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GENERALIZED VISCOSITY METHOD FOR APPROXIMATING SOLUTIONS OF COUNTABLE FAMILIES OF CERTAIN NONLINEAR MAPPINGS IN REAL HILBERT SPACE

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Abstract. The purpose of this paper is to introduce a three step iterative algorithm which include a general viscosity explicit method for approximating a common solution of fixed point problem of an infinite family of k_i-demimetric mapping and a directed operator in the framework of real Hilbert space. Furthermore, we prove a strong convergence theorem for approximating a common solution of the aforementioned problems. We also show that our iterative algorithm holds for an infinite family of L-Lipschitzian and quasi-pseudocontractive mapping together with a directed operator. The iterative algorithm presented in this article is design in such a way that it solves some variational inequality problem and no compactness condition is impose on our scheme and mapping. Finally, we give applications of our main result to variational inclusion and equilibrium problems. Our result complements and extends some related result in literature.

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Key words. Generalized viscosity; quasi-pseudocontractive mappings; demimetric mappings; directed operators; iterative scheme; Fixed point problem.

1. INTRODUCTION

Let C be a nonempty, closed and convex subset of a real Hilbert space H. A point $p \in C$ is called a fixed point of T if $Tp = p$. We denote by $F(T)$ the set of all fixed points of T. A nonlinear mapping $T : C \to C$ is said to be:

(i) Nonexpansive if $||Tx - Ty|| \leq ||x - y||, \forall\ x, y \in C$;

(ii) Quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - x^*|| \leq ||x - x^*||, \forall\ x \in C \text{ and } x^* \in F(T);$$

(iii) Firmly nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2, \forall\ x, y \in C;$$

(iv) Firmly quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 - ||(I - T)x||^2, \forall\ x \in C \text{ and } x^* \in F(T);$$

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(v) Strictly pseudo-contractive if there exists $k \in [0,1)$ such that

$$||Tx - Ty||^2 \leq ||x-y||^2 + k||(I-T)x - (I-T)y||^2, \forall\, x, y \in C;$$

(vi) Directed if $F(T) \neq \emptyset$ and $\langle Tx - x^*, Tx - x\rangle \leq 0, \forall\, x \in C$ and $x^* \in F(T)$;
(vii) Demicontractive if $F(T) \neq \emptyset$ and there exist $k \in [0,1)$ such that

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + k||Tx - x||^2, \forall\, x \in C \text{ and } x^* \in F(T).$$

Remark 1.1. Bauschke and Combettes [7] gave the definition of directed mapping as follows. A map $T : C \to C$ is directed if

$$||Tx - x^*||^2 \leq ||x - x^*||^2 - ||Tx - x||^2, \forall\, x \in C \text{ and } x^* \in F(T).$$

This implies that the class of directed mapping coincides with that of firmly quasi-nonexpansive mapping (see [21]).
Let $T : H \to H$ be a mapping, then the following statements are equivalent:
(i) T is directed;
(ii) there holds the relation:

$$||Tx - p||^2 \leq ||x - p||^2 - ||x - Tx||^2,\ \forall\, p \in F(T),\ x \in H.$$

Definition 1.2. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $k \in (-\infty, 1)$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called k-demimetric if for any $x \in C$ and $p \in F(T)$

$$\langle x - p, x - Tx\rangle \geq \frac{1-k}{2}||x - Tx||^2. \tag{1}$$

Definition 1.3. A mapping $T : H \to H$ is demiclosed at a point $z \in H$ if the weak convergence of any sequence $\{x_k\}$ to some point x^* and the strong convergence $\{T(x_k)\}$ to z implies that $T(x^*) = z$.

Definition 1.4. An operator $T : C \to C$ is said to be quasi-pseudocontractive if $F(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||Tx - x||^2\ \forall\, x \in C \text{ and } x^* \in F(T). \tag{2}$$

Definition 1.5. A mapping $T : C \to C$ is said to be L-Lipschitzian if there exist some $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \forall\, x, y \in C. \tag{3}$$

It is very clear that the class of quasi-pseudocontractive mappings include the class of demicontractive mappings which contains the class of nonexpansive, quasi-nonexpansive and pseudo-contractive mappings.
Directed operators are important because they include many types of nonlinear operators such as nonexpansive mapping and quasi-nonexpansive mappings. The subgradient projection T of a continuous convex function $f : H \to \mathbb{R}$ is a directed operator.

