MATHEMATICA, 66 (89), N° 1, 2024, pp. 18–31

RECURRENT SETS FOR ENDOMORPHISMS OF TOPOLOGICAL GROUPS

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Abstract. This paper studies topological definitions of chain recurrence and shadowing for continuous endomorphisms of topological groups generalizing the relevant concepts for metric spaces. It is proved that in this case the sets of chain recurrent points and chain transitive component of the identity are topological subgroups. Furthermore, we show that some dynamical properties are induced by the original system on quotient spaces. These results link an algebraic property to a dynamical property.

MSC 2020. 37B20, 54H11.

Key words. Chain recurrence, chain component, shadowing property, topological group, topological entropy.

1. INTRODUCTION

One of the main problems in discrete and continuous dynamical systems is the description of the orbit structure for a system from a topological point of view. A discrete dynamical system usually consists of a compact metric space X and a continuous function f from X to itself. A number of properties of interest in such systems are defined in purely topological terms, for example recurrence, non-wandering points and transitivity. Recently, Good and Macias [18] defined other properties for dynamical systems in purely topological terms, for example sensitive dependence on initial conditions, chain transitivity and recurrence, shadowing, and positive expansiveness. In the presence of compactness, existence of an unique uniformity, allows us to mimic existing metric proofs. The uniform approach has been studied in a number of cases: Hood [19] defined topological entropy for uniform spaces; Morales and Sirvent [20] considered positively expansive measures for measurable functions on uniform spaces, extending results from the literature; Devaney chaos for uniform spaces is considered in [12]; Auslander et al. [6] generalized many known

The authors thank the referee for his helpful comments and suggestions.

Some of the results included in this paper were obtained during visit of S. A. Ahmadi at the Abdus Salam International Centre for Theoretical Physics (ICTP). Support of this institution is widely acknowledged.

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DOI: 10.24193/mathcluj.2024.1.02

results about equicontinuity to the uniform spaces; Das et al. [13] generalized spectral decomposition theorem to the uniform spaces; The authors of [3] generalized concepts of entropy points, expansivity and the shadowing property for dynamical systems on uniform spaces and obtained a relation between the topological shadowing property and the positive uniform entropy. For more results on properties of dynamical systems in purely topological terms, one is referred to [2, 14, 22, 26–28].

Motivated by these ideas we show that if the underlying set of a dynamical system is an abelian topological group, then, surprisingly, dynamical objects of a dynamical system exhibit some algebraic properties. Recurrence behaviour is one of the most important concepts in topological dynamics [1, 4, 25]. We are going to investigate the properties of recurrent sets of a continuous endomorphism of a topological group as a discrete dynamical system. Every topological group is a uniform space in a natural way. Specifically, a uniform group structure on a topological group is defined by the collection of sets

$$\{(x,y) \mid xy^{-1} \in E\}; \quad E \in \mathfrak{B}_e,$$

where $\mathfrak{B}_e$ is a system of symmetric neighbourhoods of the identity e in G. We make a standardized assumption that all topological groups are abelian and compact, and f is a continuous endomorphism on G, although some of the results apply to more general settings.

A fixed point of dynamical system f, exhibits the simplest type of recurrence. We denote by $\mathrm{Fix}(f)$ the set of all fixed points of f. A point carried back to itself by a dynamical system f exhibits the next most elementary type of recurrence. For some $m \in \mathbb{N}$, a point $x \in G$ is called *m-periodic* if $f^m(x) = x$. We denote by $\mathrm{Per}_m(f)$ the set of all m-periodic points of f and we set $\mathrm{Per}(f) = \bigcup_{m=1}^{\infty} \mathrm{Per}_m(f)$. A point $x \in G$ is *non-wandering* if for each neighbourhood U of x, there exists $n \in \mathbb{N}$ such that $U \cap f^n(U) \neq \emptyset$. We denote by $\Omega(f)$ the set of all non-wandering points of f.

For $D \in \mathcal{B}_e$, a *D-pseudo-orbit* or *D-chain* of f is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $f(x_n)x_{n+1}^{-1} \in D$ for $n \in \mathbb{N}$. We use the symbol $\mathcal{O}_E(f,x,y)$ for the set of E-chains $\{x_0, x_1, \ldots, x_n\}$ of f with $x_0 = x$ and $x_n = y$. For $x, y \in G$, we write $x \overset{E}{\rightsquigarrow} y$ if $\mathcal{O}_E(f,x,y) \neq \emptyset$ and we write $x \rightsquigarrow y$ if $\mathcal{O}_E(f,x,y) \neq \emptyset$ for each $E \in \mathfrak{B}_e$. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set $\{x \in G \mid x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and it is denoted by $\mathrm{CR}(f)$. Denote by $\mathrm{CC}(f)$ the *chain component* of f containing the identity e, i.e., $\mathrm{CC}(f) = \{x \in G \mid x \leftrightsquigarrow e\}$ [17]. Clearly,

$$\mathrm{Fix}(f) \subseteq \mathrm{Per}_m(f) \subseteq \mathrm{Per}(f) \subseteq \Omega(f) \subseteq \mathrm{CR}(f).$$

This paper is organized as follows. Section 2 describes which recurrent sets are topological subgroups. In Section 3 we prove that some recurrent subgroups are invariant under canonical map. In Section 4 we prove a corollary of the addition theorem which states that the entropy is additive in the appropriate sense with respect to invariant subgroups.

2. RECURRENT SUBGROUPS

This section is devoted to the algebraic properties of recurrent sets. Our following results show that when the underlying set of a dynamical system is a topological group, then most of the well-known recurrent sets are also topological subgroups of the underlying topological group. We seek a definition of recurrence so that the set $\mathfrak{R}(f)$ of recurrent points with respect to an endomorphism f has the following desirable properties:

(R1) $\mathfrak{R}(f)$ is a subgroup;
(R2) The set $\mathfrak{R}(f)$ is forward invariant with respect to f, i.e., $f(\mathfrak{R}(f)) \subseteq \mathfrak{R}(f)$.
(R3) $\mathfrak{R}(f)$ is closed;
(R4) $\mathfrak{R}(f)$ is invariant under topological conjugacy, i.e., if $f : G \to G$ and $g : H \to H$ are two continuous endomorphisms on topological groups and $\phi : G \to H$ is a continuous isomorphism with continuous inverse such that $\phi \circ f = g \circ \phi$, then $\mathfrak{R}(g) = \phi(\mathfrak{R}(f))$;
(R5) $\mathfrak{R}(f)$ is invariant under canonical mapping, i.e., if $\tilde{f} : G/H \to G/H$ is the canonical mapping induced by f, then $\mathfrak{R}(\tilde{f}) = \{\mathfrak{R}(f)\}$.

