability is one for which the effect of perturbations can be studied. In fact, some types of stabilities, such as the Lyapunov stability for instance, are defined in terms of the behavior of solutions under perturbations (see [7]-[16]). The usual technical in the literature for the stability analysis of perturbed nonlinear systems is based on the stability of the associated nominal systems [11–13]. Furthermore the practical stability, in the sense introduced in [2, 3], is very important and very useful for analyzing the stability or for designing practical controllers of dynamical systems. The practical stability only needs to stabilize a system into a region of phase space, namely the system may oscillate close to the state, in which the performance is still acceptable (see [5,6,18]). The uncertainties were represented by an additive term on the right-hand side of the state equation and the origin is not supposed to be an equilibrium point of the system. One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. Here we define a new method for stability in terms of the behavior of solutions using the transition matrix

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of the nominal system. Being formulated in terms of integral inequalities of Gronwall type, it is a type of stability which is easy to verify in practice, and it extends the class of systems for which the effect of perturbations can be measured [1, 4, 14, 15, 17, 19, 20]. We will provide some sufficient conditions for exponential stability of perturbed systems under the assumption that the unperturbed linear system is exponentially stable. Our approach uses a new integral inequality depending on a small parameter which allows us to conclude the global practical uniform asymptotic stability of the system in presence of perturbations. The effectiveness of the proposed method is shown throughout some numerical examples in the plane.

2. PRELIMINARY

It is well known that some differential equations can be solved explicitly. Unfortunately, there is no general recipe for solving a given differential equation. Moreover, finding explicit solutions is in general impossible unless the equation is of a particular form. Given two solutions to a dynamical system with initial conditions that are close at the same value of time, these solutions will remain close over the entire time interval and not just at the initial time. The qualitative behavior of the solutions of perturbed nonlinear systems of differential equations is often studied by considering some integral inequalities. Consider the time-varying differential equation described by the following:

$$\dot{x} = F(t, x),$$

where $F : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function and globally Lipschitz uniformly on t with respect to x such that $F(t, 0) = 0,\ \forall t \geq 0$. The associated perturbed systems is given by:

$$\dot{y} = F(t, y) + h(t, y, \varepsilon),$$

where $t \in \mathbb{R}^+$, $\varepsilon > 0$, $h : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}_+^* \longrightarrow \mathbb{R}^n$ is a continuous function which represents the perturbation term satisfying the following condition: $\|h(t, y, \varepsilon)\| \leq w(t, \varepsilon),\ \ \forall y \in \mathbb{R}^n,\ \forall t \geq 0$, where $w(.,.)$ is a nonnegative continuous function. Let consider the following generalized Lipschitz condition on $F(.,.)$: there exists a continuous function $v : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}^+$, such that,

$$\|F(t, y) - F(t, x)\| \leq v(t, \varepsilon)\|y - x\|,\ \ \forall y \in \mathbb{R}^n,\ \ \forall x \in \mathbb{R}^n,\ \ \forall t \geq 0.$$

One can find some estimations on the solutions of the perturbed equation with respect to the solutions of the original unperturbed system. Let $x(t_0) = x_0$ and $y(t_0) = y_0 = x_0(\varepsilon)$. The solution of the perturbed equation is given by:

$$y(t) = x_0(\varepsilon) + \int_0^t \big(F(\tau, y(\tau)) + h(\tau, y(\tau), \varepsilon)\big)\mathrm{d}\tau.$$

Thus, $\forall t_0 \geq 0,\ \forall x_0, y_0$, one has

$$\|y(t) - x(t)\| \leq \|x_0(\varepsilon) - x_0\| + \int_0^t \|F(\tau, y(\tau)) + h(\tau, y(\tau), \varepsilon) - F(\tau, x(\tau))\|\mathrm{d}\tau.$$

It follows that,

$$\|y(t)-x(t)\| \leq \|x_0(\varepsilon)-x_0\| + \int_0^t \Big(v(\tau,\varepsilon)\|y(\tau)-x(\tau)\| + w(\tau,\varepsilon)\Big)\mathrm{d}\tau.$$

The last inequality shows that one can examine, using a certain integral inequality, the difference between the respective solutions of the unperturbed system and the perturbed one in order to investigate the asymptotic behavior of solutions. The following generalized integral inequality of Gronwall type permits to solve this problem for different classes of perturbed systems.

2.1. Integral inequalities. First, we give a new integral inequality which will be useful for our study. This result extends the Gronwall-Bellman inequality given in [15].

Lemma 2.1. *Let u, v and w nonnegative continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+^*$ satisfying the inequality*

$$u(t,\varepsilon) \leq c(\varepsilon) + \int_a^t (u(\tau,\varepsilon)v(\tau,\varepsilon) + w(\tau,\varepsilon))\mathrm{d}\tau, \tag{1}$$

where a, ε and $c(\varepsilon)$ are nonnegative constants. Then,

$$u(t,\varepsilon) \leq c(\varepsilon)\mathrm{e}^{\int_a^t v(\tau,\varepsilon)\mathrm{d}\tau} + r\mathrm{e}^{\int_a^t (v(\tau,\varepsilon)+\frac{w(\tau,\varepsilon)}{r})\mathrm{d}\tau} \quad \forall t \geq a, \forall r > 0. \tag{2}$$

Proof. From (1) and the inequality $x < \mathrm{e}^x$, we have for all $r > 0$ and $t \geq$ a

$$0 \leq u(t,\varepsilon) < c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + \int_a^t u(\tau,\varepsilon)v(\tau,\varepsilon)\mathrm{d}\tau, \tag{3}$$

which implies

$$\frac{u(t,\varepsilon)}{c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + \int_a^t u(\tau,\varepsilon)v(\tau,\varepsilon)\mathrm{d}\tau} \leq 1.$$