Recently Chang et. al. [11] considered the split equality fixed point problem for quasi-pseudo-contractive mappings and employed the following iterative scheme to prove a strong convergence theorem imposing the compactness condition on this class of mapping.

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n); \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)((1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T))u_n; \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n); \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)((1-\xi_n)I + \xi_n S((1-\eta_n)I + \eta_n S))v_n; \end{cases}$$

where H_1, H_2 and H_3 are three Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two linear bounded operators with adjoint A^* and B^* respectively. $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are two L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$, $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Using their iterative scheme, they [11] proved a strong convergence result imposing a compactness condition, (see [11, Theorem 3.2] for details).

Apart from the work of Chang et. al. [11], many authors have introduced different iterative algorithms being Halpern, Viscosity, Mann, Kranoselski, Parallel, Cylic, Hybrid to mention a few to approximate solutions of fixed point problems both in Hilbert spaces and Banach spaces, (see [1–6, 9, 14, 15, 20] and the references contained in).

In 2016, Takahashi [20] introduced a Halpern type algorithm for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problem for a finite family of inverse strongly monotone mappings in a real Hilbert space. He proved the following theorem:

Theorem 1.6 ([20]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\{k_1, ...k_M\} \subset (-\infty, 1)$ and $\{\mu_1, ...\mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Assume that $\bigcap_{j=1}^{M} F(T_j) \cap (\bigcap_{i=1}^{N} VI(C, B_i)) \neq \emptyset$. Let $\{\mu_n\}$ be a sequence in C such that $u_n \to u$. For $x_1 = x \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n; \\ w_n = \sum_{i=1}^{N} \sigma_i P_C(I - \eta_n B_i)x_n; \\ x_{n+1} = \delta_n u_n + (1-\delta_n)(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)), \ \forall\, n \in \mathbb{N}; \end{cases}$$

where $a, b, c \in \mathbb{R}, \{\lambda_n\}, \{\eta_n\} \subset (0, \infty), \{\xi_1, ..., \xi_M\}, \{\sigma_1, ..., \sigma_N\} \subset (0, 1)$ and $\{\alpha_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ satisfying the following conditions:

(i) $0 < a \leq \lambda_n \leq \min\{1 - k_1, ..., 1 - k_M\}, 0 < b \leq \eta_n \leq 2\min\{\mu_1, ..., \mu_n\}$;

(ii) $\sum_{j=1}^{\infty} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(iii) $0 < c < \alpha_n,\ \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;
(iv) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in \bigcap_{j=1}^{\infty} F(T_j) \cap (\bigcap_{i=1}^{N} VI(C, B_i))$ *where* $z_0 = P_{\bigcap_{j=1}^{\infty} F(T_j)\cap(\bigcap_{i=1}^{N} VI(C,B_i))} u$ *and* $VI(C, B)$ *is the solution set of variational inequality problem.*

The viscosity approximation method introduced by Moudafi [16] in 2000 is design in such a way that it solves some variational inequality problem. Since the inception of viscosity iterative method, different authors have employed it to approximate solutions of fixed point problem and other related optimization problems (see [1–4, 13, 22, 23] and the references contained in).

In 2005, Xu et. al. [23] combined the viscosity iterative scheme together with the implicit midpoint method to approximate a solution of a fixed point problem of a nonexpansive mapping in the framework of real Hilbert space. They proved a strong convergence theorem using the following iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}),\ \forall\ n \geq 1, \tag{4}$$

where f is a contraction and $\{\alpha_n\} \subset (0, 1)$. They also proved that iterative algorithm solves some variational inequality problem:

$$\langle (f - 1)x^*, z - x^* \rangle \leq 0,\ \forall\ z \in F(T). \tag{5}$$

Moreso, Alghamdi et. al. [5] introduced the implicit midpoint rule for nonexpansive mappings and obtained a weak convergence result using an implicit algorithm to approximate the solution of the implicit midpoint rule for nonexpansive mappings in the framework of Hilbert space.