Definition 2.1. We say that a subset $\mathfrak{R}(f)$ of G with respect to an endomorphism f is a *recurrent subgroup* if it satisfies properties (R1)–(R4).

Closely related to fixed points are the eventually fixed points, which are the points that reach a fixed point after finitely many iterations. More explicitly, a point x is said to be an *eventually fixed point* of a map f if there exists some $k \in \mathbb{N}$ such that $f^k(x) \in \mathrm{Fix}(f)$. A point x is said to be an *eventually m-periodic point* if $f^k(x) \in \mathrm{Per}_m(f)$ for some $k \in \mathbb{N}$. Denote by $\mathrm{EFix}(f)$ and $\mathrm{EPer}_m(f)$, the set of all eventually fixed point and the set of eventually m-periodic points of f, respectively. Also we set $\mathrm{EPer}(f) = \bigcup_{m=1}^{\infty} \mathrm{EPer}_m(f)$.

Proposition 2.2. *Let f be a continuous endomorphism of a topological group G. Then*

(i) $\mathrm{Fix}(f)$ *is a recurrent subgroup.*
(ii) $\mathrm{Per}_m(f)$ *is a recurrent subgroup.*
(iii) $\overline{\mathrm{Per}(f)}$ *is a recurrent subgroup.*
(iv) $\overline{\mathrm{EFix}(f)}$ *is a recurrent subgroup.*
(v) $\overline{\mathrm{EPer}_m(f)}$ *is a recurrent subgroup.*
(vi) $\overline{\mathrm{EPer}(f)}$ *is a recurrent subgroup.*

Proof. (i) For any $x, y \in \mathrm{Fix}(f)$, it can be verified that

$$f(xy^{-1}) = f(x)f(y^{-1}) = xy^{-1},$$

implying that $\mathrm{Fix}(f)$ is a subgroup. Clearly, $\mathrm{Fix}(f)$ is closed, f-invariant and invariant under algebraic topological conjugacy.

(ii) The proof is similar to part (i).

(iii) For any $x, y \in \mathrm{Per}(f)$, there exist $m, n \in \mathbb{N}$ such that $f^m(x) = x$ and $f^n(y) = y$, implying that $f^{mn}(xy^{-1}) = (f^m)^n(x)(f^n)^m(y^{-1}) = xy^{-1}$. Then, $xy^{-1} \in \mathrm{Per}(f)$. This implies that $\mathrm{Per}(f)$ is a subgroup, and so is $\overline{\mathrm{Per}(f)}$. Clearly, $\mathrm{Per}(f)$ is f-invariant, and so is $\overline{\mathrm{Per}(f)}$ by continuity. Let $f : G \to G$ and $g : H \to H$ be two continuous endomorphisms and let $\phi : G \to H$ be a continuous automorphism with continuous inverse such that $\phi \circ f = g \circ \phi$. Clearly, $\phi(\mathrm{Per}(f)) \subset \mathrm{Per}(g)$. This implies that $\phi(\overline{\mathrm{Per}(f)}) \subset \overline{\phi(\mathrm{Per}(f))} \subset \overline{\mathrm{Per}(g)}$. For the reverse inclusion, let $x \in \overline{\mathrm{Per}(g)}$. Then there exists a net x_λ in $\mathrm{Per}(g) = \phi(\mathrm{Per}(f))$ with $x_\lambda \to x$. For each λ, there exists a point $z_\lambda \in \mathrm{Per}(f)$ such that $x_\lambda = \phi(z_\lambda)$. This, together with $x_\lambda \to x$, implies that $z_\lambda \to \phi^{-1}(x)$. Therefore, $\phi^{-1}(x) \in \overline{\mathrm{Per}(f)}$. This implies that $\phi(\overline{\mathrm{Per}(f)}) = \overline{\mathrm{Per}(g)}$.

(iv) For any $x, y \in \mathrm{EFix}(f)$, there exist $m, n \in \mathbb{N}$ such that $f^m(x), f^n(y) \in \mathrm{Fix}(f)$, implying that $f^{mn}(xy^{-1}) = f^m(x)f^n(y^{-1}) \in \mathrm{Fix}(f)$. Then, $xy^{-1} \in \mathrm{EFix}(f)$. This implies that $\mathrm{EFix}(f)$ is a subgroup, and so is $\overline{\mathrm{EFix}(f)}$. Clearly, $\mathrm{EFix}(f)$ is f-invariant, and so is $\overline{\mathrm{EFix}(f)}$ by continuity. Let $f : G \to G$ and $g : H \to H$ be two continuous endomorphisms and let $\phi : G \to H$ be a continuous automorphism with continuous inverse such that $\phi \circ f = g \circ \phi$. If $\phi(x) \in \phi(\mathrm{EFix}(f))$, then there exists positive integer m such that $f^m(x) \in \mathrm{Fix}(f)$. Thus, $\phi(f^m(x)) \in \phi(\mathrm{Fix}(f)) = \mathrm{Fix}(g)$, implying that $g^m(\phi(x) \in \mathrm{Fix}(g)$. Therefore, $\phi(\mathrm{EFix}(f)) \subset \mathrm{EFix}(g)$. This implies that $\phi(\overline{\mathrm{EFix}(f)}) \subset \overline{\phi(\mathrm{EFix}(f))} \subset \overline{\mathrm{EFix}(g)}$. For the reverse inclusion, let $x \in \overline{\mathrm{EFix}(g)}$. Then there exists a net x_λ in $\mathrm{EFix}(g) = \phi(\mathrm{EFix}(f))$ with $x_\lambda \to x$. For each λ, there exists a point $z_\lambda \in \mathrm{EFix}(f)$ such that $x_\lambda = \phi(z_\lambda)$. This, together with $x_\lambda \to x$, implies that $z_\lambda \to \phi^{-1}(x)$. Therefore, $\phi^{-1}(x) \in \overline{\mathrm{EFix}(f)}$. This implies that $\phi(\overline{\mathrm{EFix}(f)}) = \overline{\mathrm{EFix}(g)}$.