Multiplying the last inequality by $v \geq 0$, we obtain

$$\frac{u(t,\varepsilon)v(t,\varepsilon) + w(t,\varepsilon)\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}}{c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + \int_a^t u(\tau,\varepsilon)v(\tau,\varepsilon)\mathrm{d}\tau} \leq v(t,\varepsilon) + \frac{w(t,\varepsilon)\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}}{c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}}. \tag{4}$$

Now, define for any $\varepsilon > 0$ and $t \geq a$

$$\begin{aligned} f_\varepsilon(t) = &\int_a^t v(\tau,\varepsilon)\mathrm{d}\tau + \log\Big(c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}\Big) \\ &- \log\left(c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + \int_a^t u(\tau,\varepsilon)v(\tau,\varepsilon)\mathrm{d}\tau\right). \end{aligned}$$

The function f_ε is defined, continuous and differentiable on $[a,+\infty)$, and

$$f'_\varepsilon(t) = v(t,\varepsilon) + \frac{w(t,\varepsilon)\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}}{c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}} - \frac{w(t,\varepsilon)\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + u(t,\varepsilon)v(t,\varepsilon)}{c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + \int_a^t u(\tau,\varepsilon)v(\tau,\varepsilon)\mathrm{d}\tau}$$

for all $t \geq a$. It follows by (4) that f_ε is a non increasing function, and hence

$$f_\varepsilon(t) \geq f_\varepsilon(a) = 0, \quad \forall t \geq a.$$

Then,

$$\log\left(c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + \int_a^t u(\tau,\varepsilon)v(\tau,\varepsilon)\mathrm{d}\tau\right)$$
$$\leq \int_a^t v(\tau,\varepsilon)\mathrm{d}\tau + \log\left(c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}\right) \quad \forall t \geq a,$$

hence,

$$c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau} + \int_a^t u(\tau,\varepsilon)v(\tau,\varepsilon)\mathrm{d}\tau \leq \left(c(\varepsilon) + r\mathrm{e}^{\int_a^t \frac{w(\tau,\varepsilon)}{r}\mathrm{d}\tau}\right)\mathrm{e}^{\int_a^t v(\tau,\varepsilon)\mathrm{d}\tau}.$$

Finally, we have by using the inequality (3)

$$u(t,\varepsilon) \leq c(\varepsilon)\mathrm{e}^{\int_a^t v(\tau,\varepsilon)\mathrm{d}\tau} + r\mathrm{e}^{\int_a^t (v(\tau,\varepsilon)+\frac{w(\tau,\varepsilon)}{r})\mathrm{d}\tau}.$$

□

3. STABILITY ANALYSIS

3.1. DEFINITIONS AND NOTATIONS

We consider the following nonlinear system described by

$$\dot{x} = f(t,x,\varepsilon), \quad x(t_0,\varepsilon) = x_{0,\varepsilon} \tag{5}$$

where $t \in \mathbb{R}_+$ is the time, $x \in \mathbb{R}^n$is the state, ε is a real parameter "small enough" and $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+^* \longrightarrow \mathbb{R}^n$ is continuous in (t,x,ε), locally Lipschitz in (x,ε) and uniformly in t.

Definition 3.1. (i) The equilibrium point $x^* = 0$ is said uniformly exponentially stable if there exists $c(\varepsilon) > 0$, $\lambda_1(\varepsilon) > 0$ and $\lambda_2(\varepsilon) > 0$ such that $\forall\, t_0 \geqslant 0$, $\forall\, \|x_{0,\varepsilon}\| \leqslant c(\varepsilon)$,

$$\|x(t,\varepsilon)\| \leqslant \lambda_1(\varepsilon)\mathrm{e}^{-\lambda_2(\varepsilon)(t-t_0)}\|x_{0,\varepsilon}\|.$$

(ii) The equilibrium point $x^* = 0$ is said globally uniformly exponentially stable if there exists $\lambda_1(\varepsilon) > 0$ and $\lambda_2(\varepsilon) > 0$ such that $\forall\, t_0 \geqslant 0$, $\forall\, x_{0,\varepsilon} \in \mathbb{R}^n$

$$\|x(t,\varepsilon)\| \leqslant \lambda_1(\varepsilon)\mathrm{e}^{-\lambda_2(\varepsilon)(t-t_0)}\|x_{0,\varepsilon}\|, \quad \forall t \geqslant t_0 \geqslant 0.$$

Definition 3.2. A solution of (5) is said to be globally uniformly bounded if every $\eta = \eta(\varepsilon) > 0$ there exists $c = c(\eta)$, independent of t_0, such that for all $t_0 \geqslant 0$,

$$\|x_{0,\varepsilon}\| < \eta \Longrightarrow \|x(t,\varepsilon)\| < c, \quad \forall t \geqslant t_0.$$

Definition 3.3. Let $r = r(\varepsilon) \geqslant 0$ and $B_r = \{x \in \mathbb{R}^n \ / \ \|x\| \leqslant r\}$.
(i) B_r is uniformly stable if for all $A = A(\varepsilon) > r$, there exist $\delta = \delta(A) > 0$ such that for all $t_0 \geqslant 0$,

$$\|x_{0,\varepsilon}\| < \delta \Longrightarrow \|x(t,\varepsilon)\| < A, \quad \forall\, t \geqslant t_0.$$

(ii) B_r is globally uniformly stable if it is uniformly stable and the solutions of system (5) are globally uniformly bounded.

Definition 3.4. B_r is globally uniformly exponentially stable if there exists $\gamma = \gamma(\varepsilon) > 0$ and $k = k(\varepsilon) \geqslant 0$ such that $\forall\, t_0 \in \mathbb{R}_+$ and $\forall\, x_{0,\varepsilon} =\in \mathbb{R}^n$,

$$\|x(t,\varepsilon)\| \leqslant k\|x_{0,\varepsilon}\| \, \exp(-\gamma(t-t_0)) + r, \quad \forall t \geqslant t_0.$$

System (5) is globally practically uniformly exponentially stable if there exist $r = r(\varepsilon) > 0$ such that B_r is globally uniformly exponentially stable.