Recently, Ke and Ma [14] modified the viscosity implicit midpoint rule by replacing the midpoint by any point of the interval $[x_n, x_{n+1}]$. They constructed the following generalized viscosity implicit rule for a nonexpansive mappings:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\hat{x}_{n+1}) \tag{6}$$

where f is a contraction and showed that $\{x_n\}$ defined in (6) converges strongly to a point $x^* \in F(T)$ which also solves the variational inequality (5).

We noticed that the computation of implicit midpoint methods is not an easy work in practice and its computation also require more assumptions. Based on these, Marino et. al. [15] introduced the following general viscosity explicit rule for quasi-nonexpansive mappings T in the framework of real

Hilbert space:

$$(7)\quad \begin{cases} \hat{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n; \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\hat{x}_{n+1}), \ \forall\ n \geq 1; \end{cases}$$

where f is a contraction, $\{\alpha_n\}, \{\beta_n\}$ and $\{s_n\}$ are sequences in $(0,1)$. They proved a strong convergence result which also solves variational inequality problem (5) to a solution of $F(T)$, where T is a quasi-nonexpansive mapping. Motivated by the works of Chang et. al. [11], Takahashi, [20] Xu et. al. [23] and other authors working in this direction, we introduce a three steps iterative algorithm which contains a general viscosity explicit method to approximate a common solution of an infinite family of k_i-demimetric mapping and a directed operators in the framework of real Hilbert space. We prove a strong convergence theorem to the solutions of the aforementioned problems. Our iterative algorithm also solves some variational inequality problem. Lastly, we give applications of our main result to variational inclusion and equilibrium problems. The result presented in this paper extends and complements the works of Takahashi [20], Marino [15], Alghamdi [5], Ke and Ma [14] and other related results in this direction.

2. PRELIMINARIES

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively.

LEMMA 2.1. *Let H be a real Hilbert space. Then for each $x, y \in H$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle.$$

LEMMA 2.2 ([12]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any sequenced $\{x_i\}_{i=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \rightarrow \mathbb{R},\ g(0) = 0,$$

such that for any positive integers i, j with i ¡ j, the following inequality holds:

$$||\sum_{i=1}^{\infty} \lambda_i x_i||^2 = \sum_{i=1}^{\infty} \lambda_i ||x||^2 - \lambda_i\lambda_j g(||x_i - x_j||).$$

LEMMA 2.3 ([18]). *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let k be a real number with $k \in (-\infty, 1)$ and let U be a k-demimetric mapping of C into H. Then $F(U)$ is closed and convex.*

LEMMA 2.4 ([18]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $k \in (-\infty, 1)$ and T be a k-demimetric mapping of C into*

H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1-k$ and define $S = (1-\lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.

LEMMA 2.5 ([11]). *Let H be a real Hilbert space and $T : H \to H$ be a L-Lipschitzian mapping with $L \geq 1$. Denote $K := (1-\xi)I + \xi T\big((1-\eta)I + \eta T\big)$ if $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, then the following conclusions holds.*

(1) *$F(T) = F(T((1-\eta)I + \eta T)) = F(K)$;*

(2) *If T is demiclosed at 0, then K is also demiclosed at 0;*

(3) *In addition, if $T : H \to H$ is quasi-pseudocontractive, then the mapping K is quasi-nonexpansive, that is,*

$$||Kx - u^*|| \leq ||x - u^*|| \;\forall\; x \in H \, and \; u^* \in F(T) = F(K).$$

LEMMA 2.6 ([24]). *Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n \delta_n, \;\; n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$,

(ii) $\limsup\limits_{n\to\infty} \delta_n \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\sigma_n \delta_n| < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