(v) For any $x, y \in \mathrm{EPer}_m(f)$, there exist $k, l \in \mathbb{N}$ such that $f^{m+k}(x) = f^k(x)$ and $f^{m+l}(y) = f^l(y)$, implying that $f^{m+k+l}(x) = f^{k+l}(x)$ and $f^{m+k+l}(y) = f^{k+l}(y)$. Then, $f^{m+k+l}(xy^{-1}) = f^{k+l}(xy^{-1})$. This implies that $\mathrm{EPer}_m(f)$ is a subgroup, and so is $\overline{\mathrm{EPer}_m(f)}$. Similarly, one can prove the properties (R3) and (R4) by adapting the proof of part (iii).

(vi) For any $x, y \in \mathrm{EPer}(f)$, there exist $k, l, m \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $f^{m+k}(x) = f^k(x)$ and $f^{n+l}(y) = f^l(y)$, implying that $f^{m+k+l}(x) = f^{k+l}(x)$ and $f^{n+k+l}(y) = f^{k+l}(y)$. Applying induction yields that

$$f^{2m+k+l}(x) = f^m(f^{m+k+l}(x)) = f^{m+k+l}(x) = f^{k+l}(x),$$
$$\vdots$$
$$f^{mn+k+l}(x) = f^{m(n-1)+k+l}(x) = f^{k+l}(x),$$

and

$$f^{2n+k+l}(y) = f^n(f^{n+k+l}(y)) = f^{n+k+l}(y) = f^{k+l}(y),$$
$$\vdots$$
$$f^{mn+k+l}(y) = f^{n(m-1)+k+l}(y) = f^{k+l}(y).$$

Thus, $f^{mn+k+l}(xy^{-1}) = f^{k+l}(xy^{-1})$. This implies that $\mathrm{EPer}(f)$ is a subgroup, and so is $\overline{\mathrm{EPer}(f)}$. Similarly, one can prove the properties (R3) and (R4) by adapting the proof of part (iii) . □

REMARK 2.3 (Categorical point of view). Let $\mathcal{C}$ be a category and let $\mathcal{C}[X]$ be the category of functors $\mathrm{Funct}(\mathbb{N}, \mathcal{C})$ where $\mathbb{N}$ is the obvious posetal category. Objects in $\mathcal{C}[X]$ can be identified with a pair (C, f), where C is an object $\mathcal{C}$ and $f : C \to C$ a morphism in $\mathcal{C}[X]$. A morphism $\phi : (C, f) \to (D, g)$ in $\mathcal{C}[X]$ is just a morphism $\phi : C \to D$, such that $g \circ \phi = \phi \circ f$. In this paper, the category Ab.Comp$[X]$ is studied, where Ab.Comp is the category of compact Hausdorff Abelian topological groups, and continuous endomorphisms among them. More precisely, an object in Ab.Comp$[X]$ is a pair (G, f), where G is a compact T_2 abelian group, and f is a continuous endomorphism of G. Also, a morphism $\phi : (G, f) \to (G', f')$ in Ab.Comp$[X]$ is a continuous endomorphism $\phi : G \to G'$ such that it commutes with the actions, that is, $f' \circ \phi = \phi \circ f$.

It is well-known that continuous maps among compact T_2 spaces are closed (and proper). Hence, also in view of the above lemma, a subobject of an object (G, f) in Ab.Comp$[X]$ is given by a closed (and, thus a compact) subgroup H of G such that $f(H) \subset H$, with action given by $f|_H : H \to H$. In view of this, a subfunctor $\mathfrak{R}$: Ab.Comp$[X] \to$ Ab.Comp$[X]$ of the identity functor id : Ab.Comp$[X] \to$ Ab.Comp$[X]$ is given by a correspondence that sends an object $(G, f) \to \mathfrak{R}$ such that (R1), (R2), (R3) and (R4) hold.

Note that if there is subfunctor of the subfunctor

$$\mathfrak{R} : \mathrm{Ab.Comp}[X] \to \mathrm{Ab.Comp}[X]$$

and we take an object (G, f), then $\mathfrak{R}f \subset G$ is a recurrent subgroup of G.

The following proposition shows that a chain recurrent set, which includes all the types of returning trajectories: periodic, eventually-periodic, non-wandering and so on, is a subgroup of G.

PROPOSITION 2.4. *Let f be a continuous endomorphism of a topological group G. Then* $\mathrm{CR}(f)$ *is a recurrent subgroup.*

Proof. For any $x \in \mathrm{CR}(f)$ and any $E \in \mathfrak{B}_e$, there exists an E-chain $\{x_1, x_2, \ldots, x_n\}$ with $x_1 = x_n = x$ such that $f(x_n)x_{n+1}^{-1} \in E$ for all $1 \leq i \leq n-1$, implying that $f(x_i^{-1})x_{i+1} \in E^{-1} = E$. Then, $x^{-1} \in \mathrm{CR}(f)$.

For any $x, y \in \mathrm{CR}(f)$, we show that $xy \in \mathrm{CR}(f)$. In fact, for any $E \in \mathfrak{B}_e$, take some $W \in \mathfrak{B}_e$ such that $W^2 \subset E$. Since $x, y \in \mathrm{CR}(f)$, there exist E-chains $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$ with $x_1 = x_m = x$ and $y_1 = y_n = y$ such that $f(x_i)x_{i+1}^{-1} \in W$ for $1 \leq i \leq m-1$ and $f(y_j)y_{j+1} \in W$ for $1 \leq j \leq n-1$. Choose two extended sequences $\{x_i\}_{i=1}^{mn}$ and $\{y_j\}_{j=1}^{mn}$ as follows:

$$x_{i+kn} = x_i \text{ for } 1 \leq i \leq m,\ 0 \leq k \leq n-1;$$
$$y_{j+kn} = y_j \text{ for } 1 \leq j \leq n,\ 0 \leq k \leq m-1.$$

Clearly, $x_1y_1 = x_{mn}y_{mn} = xy$ and

$$f(x_iy_i)(x_{i+1}y_{i+1})^{-1} = f(x_i)x_{i+1}^{-1}f(y_i)y_{i+1}^{-1} \in W^2 \subset E.$$

Therefore, $xy \in \mathrm{CR}(f)$, implying that $\mathrm{CR}(f)$ is a subgroup.