3.2. MAIN RESULTS

We consider the following system :

$$\begin{cases} \dot{x} = A_\varepsilon(t)x + h(t,x,\varepsilon) \\ x(t_0,\varepsilon) = x_{0,\varepsilon}, \end{cases} \tag{6}$$

where $A_\varepsilon(.)$ is an $n \times n$ continuous matrix on $\mathbb{R}_+$. We will first study the linear problem where $h(t,x,\varepsilon) = 0$

$$\begin{cases} \dot{x} = A_\varepsilon(t)x \\ x(t_0,\varepsilon) = x_{0,\varepsilon}. \end{cases} \tag{7}$$

We may write $A_\varepsilon(t) = A_0(t) + \varepsilon F(t)$ where $A_0(\cdot)$ and $F(.)$ are an $n \times n$ continuous matrix on $\mathbb{R}_+$ and ε being a small real parameter. In order to study the global exponential stability of the system (6) we shall assume throughout all the paper that the nominal unperturbed system

$$\begin{cases} \dot{x} = A_0(t)x \\ x(t_0) = x_0 \end{cases} \tag{8}$$

is globally uniformly exponentially stable, that is there exists constants $c > 0$ and $\gamma > 0$ independent of t_0 such that $\forall\, t_0 \geq 0$

$$\|R_{A_0}(t,t_0)\| \leq c\,\mathrm{e}^{-\gamma(t-t_0)} \quad \forall\, t \geq t_0 \tag{9}$$

where $R_{A_0}(t,t_0)$ denotes the state transition matrix of the system (8) [16]. We have the following result.

Theorem 3.5. *Assume the unperturbed system* (8) *is globally uniformly exponentially stable and suppose* $F(\cdot)$ *is a bounded function, then for any* $\varepsilon \in [0, \frac{\gamma}{c})$ *the system* (7) *is globally uniformly exponentially stable.*

Proof. Let $R_{A_\varepsilon}(t,t_0)$ denotes the state transition matrix for the system (7). Since the mapping $\varepsilon \mapsto R_{A_\varepsilon}(t,t_0)$ is C^∞, then one may write an asymptotic expansion of R_{A_ε} as

$$R_{A_\varepsilon}(t,t_0) = R_{A_0}(t,t_0) + \varepsilon Y_1(t) + \cdots + \varepsilon^i Y_i(t) + \dots \tag{10}$$

where the $Y_i(\cdot)$, $i \geq 1$, are matrices that can be found as follows. First, we plug equation (10) into the system

$$\begin{cases} \dfrac{\partial}{\partial t} R_{A_\varepsilon}(t,t_0) = (A_0(t) + \varepsilon F(t)) R_{A_\varepsilon}(t,t_0) \\ R_{A_\varepsilon}(t_0,t_0) = I. \end{cases}$$

It follows

$$\begin{aligned} &\dot{R}_{A_0}(t,t_0) + \varepsilon \dot{Y}_1(t) + \cdots + \varepsilon^i \dot{Y}_i(t) + \dots \\ &= (A_0(t) + \varepsilon F(t)) \left(R_{A_0}(t,t_0) + \varepsilon Y_1(t) + \cdots + \varepsilon^i Y_i(t) + \dots \right), \end{aligned}$$

or similarly

$$\begin{aligned} &\varepsilon \left(\dot{Y}_1(t) - A_0 Y_1(t) - F(t) R_{A_0}(t,t_0) \right) + \varepsilon^2 \left(\dot{Y}_2(t) - A_0 Y_2(t) - F(t) Y_1(t) \right) + \\ &\cdots + \varepsilon^i \left(\dot{Y}_i(t) - A_0 Y_i(t) - F(t) Y_{i-1}(t) \right) + \cdots = 0. \end{aligned}$$

The previous identity is verified for all $\varepsilon > 0$ if and only if

$$\dot{Y}_1(t) - A_0 Y_1(t) - F(t) R_{A_0}(t,t_0) = 0$$

and

$$\dot{Y}_i(t) - A_0 Y_i(t) - F(t) Y_{i-1}(t) = 0 \quad \forall\, i \geq 2. \tag{11}$$

On another hand, by using the identity

$$I = R_{A_\varepsilon}(t_0,t_0) = I + \varepsilon Y_1(t_0) + \cdots + \varepsilon^i Y_i(t_0) + \dots, \quad \forall\, \varepsilon > 0$$

we obtain

$$Y_1(t_0) = \cdots = Y_i(t_0) = 0.$$

To find Y_1 we solve the following system:

$$\begin{cases} \dot{Y}_1(t) = A_0 Y_1(t) + F(t) R_{A_0}(t,t_0) \\ Y_1(t_0) = 0. \end{cases}$$

It follows by the Duhamel's formula

$$Y_1(t) = \int_{t_0}^{t} R_{A_0}(t,s_1) F(s_1) R_{A_0}(s_1,t_0)\, ds_1.$$

Now we solve

$$\begin{cases} \dot{Y}_2(t) = A_0 Y_2(t) + F(t) Y_1(t) \\ Y_2(t_0) = 0, \end{cases}$$

we obtain,

$$
\begin{aligned}
Y_2(t) &= \int_{t_0}^{t} R_{A_0}(t,s_1)F(s_1)Y_1(s_1)\mathrm{d}s_1 \\
&= \int_{t_0}^{t} R_{A_0}(t,s_1)F(s_1)\int_{t_0}^{s_1} R_{A_0}(s_1,s_2)F(s_2)R_{A_0}(s_2,t_0)\mathrm{d}s_2\,\mathrm{d}s_1 \\
&= \int_{t_0}^{t}\int_{t_0}^{s_1} R_{A_0}(t,s_1)F(s_1)R_{A_0}(s_1,s_2)F(s_2)R_{A_0}(s_2,t_0)\mathrm{d}s_2\,\mathrm{d}s_1 \;.
\end{aligned}
$$