3. MAIN RESULT

LEMMA 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\{k_1, ..., k_i\} \subset (-\infty, 1)$ for $i = 1, 2, ...$ and $0 < a \leq \lambda_n \leq min\{1-k_1, ..., 1-k_i\}$. Let S_i for $i = 1, 2, ...$ be an infinite family of k_i-demimetric mappings which is demiclosed at the origin and $U : H \to H$ be a directed operator. Suppose $g \in \Pi_C$ with $\rho \in (0,1)$ and $\Gamma := \bigcap\limits_{i=1}^{\infty} F(T_i) \bigcap F(U) \neq \emptyset$. For any $x_1 \in C$, let $\{x_n\}$ be a sequence generated iteratively by*

$$\begin{cases} \hat{x}_{n+1} = \beta_{n,0}x_n + \sum\limits_{i=1}^{\infty} \beta_{n,i} T_{\alpha_i} x_n; \\ z_n = t_n x_n + (1-t_n)\hat{x}_{n+1}; \\ x_{n+1} = \gamma_n g(x_n) + (1-\gamma_n) U z_n, \;\; \forall\; n \geq 1; \end{cases} \tag{8}$$

where $\{\beta_{n,0}\}$, $\{\beta_{n,i}\}$, $\{t_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ with $\sum\limits_{i=1}^{\infty} \beta_{n,i} = 1$ and $T_{\alpha_i} = ((1-\lambda_n)I + \lambda_n S_i)$. Then, $\{x_n\}$ is bounded.

Proof. Let $p \in \Gamma$, then we have from (8) and Lemma 2.4 that

$$
\begin{aligned}
||\hat{x}_{n+1} - p|| &= ||\beta_{n,0} x_n + \sum_{i=1}^{\infty} \beta_{n,i} T_{\alpha_i} x_n - p|| \\
&\le \beta_{n,0} ||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i} ||T_{\alpha_i} x_n - p|| \\
&\le \beta_{n,0} ||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i} ||x_n - p|| \\
&\le ||x_n - p||.
\end{aligned} \tag{9}
$$

Using (8) and (9), we also have that

$$
\begin{aligned}
||z_n - p|| &\le t_n ||x_n - p|| + (1 - t_n) ||\hat{x}_{n+1} - p|| \\
&\le t_n ||x_n - p|| + (1 - t_n) ||x_n - p|| \\
&= ||x_n - p||.
\end{aligned} \tag{10}
$$

It follows from (8) and (9) that

$$
\begin{aligned}
&||x_{n+1} - p||^2 \le \gamma_n ||g(x_n) - p||^2 + (1 - \gamma_n) ||U z_n - p||^2 \\
&\le \gamma_n (||g(x_n) - g(p)|| + ||g(p) - p||)^2 + (1 - \gamma_n) ||z_n - p||^2 \\
&- (1 - \gamma_n) ||z_n - U z_n||^2 \\
&\le 2\gamma_n (||g(x_n) - g(p)||^2 + ||g(p) - p||)^2 + (1 - \gamma_n) ||z_n - p||^2 \\
&\le 2\gamma_n \rho^2 ||x_n - p||^2 + 2\gamma_n ||g(p) - p||^2 + (1 - \gamma_n) ||x_n - p||^2 \\
&\le (1 - \gamma_n (1 - 2\rho^2)) ||x_n - p||^2 + \gamma_n (1 - 2\rho^2) \frac{2}{1 - 2\rho^2} (||g(p) - p||^2).
\end{aligned}
$$

It follows from induction that

$$
||x_n - p||^2 \le \max\{||x_1 - p||^2,\ \frac{2}{1 - 2\rho^2} (||g(p) - p||^2)\},\ n \ge 1,
$$

which implies that $\{x_n\}$ is bounded. Consequently, we haves that $\{z_n\}$, $\{T_{\alpha_i} x_n\}$ and $\{U z_n\}$ are all bounded. Hence, we complete the proof. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\{k_1, ..., k_i\} \subset (-\infty, 1)$ for $i = 1, 2, ...$ and $0 < a \le \lambda_n \le \min\{1 - k_1, ..., 1 - k_i\}$. Let S_i for $i = 1, 2, ...$ be an infinite family of k_i-demimetric mappings which is demiclosed at the origin and $U : H \to H$ be a directed operator. Suppose $g \in \Pi_C$ with $\rho \in (0, 1)$ and assume that $\{\beta_{n,0}\}$, $\{\beta_{n,i}\}, \{t_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ with $\sum_{i=1}^{\infty} \beta_{n,i} = 1$ and $T_{\alpha_i} = ((1 - \lambda_n) I + \lambda_n S_i)$. Then the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_{n,0} \beta_{n,i} (1 - t_n)$.