Next assume that $E \in \mathfrak{B}_e$ and choose $\hat{E} \in \mathfrak{B}_e$ such that $\hat{E}^2 \subset E$. By the uniform continuity there exists $D \in \mathfrak{B}_e$ such that $xy^{-1} \in D$ implies $f(x)f(y)^{-1} \in \hat{E}$. Choose $\hat{D} \in \mathfrak{B}_e$ with $\hat{D}^2 \subset D$. Assume that $x^{(\lambda)}$ is a net in $\mathrm{CR}(f)$ such that $x^{(\lambda)} \to x$. Then for some λ, $x^{(\lambda)}x^{-1} \in D$. Since $x^{(\lambda)} \in \mathrm{CR}(f)$, there exists a $\hat{D}$-pseudo-orbit $\{x_0, x_1, \dots, x_n\}$ with $x_0 = x_n = x^{(\lambda)}$. Thus $\{x, x_1, \dots, x_{n-1}, x\}$ is an E-pseudo-orbit and hence $x \in \mathrm{CR}(f)$. Therefore, $\mathrm{CR}(f)$ is closed.

Fix any $E \in \mathfrak{B}_e$. Then, $\{x, f(x)\}$ is an E-pseudo-orbit from x to $f(x)$. Choose $U, V \in \mathfrak{B}_e$ such that $U^2 \subset E$ and $V \subset U \cap f^{-1}(U)$. Since $x \in \mathrm{CR}(f)$, there exists a V-pseudo-orbit $\{x_0 = x, x_1, \dots, x_{n-1}, x_n = x\}$ from x to itself. Then $\{f(x), x_2, x_3, \dots, x_n = x\}$ is an E-pseudo-orbit from $f(x)$ to x and hence $\{f(x), x_2, x_3, \dots, x_{n-1}, x, f(x)\}$ is an E-pseudo-orbit from $f(x)$ to itself. Therefore, $f(x) \in \mathrm{CR}(f)$, implying that $f(\mathrm{CR}(f)) \subset \mathrm{CR}(f)$. □

The relation '$\leftrightsquigarrow$' is an equivalence relation on $\mathrm{CR}(f)$. The equivalence classes of this relation are called *chain components*. These are compact invariant sets and cannot be decomposed into two disjoint non-empty compact invariant sets, hence serve as building blocks of the dynamics. The topology of chain recurrent set and chain components have been always in particular interest [8, 9, 21, 24].

Proposition 2.5. *Let f be a continuous endomorphism of topological group G. Then $\mathrm{CC}(f)$ is a recurrent subgroup.*

Proof. Suppose that $x, y \in \mathrm{CC}(f)$ and $E \in \mathfrak{B}_e$. Choose $W \in \mathfrak{B}_e$ such that $W^2 \subset E$. Then, $x \overset{W}{\rightsquigarrow} e$ and $y \overset{W}{\rightsquigarrow} e$, implying that there exist W-chains

$$\{x = x_0, x_1, \dots, x_m = e\} \qquad \text{and} \qquad \{y = y_0, y_1, \dots, y_n = e\}.$$

Without loss of generality, assume that $m \le n$. Clearly,

$$\{x_0^{-1}y_0, x_1^{-1}y_1, \dots x_m^{-1}y_m, y_{m+1}, \dots, y_n = e\}$$

is an E-chain from $x^{-1}y$ to e. It follows from $x, y \in \mathrm{CC}(f)$ that $e \overset{W}{\rightsquigarrow} x$ and $e \overset{W}{\rightsquigarrow} y$. Then, there exist W-chains

$$\{e = \hat{x}_0, \hat{x}_1, \dots, \hat{x}_p = x\} \qquad \text{and} \qquad \{e = \hat{y}_0, \hat{y}_1, \dots, \hat{y}_q = y\}.$$

Without loss of generality, assume that $p \le q$. Then, the sequence

$$\{\hat{y}_0, \hat{y}_1, \dots, \hat{y}_{q-p-1}, \hat{y}_{q-p}\hat{x}_0^{-1}, \hat{y}_{q-p+1}\hat{x}_1^{-1}, \dots, \hat{y}_q\hat{x}_p^{-1}\}$$

is an E-chain from e to $x^{-1}y$. Therefore, $x^{-1}y \in \mathrm{CC}(f)$, implying that $\mathrm{CC}(f)$ is a subgroup.

Let $E \in \mathfrak{B}_e$ and choose $W \in \mathfrak{B}_e$ such that $W^2 \subset E$. By uniform continuity there exists $W \supset D \in \mathfrak{B}_e$ such that $xy^{-1} \in D$ implies $f(x)f(y^{-1}) \in W$. Let $z \in \overline{\mathrm{CC}(f)}$. Then, there exists $x \in \mathrm{CC}(f)$ such that $xz^{-1} \in D$. Clearly, $x \overset{D}{\rightsquigarrow} e$ and $e \overset{D}{\rightsquigarrow} x$. This implies that there exist D-chains

$$\{x = x_0, x_1, \ldots, x_n = e\} \qquad \text{and} \qquad \{e = x'_0, x'_2, \ldots, x'_m = x\}.$$

Clearly, $\{z, x_1, x_2, \ldots, e\}$ and $\{e = x'_0, x'_1, \ldots, x'_{m-1}, z\}$ are E-chains. Then, $z \overset{E}{\rightsquigarrow} e$ and $e \overset{E}{\rightsquigarrow} z$. Therefore, $z \in \mathrm{CC}(f)$, implying that $\mathrm{CC}(f)$ is closed.