Using (11), we obtain by induction

$$
\begin{aligned}
Y_i(t) = \int_{t_0}^{t}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{i-1}} & R_{A_0}(t,s_1)F(s_1)R_{A_0}(s_1,s_2)F(s_2) \\
& \ldots R_{A_0}(s_{i-1},s_i)F(s_i)R_{A_0}(s_i,t_0)\mathrm{d}s_i\,\mathrm{d}s_{i-1}\ldots\mathrm{d}s_1 \,.
\end{aligned}
$$

It follows by (9)

$$
\begin{aligned}
\|Y_i(t)\| \le \int_{t_0}^{t}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{i-1}} & c\,\mathrm{e}^{-\gamma(t-s_1)}\|F(s_1)\|\;c\,\mathrm{e}^{-\gamma(s_1-s_2)}\;\|F(s_2)\|\ldots \\
& c\,\mathrm{e}^{-\gamma(s_{i-1}-s_i)}\;\|F(s_i)\|\;c\,\mathrm{e}^{-\gamma(s_i-t_0)}\;\mathrm{d}s_i\,\mathrm{d}s_{i-1}\ldots\mathrm{d}s_1 \\
= c^i\,\mathrm{e}^{-\gamma(t-t_0)}\int_{t_0}^{t}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{i-1}} & \|F(s_1)\|\,\|F(s_2)\|\ldots\;\|F(s_i)\|\,\mathrm{d}s_i\,\mathrm{d}s_{i-1}\ldots\mathrm{d}s_1.
\end{aligned}
$$

Since $F(.)$ is bounded, then $\exists\; k>0$ s.t. $\|F(t)\|\le k\quad\forall\; t\ge t_0$. It follows by the Cauchy formula for repeated integration

$$
\|Y_i(t)\| \le k^i\,c^i\frac{\mathrm{e}^{-\gamma(t-t_0)}}{(i-1)!}\int_{t_0}^{t}(t-s)^{i-1}\,\mathrm{d}s = k^i\,c^i\,\mathrm{e}^{-\gamma(t-t_0)}\frac{(t-t_0)^i}{i!}\quad\forall\; t\ge t_0.
$$

We obtain using (10)

$$
\begin{aligned}
\|R_{A_\varepsilon}(t,t_0)-R_{A_0}(t,t_0)\| &\le \sum_{i=1}^{+\infty}\varepsilon^i\|Y_i(t)\| \\
&\le \mathrm{e}^{-\gamma(t-t_0)}\sum_{i=1}^{+\infty}\frac{k^i\,c^i\,\varepsilon^i\,(t-t_0)^i}{i!} \\
&\le \mathrm{e}^{(-\gamma+kc\varepsilon)(t-t_0)},
\end{aligned}
$$

yielding by (9)

$$
\begin{aligned}
\|R_{A_\varepsilon}(t,t_0)\| &\le \|R_{A_0}(t,t_0)\| + \|R_{A_\varepsilon}(t,t_0)-R_{A_0}(t,t_0)\| \\
&\le c\,\mathrm{e}^{-\gamma(t-t_0)} + \mathrm{e}^{-(\gamma-kc\varepsilon)(t-t_0)} \\
&\le K\mathrm{e}^{-\gamma_\varepsilon(t-t_0)}
\end{aligned}
$$

with $K=c+1$ and $\gamma_\varepsilon=\gamma-k\,c\,\varepsilon$. Thus, choosing $\varepsilon<\gamma/(kc)$ we obtain

$$
\|x(t,\varepsilon)\| = \|R_{A_\varepsilon}(t,t_0)\;x(t_0,\varepsilon)\|
$$

$$\leq \|R_{A_\varepsilon}(t,t_0)\|\|x(t_0,\varepsilon)\|$$
$$\leq K\,\|x_{0,\varepsilon}\|\mathrm{e}^{-\gamma_\varepsilon(t-t_0)}$$

with $\gamma_\varepsilon > 0$, and the proof is completed. □

Now we study the problem (6) when $h(t,x,\varepsilon) \neq 0$. In this case we have the following Theorem.

Theorem 3.6. *Consider the system* (6) *with the following assumptions:*
($\mathcal{A}_1$) $A_\varepsilon(t)$ *can be written as:*

$$A_\varepsilon(t) = A_0(t) + \varepsilon F(t),$$

where $F(.)$ *is an* $n\times n$ *continuous and bounded matrix on* $\mathbb{R}_+$ *and* ε *being a small real parameter.*
($\mathcal{A}_2$) *The system* (8) *is globally uniformly exponentially stable.*
($\mathcal{A}_3$) *The nominal system associate to* (6) *has a unique solution.*
($\mathcal{A}_4$) *The function* h *is defined on* $\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+^*$, *continuous in* (t,x,ε) *and locally Lipschitz in* (x,ε), *uniformly in* t.
($\mathcal{A}_5$) *There exists continuous positive functions* ϕ *and* λ_ε *verifying:*

$$\|h(t,x,\varepsilon)\| \leqslant \phi(t)\|x\| + \lambda_\varepsilon(t) \quad \forall t\in\mathbb{R}_+. \tag{12}$$

($\mathcal{A}_6$) $\phi\in L^p(\mathbb{R}_+,\mathbb{R}_+)$ *for some* $p\in[1,+\infty)$.
($\mathcal{A}_7$) *There exists a constant* $M'>0$, *such that*

$$\lambda_\varepsilon(t) \leq M'\mathrm{e}^{-\gamma_\varepsilon t}, \tag{13}$$

with $\gamma_\varepsilon = \gamma - kc\varepsilon$ *where* $k=\sup_{t\geq 0}\|F(t)\|$ *and* γ *and* c *are given in* (9). *Then,*

$$\forall\ (t_0,x_0,\varepsilon)\in\mathbb{R}_+\times\mathbb{R}^n\times[0,\frac{\gamma}{kc}),$$

the maximal solution $x(.,\varepsilon)$ *of* (6) *such that* $x(t_0,\varepsilon)=x_{0,\varepsilon}$, *verifies:*
(*i*) *The function* $x(.,\varepsilon)$ *is defined on* $[t_0,+\infty)$.
(*ii*) *for all* $t\geq t_0$

$$\|x(t,\varepsilon)\| \leqslant L\|x_{0,\varepsilon}\|\mathrm{e}^{-\delta(t-t_0)} + N\mathrm{e}^{-\theta t},$$

where $N,L\geq 0$ *and* $\delta,\theta\in(0,\gamma_\varepsilon]$.