Then, $\{x_n\}$ generated iteratively by (8) converges strongly to a point $x^* \in \Gamma$ which solves some variational inequality problem

$$\langle g(x^*) - x^*, p - x^* \rangle \leq 0, \ \forall \ p \in \Gamma. \tag{11}$$

Proof. Let $p \in \Gamma$, then we have from Lemma 2.2 and Lemma 2.4 that

$$\begin{aligned} ||\hat{x}_{n+1} - p||^2 &\leq \beta_{n,0}||x_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}||T_{\alpha_i} x_n - p||^2 \\ &- \beta_{n,0}\beta_{n,i} g(||x_n - T_{\alpha_i} x_n||) \\ &\leq \beta_{n,0}||x_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}||x_n - p||^2 \\ &- \beta_{n,0}\beta_{n,i} g(||x_n - T_{\alpha_i} x_n||) \\ &= ||x_n - p||^2 - \beta_{n,0}\beta_{n,i} g(||x_n - T_{\alpha_i} x_n||). \end{aligned} \tag{12}$$

From (8) and (12), we have that

$$\begin{aligned} ||z_n - p||^2 &\leq t_n||x_n - p||^2 + (1 - t_n)||\hat{x}_{n+1} - p||^2 \\ &\leq t_n||x_n - p||^2 + (1 - t_n)\big[||x_n - p||^2 \\ &- \beta_{n,0}\beta_{n,i} g(||x_n - T_{\alpha_i} x_n||)\big] \\ &\leq ||x_n - p||^2 - \beta_{n,0}\beta_{n,i}(1 - t_n) g(||x_n - T_{\alpha_i} x_n||). \end{aligned} \tag{13}$$

Using (8) and (13), we have that

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq \gamma_n||g(x_n) - p||^2 + (1 - \gamma_n)||Uz_n - p||^2 \\ &\leq \gamma_n||g(x_n) - p||^2 + (1 - \gamma_n)||z_n - p||^2 \\ &- (1 - \gamma_n)||Uz_n - z_n||^2 \\ &\leq \gamma_n||g(x_n) - p||^2 + (1 - \gamma_n)\big[||x_n - p||^2 \\ &- \beta_{n,0}\beta_{n,i}(1 - t_n) g(||x_n - T_{\alpha_i} x_n||)\big] \\ &- (1 - \gamma_n)||Uz_n - z_n||^2 \\ &= \gamma_n||g(x_n) - p||^2 \\ &+ (1 - \gamma_n)||x_n - p||^2 \\ &- (1 - \gamma_n)\beta_{n,0}\beta_{n,i}(1 - t_n) g(||x_n - T_{\alpha_i} x_n||) \\ &- (1 - \gamma_n)||Uz_n - z_n||^2 \end{aligned} \tag{14}$$

We now divide our proof into two cases.

Case 1. Assume that $\{||x_n - p||\}$ is a monotonically decreasing sequence. Then $\{||x_n - p||\}$ is convergent and clearly,

$$\lim_{n \to \infty} ||x_n - p|| = \lim_{n \to \infty} ||x_{n+1} - p||.$$

From (14) and conditions (i) and (ii) of (3.2), we have that

$$(1-\gamma_n)\beta_{n,0}\beta_{n,i}(1-t_n)g(||x_n - T_{\alpha_i}x_n||)$$
$$\leq \gamma_n||g(x_n)-p||^2 + (1-\gamma_n)||x_n - p||^2 - ||x_{n+1}-p||^2.$$

Hence, we have that

(15) $$\lim_{n\to\infty} g(||x_n - T_{\alpha_i}x_n||) = 0.$$

From the property of g in Lemma 2.2, we obtain that

(16) $$\lim_{n\to\infty} ||x_n - T_{\alpha_i}x_n|| = 0.$$

Similarly, using (14), we have that

$$(1-\gamma_n)||Uz_n - z_n||^2 \leq \gamma_n||g(x_n)-p||^2 + (1-\gamma_n)||x_n-p||^2$$
$$-||x_{n+1}-p||^2 - (1-\gamma_n)\beta_{n,0}\beta_{n,i}(1-t_n)g(||x_n - T_{\alpha_i}x_n||)$$