Suppose that $x \in \mathrm{CC}(f)$ and $E \in \mathfrak{B}_e$. Choose $U, V \in \mathfrak{B}_e$ such that $U^2 \subset E$ and $V \subset U \cap f^{-1}(U)$. It follows from $x \in \mathrm{CC}(f)$ that there exists a V-pseudo-orbit $\{x_0 = x, x_1, \ldots, x_{n-1}, x_n = e\}$ from x to e. Then,

$$\{f(x), x_2, x_3, \ldots, x_n = e\}$$

is an E-pseudo-orbit from $f(x)$ to e. Again from $x \in \mathrm{CC}(f)$, it follows that there exists a V-pseudo-orbit $\{x_0 = e, x_1, \ldots, x_{n-1}, x_n = x\}$ from e to x. Then, $\{x_0, x_1, x_2, x_3, \ldots, x_n = x, f(x)\}$ is an E-pseudo-orbit from e to $f(x)$. Thus, $f(x) \in \mathrm{CC}(f)$, implying that $f(\mathrm{CC}(f)) \subset \mathrm{CC}(f)$. □

Remark 2.6. It is not difficult to check that if f is an automorphism, then $f(\mathrm{CC}(f)) = \mathrm{CC}(f)$.

3. DYNAMICS INDUCED ON QUOTIENT SPACES BY ENDOMORPHISMS

Suppose that G is a topological group with identity e, and H is a closed subgroup of G. Denote by G/H the set of all left cosets aH of H in G, and endow it with the quotient topology with respect to the canonical mapping $\pi : G \to G/H$ defined by $\pi(x) = xH$ for any $x \in G$. Then, the family $\{\pi(xE) \mid x \in G,\ E \in \mathfrak{B}_e\}$ is a local base of the space G/H at the point $xH \in G/H$, the mapping π is open, and G/H is a homogeneous T_1-space.

Let $f : G \to G$ be an endomorphism such that $f(H) \subset H$. Then f induces a map $\tilde{f} : G/H \to G/H$ such that the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & G \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/H & \xrightarrow[\ \tilde{f}\ ]{} & G/H \end{array}$$

This mapping is called a *canonical map* [5].

In the last section we introduce several recurrent subgroups $\mathfrak{R}(f)$ for a dynamical system f, which leads us to investigate the dynamic of induced mapping $\tilde{f} : G/\mathfrak{R}(f) \to G/\mathfrak{R}(f)$. We are interested in cases that $\mathfrak{R}(f)$ is invariant under canonical mapping, i.e., $\mathfrak{R}(\tilde{f}) = \{\mathfrak{R}(f)\}$.

Proposition 3.1. *Let $f : G \to G$ be a continuous automorphism and $\tilde{f} : G/\mathrm{CC}(f) \to G/\mathrm{CC}(f)$ be the map induced by f. Then,* $\mathrm{CC}(\tilde{f}) = \{\mathrm{CC}(f)\}$.

Proof. Denote $H = \mathrm{CC}(f)$. Let $xH \in \mathrm{CC}(\tilde{f})$ and $E \in \mathfrak{B}_e$. Then, $xH \overset{\pi(E)}{\rightsquigarrow} H$, implying that there exist points $x_0, x_1, \ldots, x_n \in G$ such that $x_0 = x$, $x_n \in H$ and $\tilde{f}(\pi(x_i))\pi(x_{i+1})^{-1} \in \pi(H)$ for all $0 \leq i \leq n-1$. Thus, for any $1 \leq i \leq n$, there exist $e_i \in E$ and $h_i \in H$ such that $f(x_i)x_{i+1}^{-1} = e_{i+1}h_{i+1}$. Choose $y_i = x_i h_i'$ with

$$h_0' = e, \quad h_{i+1}' = f(h_i')h_{i+1}.$$

Then, $y_0 = x$, $y_n = h_n'$, and

$$f(y_i)y_{i+1}^{-1} = f(x_i)x_{i+1}^{-1}f(h_i')(h_{i+1}')^{-1} = f(x_i)x_{i+1}^{-1}h_{i+1}^{-1} = e_{i+1} \in E,$$

implying that $x \overset{E}{\rightsquigarrow} h_n'$. From $h_n' \in H = \mathrm{CC}(f)$, it follows that $h_n' \overset{E}{\rightsquigarrow} e$. Then, $x \overset{E}{\rightsquigarrow} e$. This implies that $x \rightsquigarrow e$ due to the arbitrariness of E.

From $xH \in \mathrm{CC}(\tilde{f})$ and $H \overset{\pi(H)}{\rightsquigarrow} xH$, it follows that there exist points $x_0, x_1, \ldots, x_n \in G$ such that $x_0 \in H$, $x_n = x$ and $\tilde{f}(\pi(x_i))\pi(x_{i+1})^{-1} \in \pi(H)$ for all $0 \leq i \leq n-1$. This implies that there exist $e_i \in E$ and $h_i \in H$ such that $f(x_i)x_{i+1}^{-1} = e_{i+1}h_{i+1}$.

For any $0 \leq i \leq n$, choose $y_i = x_i h_i'$ with

$$h_n' = e, \quad h_{i-1}' = f^{-1}(h_i')h_{i+1}^{-1}.$$

Then, $y_0 = h_0'$, $y_n = x$, and

$$f(y_i)y_{i+1}^{-1} = f(x_i)x_{i+1}^{-1}f(h_i')(h_{i+1}')^{-1} = f(x_i)x_{i+1}^{-1}h_{i+1}^{-1} = e_{i+1} \in E,$$

implying that $h_0' \rightsquigarrow x$. From $h_0' \in H = \mathrm{CC}(f)$, it follows that $e \overset{E}{\rightsquigarrow} h_0'$. Therefore, $e \overset{E}{\rightsquigarrow} x$, implying that $e \rightsquigarrow x$ due to the arbitrariness of E. Hence, $x \in \mathrm{CC}(f)$. □

The shadowing property provides tools for fitting real trajectories nearby to approximate trajectories [11]. The following definition generalizes the relevant concept for metric spaces to topological groups.

Definition 3.2. We say that a D-pseudo orbit of f is *E-shadowed* by a point x in G if $f^n(x)x_n^{-1} \in E$ for any $n \in \mathbb{N}$. A continuous endomorphism $f : G \to G$ has the *shadowing property* if for any $E \in \mathfrak{B}_e$, there exists some $D \in \mathfrak{B}_e$ such that every D-pseudo orbit of f can be E-shadowed by some point in G.

Lemma 3.3. *Let $f : G \to G$ be a continuous endomorphism with the shadowing property. If H is an f-invariant subgroup of G, then for any $E \in \mathfrak{B}_e$, there exists $D \in \mathfrak{B}_e$ such that every DH-pseudo-orbit can be EH-shadowed by some point in G.*

Proof. Fix any $E \in \mathfrak{B}_e$. The shadowing property implies that there exists $D \in \mathfrak{B}_e$ such that every D-pseudo-orbit can be E-shadowed by some point in G.