In order to prove Theorem 3.6 we need the following lemma from [14].

Lemma 3.7. *Let* $\phi\in L^p(\mathbb{R}_+,\mathbb{R}_+)$ *where* $p\in(1,+\infty)$. *We denote by* $\|\phi\|_p$ *the p-norme of* ϕ. *Then,* $\forall t\geq 0$, $s\geq 0$ *and* $t\geq t_0$

$$\int_{t_0}^{t}\phi(\tau)\mathrm{d}\tau \leqslant N + L(t-t_0)$$

where $N=\int_0^s\phi(\tau)\mathrm{d}\tau + \frac{M_s}{p}$ *and* $L=\frac{p-1}{p}M_s$ *with* $M_s=\|\phi_{|[s,+\infty[}\|_p$.

Proof. (of Theorem 3.6).

(i) The system (6) can be written

$$\dot{x}(t,\varepsilon) = f(t, x(t,\varepsilon), \varepsilon),$$

where

$$f(t,x,\varepsilon) = A_\varepsilon(t)x(t,\varepsilon) + h(t, x(t,\varepsilon), \varepsilon).$$

The function f is continuous in (t,x,ε), locally Lipschitz in (x,ε) and uniformly in t then the standard Cauchy-Lipschitz Theorem $\forall\ (t_0, x_0, \varepsilon) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+^*$ asserts that there exists a unique maximal solution $x(.,\varepsilon)$ of (6) such that $x(t_0,\varepsilon) = x_{0,\varepsilon}$.

Next, we prove that $x(.,\varepsilon)$ is defined on $[t_0, +\infty)$. Supposed by contradiction that there exists $T_{max} > t_0$ such that $x(.,\varepsilon)$ is defined on $[t_0, T_{max})$. Then, for all $t \in [t_0, T_{max})$

$$\|\dot{x}(t,\varepsilon)\| \leq (M_1 + M_2)\|x(t,\varepsilon)\| + M_3,$$

where

$$M_1 = \sup_{[t_0, T_{max}]} \|A_\varepsilon(t)\|,$$
$$M_2 = \sup_{[t_0, T_{max}]} \|\phi(t)\|,$$
$$M_3 = \sup_{[t_0, T_{max}]} \|\lambda_\varepsilon(t)\|.$$

It follows that

$$\| \int_{t_0}^{t} \dot{x}(s,\varepsilon)\mathrm{d}s \| \leq \int_{t_0}^{t} \left((M_1 + M_2)\|x(s,\varepsilon)\| + M_3\right) \mathrm{d}s,$$

hence

$$\|x(t,\varepsilon)\| \leq \|x(t_0,\varepsilon)\| + \int_{t_0}^{t} \left((M_1 + M_2)\|x(s,\varepsilon)\| + M_3\right) \mathrm{d}s.$$

Using Lemma 2.1, we obtain for all $t \in [t_0, T_{max})$

$$\|x(t,\varepsilon)\| \leq \|x(t_0,\varepsilon)\| \mathrm{e}^{\int_{t_0}^{t}(M_1+M_2)\mathrm{d}s} + \mathrm{e}^{\int_{t_0}^{t}(M_1+M_2+M_3)\mathrm{d}s} \leq M_4,$$

with

$$M_4 \ = \ \|x(t_0,\varepsilon)\| \mathrm{e}^{(M_1+M_2)T_{max}} + \mathrm{e}^{(M_1+M_2+M_3)T_{max}}.$$

Consequently, $x(.,\varepsilon)$ remains within the compact B_{M_4}, which contradicts that $T_{max} < +\infty$. We conclude that

$$T_{max} = +\infty.$$

(ii) We can write the solution $x(t,\varepsilon)$ of (6) as

$$x(t,\varepsilon) = R_{A_\varepsilon}(t,t_0)x(t_0,\varepsilon) + \int_{t_0}^{t} R_{A_\varepsilon}(t,s)h(s, x(s,\varepsilon), \varepsilon)\mathrm{d}s,$$

where $R_{A_\varepsilon}(t,t_0)$ is the transition matrix of the system (7). Then, we have

$$\|x(t,\varepsilon)\| \le \|R_{A_\varepsilon}(t,t_0)\|\|x(t_0,\varepsilon)\| + \int_{t_0}^{t} \|R_{A_\varepsilon}(t,s)\|\|h(s,x(s,\varepsilon),\varepsilon)\|\mathrm{d}s.$$

By using the assumption $(\mathcal{A}_2)$ and Theorem 3.5, we get

$$\|x(t,\varepsilon)\| \le K\|x_{0,\varepsilon}\|\mathrm{e}^{-\gamma_\varepsilon(t-t_0)} + \int_{t_0}^{t} K\mathrm{e}^{-\gamma_\varepsilon(t-s)}\|\|h(s,x(s,\varepsilon),\varepsilon)\|\mathrm{d}s.$$

From the inequality (12) we deduce that

$$\mathrm{e}^{\gamma_\varepsilon t}\|x(t,\varepsilon)\| \leqslant K\|x_{0,\varepsilon}\|\mathrm{e}^{\gamma_\varepsilon t_0} + \int_{t_0}^{t} \left(K\mathrm{e}^{\gamma_\varepsilon s}\phi(s)\|x(s,\varepsilon)\| + K\mathrm{e}^{\gamma_\varepsilon s}\lambda_\varepsilon(s)\right)\mathrm{d}s.$$