On using conditions (i) and (ii) of (8) and (15), we haves that

(17) $$\lim_{n\to\infty} ||Uz_n - z_n|| = 0.$$

We obtain from (8) and (16) that

(18) $$||\hat{x}_{n+1} - x_n|| \leq \sum_{i=1}^{\infty} \beta_{n,i}||T_{\alpha_i}x_n - x_n|| \to 0, \text{ as } n \to \infty.$$

From (8) and (18), we obtain that

(19) $$||z_n - x_n|| \leq (1-t_n)||\hat{x}_{n+1} - x_n|| \to 0, \text{ as } n \to \infty.$$

Now, using (17) and (19), we obtain that

(20) $$\lim_{n\to\infty} ||Uz_n - x_n|| = 0.$$

We obtain from (8), condition (i) of (8) and (20) that

(21) $$||x_{n+1} - x_n|| \leq \gamma_n||g(x_n) - x_n|| + (1-\gamma_n)||Uz_n - x_n||.$$

Hence, we have that

(22) $$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0.$$

Since $\{x_n\}$ and $\{z_n\}$ are bounded, there exist subsequences $\{x_{n_k}\}$ and $\{z_{n_k}\}$ which converges weakly to x^*. From (16) and the demiclosedness principle, we have that $x^* \in \bigcap_{i=1}^{\infty} F(T_{\alpha_i})$. Similarly, using (17) and the demiclosedness principle, we have that $x^* \in F(U)$. Hence, we have that $x^* \in \Gamma$.
Next, we show that $\{x_n\}$ converges strongly to x^*.
Since U is a directed operator and from (8), we have that

$$||x_{n+1} - x^*||^2 \leq \gamma_n^2||g(x_n) - x^*||^2 + (1-\gamma_n)^2||Uz_n - x^*||^2$$
$$+ 2\gamma_n(1-\gamma_n)\langle g(x) - x^*, Uz_n - x^*\rangle$$
$$= \gamma_n^2||g(x_n) - x^*||^2 + 2\gamma_n(1-\gamma_n)\langle g(x_n) - g(x^*), Uz_n - x^*\rangle$$

$$
\begin{aligned}
&+ 2\gamma_n(1-\gamma_n)\langle g(x^*) - x^*, Uz_n - x^*\rangle + (1-\gamma_n)^2||Uz_n - x^*||^2\\
&\le \gamma_n^2||g(x_n) - x^*||^2 + \gamma_n(1-\gamma_n)\big[||g(x_n) - g(x^*)||^2 + ||Uz_n - x^*||^2\big]\\
&+ 2\gamma_n(1-\gamma_n)\langle g(x^*) - x^*, Uz_n - x^*\rangle + (1-\gamma_n)||Uz_n - x^*||^2\\
&\le \gamma_n^2||g(x_n) - x^*||^2 + \gamma_n(1-\gamma_n)\rho^2||x_n - x^*||^2 + \gamma_n(1-\gamma_n)||Uz_n - x^*||^2\\
&+ 2\gamma_n(1-\gamma_n)\langle g(x^*) - x^*, Uz_n - x^*\rangle + (1-\gamma_n)^2||Uz_n - x^8||^2\\
&\le (1-\gamma_n)||z_n - x^*||^2 + \gamma_n(1-\gamma_n)\rho^2||x_n - x^*||^2\\
&+ \gamma_n\big[\gamma_n||g(x_n) - x^*||^2 + 2(1-\gamma_n)\langle g(x^*) - x^*, Uz_n - x^*\rangle\big]\\
&\le (1-\gamma_n)||x_n - x^*||^2 + 2(1-\gamma_n)\rho^2||x_n - x^*||^2\\
&+ \gamma_n\big[\gamma_n||g(x_n) - x^*||^2 + 2(1-\gamma_n)\langle g(x^*) - x^*, Uz_n - x^*\rangle\big].
\end{aligned}
$$