Given any fixed DH-pseudo-orbit $\{y_n\}$, then $f(y_n)y_{n+1}^{-1} \in DH$ for all $n \in \mathbb{N}$. This implies that for any $n \in \mathbb{N}$, there exists $d_n \in D$ and $h_n \in H$ such that $f(y_n)y_{n+1}^{-1}h_n^{-1} = d_n$.

Choose the sequence $h'_n$with $h'_{n+1} = f(h'_n)h_n$ and take $x_n = y_nh'_n$. Then,

$$\begin{aligned} f(x_n)x_{n+1} &= f(y_nh'_n)(y_{n+1}h'_{n+1})^{-1} \\ &= f(y_n)y_{n+1}^{-1}f(h'_n)(h'_{n+1})^{-1} \\ &= f(y_n)y_{n+1}h_n^{-1} = d_n \in D, \end{aligned}$$

implying that $\{x_n\}$ is a D-pseudo-orbit of f. Thus, there exists $y \in G$ such that $f^n(y)x_n^{-1} \in E$ for $n = 0, 1, 2, \ldots$. Therefore, $f^n(y)(y_nh'_n)^{-1} \in E$. This implies that $f^n(y)y_n^{-1} \in EH$. □

THEOREM 3.4. *Let $f : G \to G$ be a continuous endomorphism with the shadowing property and let H be a an f-invariant subgroup of G. Then, the canonical mapping $\tilde{f} : G/H \to G/H$ has the shadowing property.*

Proof. Fix any $E \in \mathfrak{B}_e$ and take $D \in \mathfrak{B}_e$ such that every D-pseudo-orbit can be E-shadowed by some point in G. Assume that $\{\pi(x_n)\}_{n=1}^{\infty} = \{x_nH\}_{n=1}^{\infty}$ is a $\pi(D)$-pseudo-orbit of $\tilde{f}$. Then, $\tilde{f}(x_nH)(x_{n+1}H)^{-1} \in \pi(D)$, implying that $f(x_n)x_{n+1}^{-1}H \in DH$. Thus, $f(x_n)x_{n+1}^{-1} \in DH$. This implies that $\{x_n\}_{n=1}^{\infty}$ is a DH-pseudo-orbit of f and by Lemma 3.3 there exists $x \in G$ such that $f^n(x)x_n^{-1} \in EH$. This implies that $\tilde{f}^n(\pi(x))\pi(x_n)^{-1} \in \pi(E)$. □

COROLLARY 3.5. *Let $f : G \to G$ be a continuous automorphism with the shadowing property. Then $\tilde{f} : G/H \to G/H$ has the shadowing property for any choice of H as a recurrent subgroup from Proposition 2.2.*

PROPOSITION 3.6. *Let $f : G \to G$ be a continuous automorphism with the shadowing property. Then, $\Omega(f) = \mathrm{CR}(f)$.*

Proof. Clearly, $\Omega(f) \subset \mathrm{CR}(f)$. Suppose that $x \in \mathrm{CR}(f)$ and U is an open neighbourhood of x. Choose some $E \in \mathfrak{B}_e$ such that $Ex \in U$. By the shadowing property there exists $D \in \mathfrak{B}$ such that every D-pseudo-orbit is E-shadowed by some point in G. From $x \in \mathrm{CR}(f)$, it follows that there exists a D-pseudo-orbit $\{x_0 = x, x_1, \ldots, x_n = x\}$ from x to itself. If we extend this sequence to an infinite D-pseudo-orbit, then this full pseudo-orbit is E-shadowed by some point $z \in G$. Thus, $zx^{-1}, f(z)x_1^{-1}, \ldots, f^n(z)x^{-1} \in E$, implying that $z, f^n(z) \in Ex \subset U$. Therefore, $f^n(U) \cap U \neq \emptyset$ and so $x \in \Omega(f)$. □

4. TOPOLOGICAL ENTROPY

Let f be a continuous endomorphism on the topological group G and K be a compact subset of G. Given $E \in \mathfrak{B}_e$ and $n \in \mathbb{N}$, a subset $A \subseteq K$ is called an (n, E, f)-*spanning set* for K if

$$K \subseteq \bigcup_{x \in A} \left(\bigcap_{i=0}^{n-1} F^{-i}(E) \right) x,$$

or equivalently, if for any $x \in K$, there exists $y \in A$ such that $f^i(x) f^i(y^{-1}) \in E$ for all $i = 0, 1, 2, \ldots, n-1$.

By compactness, there exists a finite (n, E, f)-spanning set for K. Let $\mathrm{span}(n, E, K)$ be the minimum cardinality of all (n, E, f)-spanning sets for K.

A subset $A \subseteq K$ is called an (n, E, f)-*separated set* for K if for any pair of distinct points x and y in A, there exists $0 \leq i \leq n-1$ such that $f^i(x) f^i(y^{-1}) \notin E$. Again by compactness of K, every (n, E, f)-separated set for K is finite.

Let $\mathrm{sep}(n, E, K)$ be the maximum cardinality of all (n, E, f)-separated set for K.

For any $U \in \mathfrak{B}_e$, define

$$\mathrm{span}(E, K) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{span}(n, E, K);$$
$$\mathrm{sep}(E, K) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(n, E, K).$$

Then, define the following quantities for a uniformly continuous map f:

$$h_{\mathrm{span}}(f, K) = \sup \left\{ \mathrm{span}(E, K) \mid E \in \mathfrak{B}_e \right\};$$
$$h_{\mathrm{span}}(f) = \sup \left\{ h_{\mathrm{span}}(f, K) \mid K \in \mathcal{K}(G) \right\};$$
$$h_{\mathrm{sep}}(f, K) = \sup \left\{ \mathrm{sep}(E, K) \mid E \in \mathfrak{B}_e \right\};$$
$$h_{\mathrm{sep}}(f) = \sup \left\{ h_{\mathrm{sep}}(f, K) \mid K \in \mathcal{K}(G) \right\};$$

where $\mathcal{K}(G)$ is the set of all nonempty compact subsets of G.