Denote

$$u(t,\varepsilon) = \mathrm{e}^{\gamma_\varepsilon t}\|x(t,\varepsilon)\|,$$

it follows

$$u(t,\varepsilon) \leqslant Ku(t_0,\varepsilon) + \int_{t_0}^{t} \left(Ku(s,\varepsilon)\phi(s) + K\mathrm{e}^{\gamma_\varepsilon s}\lambda_\varepsilon(s)\right)\mathrm{d}s.$$

Applying Lemma 2.1, we get

$$u(t,\varepsilon) \leqslant u(t_0,\varepsilon)\mathrm{e}^{\int_{t_0}^{t} K\phi(s)\mathrm{d}s} + r\mathrm{e}^{\int_{t_0}^{t} K\frac{\lambda_\varepsilon(s)\mathrm{e}^{\gamma_\varepsilon s}}{r} + K\phi(s)\mathrm{d}s} \quad \forall t \ge t_0,\ \forall r > 0.$$

Since $\|x(t,\varepsilon)\| = \mathrm{e}^{-\gamma_\varepsilon t}u(t,\varepsilon)$, we obtain

$$\|x(t,\varepsilon)\| \leqslant K\|x_{0,\varepsilon}\|\mathrm{e}^{\int_{t_0}^{t} K\phi(s)\mathrm{d}s - \gamma_\varepsilon(t-t_0)} + r\mathrm{e}^{\int_{t_0}^{t} K\phi(s) + \frac{K\lambda_\varepsilon(s)\mathrm{e}^{\gamma_\varepsilon s}}{r}\mathrm{d}s - \gamma_\varepsilon t}. \tag{14}$$

Let

$$M' = \sup_{t\ge 0}[\mathrm{e}^{\gamma_\varepsilon t}\lambda_\varepsilon(t)] \ \text{ and } \ M_s = \left(\int_{s}^{+\infty} \phi^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}}.$$

We have by the assumptions $(\mathcal{A}_6)$ and $(\mathcal{A}_7)$, that M' and $M_s \in \mathbb{R}_+$. It follows that

$$\int_{t_0}^{t} \frac{K\mathrm{e}^{\gamma_\varepsilon s}\lambda_\varepsilon(s)}{r}\mathrm{d}s \leqslant \frac{KM'}{r}t \quad \forall t \ge t_0. \tag{15}$$

Moreover since $\phi \in L^p(\mathbb{R}_+,\mathbb{R}_+)$, then

$$\int_{t}^{+\infty} \phi^p(s)\mathrm{d}s \underset{t\to+\infty}{\longrightarrow} 0,$$

and hence there exists $s \ge 0$ such that

$$M_s < \frac{\gamma_\varepsilon}{K}\frac{p}{p-1}.$$

By using Lemma 3.7, we obtain for all $t \ge 0$

$$\int_{t_0}^{t} \phi(s)\mathrm{d}s \leqslant \int_{0}^{s} \phi(\tau)\mathrm{d}\tau + \frac{M_s}{p} + M_s\frac{p-1}{p}(t-t_0). \tag{16}$$

From (15) and (16), we get

$$\begin{aligned}&\int_{t_0}^{t} K\phi(s)\mathrm{d}s-\gamma_\varepsilon(t-t_0)\\&\le K\left(\int_0^s\phi(s)\mathrm{d}s+\frac{M_s}{p}\right)+\left(KM_s\frac{p-1}{p}-\gamma_\varepsilon\right)(t-t_0),\end{aligned}$$

and

$$\begin{aligned}&\int_{t_0}^{t} K\phi(s)\mathrm{d}s+\frac{K\lambda_\varepsilon(s)\mathrm{e}^{\gamma_\varepsilon s}}{r}\mathrm{d}s-\gamma_\varepsilon t\\&\le\left(-\gamma_\varepsilon+KM_s\frac{p-1}{p}+K\frac{M'}{r}\right)t+K\left(\int_0^s\phi(\tau)\mathrm{d}\tau+\frac{M_s}{p}\right),\end{aligned}$$

hence

$$\begin{aligned}&\|x(t,\varepsilon)\|\le K\mathrm{e}^{K(\int_0^s\phi(\tau)\mathrm{d}\tau+\frac{M_s}{p})}\|x_{0,\varepsilon}\|\mathrm{e}^{-(\gamma_\varepsilon-KM_s\frac{p-1}{p})(t-t_0)}\\&+r\mathrm{e}^{-\left(\gamma_\varepsilon-KM_s\frac{p-1}{p}-\frac{KM'}{r}\right)t+K(\int_0^s\phi(\tau)\mathrm{d}\tau+\frac{M_s}{p})}.\end{aligned}$$

Taking

$$\begin{aligned}r&>\frac{M'}{\frac{\gamma_\varepsilon}{K}-\frac{p-1}{p}M_s},\\L&=K\,\mathrm{e}^{K\left(\frac{M_s}{p}+\int_0^s\phi(\tau)\mathrm{d}\tau\right)},\\N&=r\,\mathrm{e}^{K\left(\frac{M_s}{p}+\int_0^s\phi(\tau)\mathrm{d}\tau\right)}=\frac{r\,L}{K},\\\delta&=\gamma_\varepsilon-K\frac{p-1}{p}M_s\in(0,\gamma_\varepsilon],\\\theta&=\gamma_\varepsilon-K\frac{p-1}{p}M_s-\frac{KM'}{r}\in(0,\delta).\end{aligned}$$

We deduce

$$\|x(t,\varepsilon)\|\le L\,\|x_{0,\varepsilon}\|\,\mathrm{e}^{-\delta(t-t_0)}+N\mathrm{e}^{-\theta t}\quad\forall t\ge t_0.$$

□

Corollary 3.8. *Under the same assumptions of Theorem* 3.6*, we have* $\forall r>\frac{M'}{\frac{\gamma_\varepsilon}{K}-\frac{p-1}{p}M_s}$, $\forall t\geqslant t_0$, $\forall x_{0,\varepsilon}\in\mathbb{R}^n\setminus B_r$*:*

$$\|x(t,\varepsilon)\|\leqslant P\,\|x_{0,\varepsilon}\|\,\mathrm{e}^{-\theta(t-t_0)},$$

where $P>0$ *and* $\theta\in(0,\gamma_\varepsilon)$*.*

Proof. Theorem 3.6 implies

$$\|x(t,\varepsilon)\|\le L\,\|x_{0,\varepsilon}\|\,\mathrm{e}^{-\delta(t-t_0)}+N\mathrm{e}^{-\theta t}\quad\forall t\ge t_0.$$