THEOREM 4.1 ([16]). *Let $f : G \to G$ be a continuous map on a topological group G. Then, $h_{\mathrm{span}}(f) = h_{\mathrm{sep}}(f) = h_{\mathrm{top}}(f)$.*

The entropy-carrying sets of a continuous map on a compact space is always of particular interest. Adapting the techniques in [23], the following is proved.

THEOREM 4.2. *Let f be a continuous endomorphism on a Hausdorff compact topological group G. Then, $h_{\mathrm{top}}(f|_{\Omega(f)}) = h_{\mathrm{top}}(f)$.*

Proof. Fix any $m \in \mathbb{N}$ and $E \in \mathfrak{B}_e$, and let $F(m, E, \Omega(f))$ be an (m, E, f)-spanning set for $\Omega(f)$ with cardinality $\mathrm{span}(m, E, \Omega(f))$.

Let

$$U = \left\{ x \in G \mid \begin{array}{l} \exists y \in F(m, E, \Omega(f)) \text{ such that } f^i(x) f^i(y^{-1}) \in E \\ \text{for all } 0 \leq i \leq m \end{array} \right\}.$$

Claim 1. U is an open neighborhood of $\Omega(f)$.

Choose $W \in \mathfrak{B}_e$ such that $W^2 \subset E$. Then, there exists $D \in \mathfrak{B}_e$ such that $xy^{-1} \in D$ implies $f^i(x)f^i(y^{-1}) \in W$ for all $0 \leq i \leq m$. For any fixed $x \in U$ and any $y \in Dx$, it is clear that $x^{-1}y \in D$, implying that $f^i(x^{-1})f^i(y) \in W$ for all $0 \leq i \leq m$.

From $x \in U$, it follows that there exists $z \in F(m, E, \Omega(f))$ such that $f^i(x)f^i(z^{-1}) \in W$ for all $0 \leq i \leq m$. Therefore, $f^i(y)f^i(z^{-1}) \in E$. This implies that $Dx \subset U$.

Since $U^c = G \backslash U$ is compact and all points in U^c are wandering, there exists $V \subset E$ such that for any $y \in U^c$ and any $n \in \mathbb{N}$,

$$f^n(Vy) \cap Vy = \emptyset.$$

Now let $F(m, V, U^c)$ be an (m, V, f)-spanning set for U^c with cardinality $\mathrm{span}(m, V, U^c)$ and let $F_m = F(m, E, \Omega(f)) \cup F(m, V, U^c)$. Since F_m is an (m, E, f)-spanning set for G, we obtain that $|F_m| \geq \mathrm{span}(m, E, G)$.

For any $l \in \mathbb{N}$, define $\phi_l : G \to F_m^i$ by $\phi_l(x) = (y_0, y_1, \ldots, y_{l-1})$, where

$$y_j \in \begin{cases} F(m, E, \Omega(f)) \cap E^{-1} f^{jm}(x), & f^{jm}(x) \in U, \\ F(m, V, U^c) \cap V^{-1} f^{jm}(x), & f^{jm}(x) \in U^c. \end{cases}$$

If $\phi_l(x) = (y_0, y_1, \ldots, y_{l-1})$ for some $x \in G$, then a point $y_j \in F(m, V, U^c)$ can not be repeated in this l-tuple. Because Ey_j's are wandering for any choice of $y_j \in F(m, V, U^c)$.

Choose $n > \mathrm{span}(m, V, U^c)$. Let $H(n, E^2, G)$ be an (n, E^2, f)-separated set for G with cardinality $\mathrm{sep}(n, E^2, G)$ and let l be a positive integer with $(l-1)m < n \leq lm$.

Claim 2. The map ϕ_l is one to one on $H(n, E^2, G)$.

Suppose that there exists $x, y \in H(n, E^2, G)$ such that $\phi_l(x) = \phi_l(y) = (y_0, y_1, \ldots, y_{l-1})$. For $0 \leq i < m$ and $0 \leq j < l$, we have

$$f^{i+jm}(x)f^{i+jm}(y^{-1}) = f^{i+jm}(x)f^i(y_j^{-1})f^{i+jm}(y^{-1})f^i(y_j) \in E^2.$$

By the choice of l, it follows that for any $0 \leq i < n$, $f^i(x)f^i(y^{-1}) \in E^2$. This, together with $x, y \in H(n, E^2, G)$, implies that $x = y$.

Claim 3. Let $p = \mathrm{span}(m, E, \Omega(f))$ and $q = \mathrm{span}(m, V, U^c)$. Then,

$$|\phi_l(H(n, E^2, G))| \leq (q+1)! l^q p^l.$$

Let I_k be the set of l-tuples in $\phi_l(H(n, E^2, G))$ such that the numbers of components y_s which belongs to $F(m, V, U^c)$ is k. Since $y_k \in F(m, V, U^c)$ can not be repeated in $\phi_l(x)$, then $k \leq q$.

For I_k, there exist $\binom{q}{k}$ ways of picking these k points $y_k \in F(m, V, U^c)$, there exist $\frac{l!}{(l-k)!}$ ways of arranging these choice among the points in the ordered l-tuples. Meanwhile, there exist at most p^{l-k} ways of picking the remaining y_s from $F(m, E, \Omega)$. Thus,

$$|I_k| \leq \binom{q}{k} \frac{l!}{(l-k)!} p^l.$$

From $\binom{q}{k} \leq q!$ and $\frac{l!}{(l-k)!} \leq l^k$, it follows that

$$|\phi_l(H(n, E^2, G))| \leq \sum_{k=0}^{q} \binom{q}{k} \frac{l!}{(l-k)!} p^l \leq (q+1)! l^q p^l.$$

Now, applying Claims 2 and 3 yields that

$$\operatorname{sep}(n, E^2, G) = |\phi_l(H(n, E^2, G))| \leq (q+1)! l^q p^l,$$

where $p = \operatorname{span}(n, E, \Omega(f))$ and $q = \operatorname{span}(m, V, U^c)$. This implies that

$$\begin{aligned}
h_{\text{sep}}(f) &\leq \limsup_{E \in \mathfrak{B}_e} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{sep}(n, E^2, G) \\
&\leq \limsup_{E \in \mathfrak{B}_e} \limsup_{l \to \infty} \frac{1}{(l-1)m} [\log((q+1)!) + q \log(l) + l \log(p)] \\
&\leq \limsup_{E \in \mathfrak{B}_e} \frac{1}{m} \limsup_{n \to \infty} \log p \\
&= \limsup_{E \in \mathfrak{B}_e} \frac{1}{m} \log \operatorname{span}(n, E, \Omega).
\end{aligned}$$

Therefore, $h_{\text{top}}(f) = h_{\text{sep}}(f) \leq h_{\text{top}}(f|_{\Omega(f)})$. □

COROLLARY 4.3. *Let $f : G \to G$ be a continuous endomorphism. Then, $h_{\text{top}}(f|_{\text{CR}(f)}) = h_{\text{top}}(f)$.*

The Addition Theorem states that the entropy is additive in appropriate sense with respect to invariant subgroups [15]. Bruno and Virili [10] proved the addition theorem in the case of locally compact totally disconnected topological groups.