Let $r > 0$, then for all $x_{0,\varepsilon} \in \mathbb{R}^n \setminus B_r$ we have

$$\begin{aligned}\|x(t,\varepsilon)\| &\leq L\|x_{0,\varepsilon}\|\mathrm{e}^{-\delta(t-t_0)} + \frac{N}{r} r\mathrm{e}^{-\theta(t-t_0)} \\ &\leq (L + \frac{N}{r})\|x_{0,\varepsilon}\|\mathrm{e}^{-\theta(t-t_0)}.\end{aligned}$$

Take $P = L + \frac{N}{r} > 0$, we obtain

$$\|x(t,\varepsilon)\| \leq P\,\|x_{0,\varepsilon}\,\|\mathrm{e}^{-\theta(t-t_0)}.$$

□

Remark 3.9. Take the limit when $r \to \frac{M'}{\frac{\gamma_\varepsilon}{K} - \frac{p-1}{p}M_s}$ in Theorem 3.6, we obtain

$$\|x(t,\varepsilon)\| \leqslant L\|x_{0,\varepsilon}\|\mathrm{e}^{-\theta(t-t_0)} + N \quad \forall t \geq t_0 \geq 0, \tag{17}$$

with

$$N = \frac{M'}{\frac{\gamma_\varepsilon}{K} - \frac{p-1}{p}M_s}\mathrm{e}^{K\left(\frac{M_s}{p} + \int_0^s \phi(\tau)\mathrm{d}\tau\right)}.$$

In particular, if we choose $p = 1$, we find

$$\|x(t,\varepsilon)\| \leqslant L\|x_{0,\varepsilon}\|\mathrm{e}^{-\theta(t-t_0)} + N \quad \forall t \geq t_0 \geq 0, \tag{18}$$

with

$$L = K\,\mathrm{e}^{K\|\phi\|_1}$$

and

$$N = \frac{K\,M}{\gamma_\varepsilon}\mathrm{e}^{K\,\|\phi\|_1}.$$

The estimates (17) and (18) imply that the system (6) is globally uniformly practically asymptotically stable in the sense that the ball B_N is globally uniformly asymptotically stable.

4. NUMERICAL RESULTS

In what follows, we give some numerical examples to illustrate our theoretical study. The first example deals with the system (7) and the second example is in concern with the system (6). All the illustrations have been performed with the software Matlab.

Example 4.1. Consider the following system

$$\begin{cases} \dot{x}_1 = -x_1 - tx_2 + \varepsilon\,(x_1 - x_2) \\ \dot{x}_2 = -x_2 + tx_1 + \varepsilon\,(x_1 + x_2) \\ x_{0,\varepsilon} = (1, 2). \end{cases} \tag{19}$$

The system (19) can be written as

$$\dot{x} = A_\varepsilon(t)\,x,$$

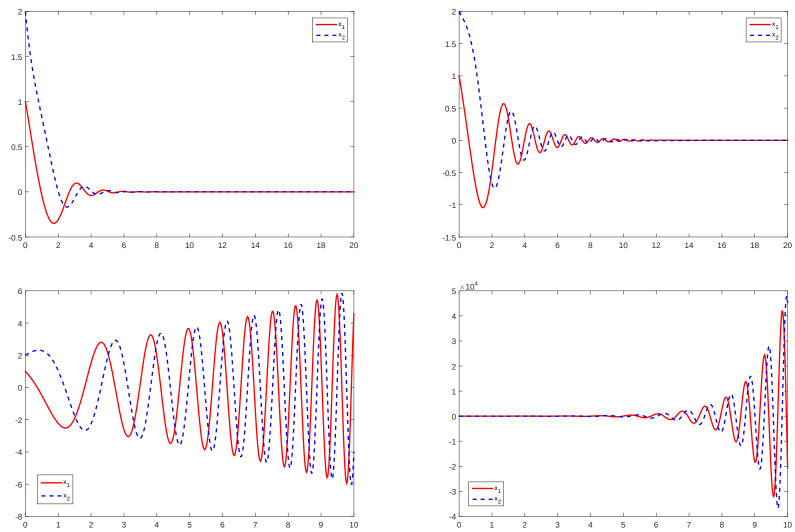

Fig. 4.1 – Time evolution of the states x_1 and x_2 of the system (19) with $t_0 = 0$ and various values of ε. From left to right and top to bottom: $\varepsilon = 0.01$, $\varepsilon = 0.5$, $\varepsilon = 1.1$ and $\varepsilon = 2$.

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A_\varepsilon(t) = A_0 + \varepsilon F(t)$$

with

$$A_0 = \begin{pmatrix} -1 & -t \\ t & -1 \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

A straightforward computation shows that the transition matrix R_{A_0} is given by

$$R_{A_0}(t, t_0) = \mathrm{e}^{-(t-t_0)} \begin{pmatrix} \cos\left(\frac{1}{2}(t^2 - t_0^2)\right) & -\sin\left(\frac{1}{2}(t^2 - t_0^2)\right) \\ \sin\left(\frac{1}{2}(t^2 - t_0^2)\right) & \cos\left(\frac{1}{2}(t^2 - t_0^2)\right) \end{pmatrix}.$$

Hence, we have

$$\|R_{A_0}(t, t_0)\| = c\,\mathrm{e}^{-\gamma(t-t_0)}$$

with $\gamma = c = 1$ and $\|\cdot\|$ denotes the Euclidean norm. Since $F(\cdot)$ is bounded, then we deduce using Theorem 3.5 that for any $\varepsilon \in [0, 1)$ the system (19) is globally uniformly exponentially stable. Figure 4.1 shows the time evolution of the states x_1 and x_2 of system (19) with $t_0 = 0$ and for various values of ε. One can notice that the solutions are stable if $\varepsilon \in [0, 1)$ as predicted by theory.

Example 4.2. Consider the following system

$$\begin{cases} \dot{x}_1 = -x_1 + \varepsilon x_2 + \frac{1}{(1+t^2)^2}\frac{x_1^2}{1+\sqrt{x_1^2+x_2^2}} + \frac{\varepsilon \mathrm{e}^{-2t}}{1+x_1^2} \\ \dot{x}_2 = -x_2 + \frac{\varepsilon}{1+t}x_1 + \frac{t}{(1+t^2)^2}\frac{x_2^2}{1+\sqrt{x_1^2+x_2^2}} \\ x_{0,\varepsilon} = (1,2), \end{cases} \tag{20}$$

which can be written as

$$\dot{x} = A_\varepsilon(t)x + h(t,x,\varepsilon),$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A_\varepsilon(t) = A_0 + \varepsilon F(t)$$

with

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} 0 & 1 \\ \frac{1}{1+t} & 0 \end{pmatrix}$$

and

$$h(t,x,\varepsilon) = \begin{pmatrix} h_1(t,x,\varepsilon) \\ h_2(t,x,\varepsilon) \end{pmatrix}$$

with

$$\begin{cases} h_1(t,x,\varepsilon) = \frac{1}{(1+t^2)^2}\frac{x_1^2}{1+\sqrt{x_1^2+x_2^2}} + \frac{\varepsilon \mathrm{e}^{-2t}}{1+x_1^2} \\ h_2(t,x,\varepsilon) = \frac{t}{(1+t^2)^2}\frac{x_2^2}{1+\sqrt{x_1^2+x_2^2}}. \end{cases}$$

The system

$$\dot{x} = A_\varepsilon(t)x$$

is globally uniformly asymptotically stable. Indeed, $F(\cdot)$ is bounded and the the transition matrix R_{A_0} satisfies:

$$R_{A_0}(t,t_0) = \begin{pmatrix} \mathrm{e}^{-(t-t_0)} & 0 \\ 0 & \mathrm{e}^{-(t-t_0)} \end{pmatrix},$$

thus

$$\|R_{A_0}(t,t_0)\| = \mathrm{e}^{-(t-t_0)}.$$

On the other hand,

$$\begin{aligned} \|h(t,x,\varepsilon)\|^2 &= h_1^2(t,x,\varepsilon) + h_2^2(t,x,\varepsilon) \\ &\le \frac{1}{(1+t^2)^3}(x_1^2+x_2^2) + \varepsilon\,(2+\varepsilon)\,\mathrm{e}^{-2t}. \end{aligned}$$

By using the classic inequality

$$\sqrt{a^2+b^2} \le a+b,$$

we get

$$\|h(t,x,\varepsilon)\|^2 \leq \phi(t)\|x(t)\| + \lambda(t,\varepsilon),$$

where

$$\phi(t) = \frac{1}{(1+t^2)^{\frac{3}{2}}}$$

and

$$\lambda(t,\varepsilon) = \sqrt{\varepsilon\,(2+\varepsilon)}\ \mathrm{e}^{-t}.$$

The functions ϕ and λ are continuous, positive and bounded on $[0,+\infty)$. Moreover

$$\phi \in L^p(\mathbb{R}_+,\mathbb{R}_+) \quad \forall\, p \in [1,+\infty).$$

To estimate $\|\phi\|_p$, we use the inequality:

$$\phi^p(t) \leq \phi(t) \quad \forall\, t \geq 0,$$

since $\|\phi\|_\infty = 1$, then

$$\int_0^{+\infty} \phi^p(t)\mathrm{d}t \leq \int_0^{+\infty} \phi(t)\mathrm{d}t = 1,$$

hence

$$\|\phi\|_p \leq 1 \quad \forall p \geq 1.$$

Consequently, one can apply Theorem 3.6 to prove the following results:

(i) there exist a unique maximal solution $x(.,\varepsilon)$ of (20) defined on $[0,+\infty)$.

(ii) $\forall\, p \geq 1$, $\forall\, \varepsilon \in (0,\frac{1}{p})$, $\forall\, t \geq t_0$

$$\|x(t,\varepsilon)\| \leq \mathrm{e}^{\frac{1}{p}}\|x_{0,\varepsilon}\|\mathrm{e}^{-(\frac{1}{p}-\varepsilon)(t-t_0)} + r\ \mathrm{e}^{-(\frac{1}{p}-\varepsilon-\frac{1}{r})\,t+\frac{1}{p}},$$

where $r > 1/(1/p-\varepsilon)$ is any arbitrary real number. In particular, we have for $p=1$ and $r \to 1/(1-\varepsilon)$

$$\|x(t,\varepsilon)\| \leqslant e\|x_{0,\varepsilon}\|\mathrm{e}^{-(1-\varepsilon)(t-t_0)} + \frac{e}{1-\varepsilon}\cdot \tag{21}$$

The estimate (21) implies that the system (20) is globally uniformly practically asymptotically stable in the sense that the ball $B_{\frac{e}{1-\varepsilon}}$ is globally uniformly asymptotically stable. Figure 4.2 shows the time evolution of the states x_1 and x_2 of the system (20) for $t_0 = 0$ and $\varepsilon = 0.1$.

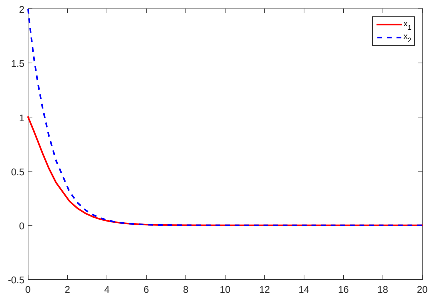

Fig. 4.2 – Time evolution of the states x_1 and x_2 of system (20) with $t_0 = 0$ and $\varepsilon = 0.1$.

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