THEOREM 4.4 ([10]). *Let G be a locally compact totally disconnected group, and $f : G \to G$ be a continuous endomorphism, and H be a compact f-invariant subgroup of G. Then,*

$$h_{\text{top}}(f) \geq h_{\text{top}}(\tilde{f}) + h_{\text{top}}(f|_H).$$

COROLLARY 4.5. *Let G be a locally compact totally disconnected group, and $f : G \to G$ a continuous endomorphism such that $h_{\text{top}}(f) < \infty$. Let $\tilde{f} : G/\text{CR}(f) \to G/\text{CR}(f)$ be the map induced by f. Then, $h_{\text{top}}(\tilde{f}) = 0$.*

REFERENCES

[1] S. A. Ahmadi, *On the topology of the chain recurrent set of a dynamical system*, Appl. Gen. Topol., **15** (2014), 167–174.

[2] S. A. Ahmadi, X. Wu and G. Chen, *Topological chain and shadowing properties of dynamical systems on uniform spaces*, Topology Appl., **275** (2020), 1–11.

[3] S. A. Ahmadi, X. Wu, Z. Feng, X. Ma and T. Lu, *On the entropy points and shadowing in uniform spaces*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **28** (2018), 1–10.

[4] J. M. Alongi and G. S. Nelson, *Recurrence and topology*, Graduate Studies in Mathematics, Vol. 85, American Mathematical Society, Providence, RI, 2007.

[5] A. Arhangel'skii and M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, Vol. 1, Atlantis Press, Amsterdam, 2008.

[6] J. Auslander, G. Greschonig and A. Nagar, *Reflections on equicontinuity*, Proc. Amer. Math. Soc., **142** (2014), 3129–3137.

[7] R. W. Bagley and M. R Peyrovian, *A note on compact subgroups of topological groups*, Bull. Aust. Math. Soc., **33** (1986), 273–278.

[8] C. J. B. Barros and J. A. Souza, *Attractors and chain recurrence for semigroup actions*, J. Dyn. Differ. Equations, **22** (2010), 723–740.

[9] C. Bonatti, S. Crovisier, N. Gourmelon and R. Potrie, *Tame dynamics and robust transitivity chain-recurrence classes versus homoclinic classes*, Trans. Amer. Math. Soc., **366** (2014), 4849–4871.

[10] A. G. Bruno and S. Virili, *Topological entropy in totally disconnected locally compact groups*, Ergodic Theory Dynam. Systems, **37** (2017), 2163–2186.

[11] B. Carvalho and D. Kwietniak, *On homeomorphisms with the two-sided limit shadowing property*, J. Math. Anal. Appl., **420** (2014), 801–813.

[12] T. Ceccherini-Silberstein and M. Coornaert, *Sensitivity and Devaney's chaos in uniform spaces*, J. Dyn. Control Syst., **19** (2013), 349–357.

[13] T. Das, K. Lee, Richeson and J. Wiseman, *Spectral decomposition for topologically anosov homeomorphisms on noncompact and non-metrizable spaces*, Topology Appl., **160** (2013), 149–158.

[14] T. T. Devi and K. B. Mangang, *Positive expansivity, chain transitivity, rigidity, and specification on general topological spaces*, Bull. Korean Math. Soc., **59** (2022), 319–343.

[15] D. Dikranjan and A. G. Bruno, *Entropy on abelian groups*, Adv. Math., **298** (2016), 612–653.

[16] D. Dikranjan, M. Sanchis and S. Virili, *New and old facts about entropy in uniform spaces and topological groups*, Topology Appl., **159** (2012), 1916–1942.

[17] B. M. Garay, *Uniform persistence and chain recurrence*, J. Math. Anal. Appl., **139** (1989), 372–381.

[18] C. Good and S. Macías, *What is topological about topological dynamics?*, Discrete Contin. Dyn. Syst., **38** (2018), 1007–1031.

[19] B. M. Hood, *Topological entropy and uniform spaces*, J. Lond. Math. Soc. (2), **8** (1974), 633–641.

[20] C. A. Morales and V. Sirvent, *Expansivity for measures on uniform spaces*, Trans. Amer. Math. Soc., **368** (2016), 5399–5414.

[21] P. Oprocha, *Chain recurrence in multidimensional time discrete dynamical systems*, Discrete Contin. Dyn. Syst., **20** (2008), 1039–1056.

[22] F. Pirfalak, S. A. Ahmadi, X. Wu and N. Kouhestani, *Topological Average Shadowing Property on Uniform Spaces*, Qual. Theory Dyn. Syst., **20** (2021), 1–15.

[23] C. Robinson, *Dynamical systems. Stability, symbolic dynamics, and chaos*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[24] N. Shekutkovski, *One property of components of a chain recurrent set*, Regul. Chaotic Dyn., **20** (2015), 184–188.

[25] S. D. Silvestrov and J. Tomiyama, *Topological dynamical systems of type I*, Expo. Math., **20** (2002), 117–142.
[26] X. Wu, Y. Luo, X. Ma and T. Lu, *Rigidity and sensitivity on uniform spaces*, Topology Appl., **252** (2019), 145–157.
[27] X. Wu, X. Ma, Z. Zhu and T. Lu, *Topological ergodic shadowing and chaos on uniform spaces*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **28** (2018), 1–9.
[28] X. Wu, S. Liang, X. Ma, T. Lu and S. A. Ahmadi, *The mean sensitivity and mean equicontinuity in uniform spaces*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **30** (2020), 1–11.

Received September 17, 2022
Accepted June 17, 2023

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