This also implies that

$$
||x_{n+1} - x^*||^2 \le (1-\varphi_n)\mu_n + \varphi_n\delta_n, \ \forall\ n \ge 0; \tag{23}
$$

where $\varphi_n = \gamma_n(1-(1-\gamma_n)\rho^2)$ and

$$
\delta_n = \frac{2(1-\gamma_n[\langle g(x^*) - x^*, Uz_n - x^*\rangle])}{1-(1-\gamma_n)\rho^2} + \frac{\gamma_n[||g(x_n) - x^*]}{1-(1-\gamma_n)\rho^2}.
$$

We now verify that

$$
\limsup_{n\to\infty}(\langle g(x^*) - x^*, Uz_n - x^*\rangle) \le 0.
$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_k}\}$ such that

$$
\limsup_{n\to\infty}(\langle g(x^*) - x^*, Uz_n - x^*\rangle) = \limsup_{n_k\to\infty}(\langle g(x^*) - x^*, x_{n_k} - x^*\rangle).
$$

Using (20), we have that

$$
\begin{aligned}
\limsup_{n_k\to\infty}(\langle g(x^*) - x^*, Uz_n - x^*\rangle) &= \limsup_{n_k\to\infty}(\langle g(x^*) - x^*, x_{n_k} - x^*\rangle)\\
&= -\liminf_{n_k\to\infty}(\langle (I-g)x^*, x_{n_k} - x^*\rangle).
\end{aligned} \tag{24}
$$

Since $\{x_{n_k}\}$ converges weakly to an element $p \in \Gamma$, we have that

$$
-\lim_{n_k\to\infty}(\langle (I-g)x^*, x_{n_k} - x^*\rangle) = -(\langle (I-g)x^*, p - x^*\rangle). \tag{25}
$$

Since $\omega_\omega(x_{n_k}, y_{n_k}) \subset \Gamma$ and (x^*, y^*) is the solution of the variational inequality problem (11). Hence, from (24) and (25), we obtain that

$$
\limsup_{n\to\infty}(\langle g(x^*) - x^*, Uz_n - x^*\rangle) \le 0. \tag{26}
$$

Therefore, Using Lemma (2.6) in (23) and condition (i) of (8), we have that

$$
\lim_{n\to\infty}(||x_n - x^*||^2) = 0,
$$

which implies that $x_n \to x^*$.

Case 2. Assume that $\{||x_n - p||\}$ is not a monotonically decreasing sequence. Then, we define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N}, k \leq n : ||x_k - p|| < ||x_{k+1} - p||\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$. From (14), we have that

$$\begin{aligned}&(1-\gamma_{\tau(n)})\beta_{\tau(n),0}\beta_{\tau(n),i}(1-t_{\tau(n)})g(||x_{\tau(n)} - T_{\alpha_i}x_{\tau(n)}||)\\ &\leq \gamma_{\tau(n)}||g(x_{\tau(n)}) - p||^2 + (1-\gamma_{\tau(n)})||x_{\tau(n)} - p||^2 - ||x_{\tau(n)+1} - p||^2.\end{aligned}$$

Hence, we have from condition (i) and (ii) of (8) that

$$\lim_{\tau(n)\to\infty} g(||x_{\tau(n)} - T_{\alpha_i}x_{\tau(n)}||) = 0. \tag{27}$$

From the property of g in Lemma 2.2, we have that

$$\lim_{\tau(n)\to\infty} ||x_{\tau(n)} - T_{\alpha_i}x_{\tau(n)}|| = 0. \tag{28}$$

Similarly from (14), we have that

$$\begin{aligned}(1-\gamma_{\tau(n)})||Uz_{\tau(n)} - z_{\tau(n)}||^2 &\leq \gamma_{\tau(n)}||g(x_{\tau(n)}) - p||^2 + (1-\gamma_{\tau(n)})||x_{\tau(n)} - p||^2\\ &-||x_{\tau(n)+1} - p||^2 - (1-\gamma_{\tau(n)})\beta_{\tau(n),0}\beta_{\tau(n),i}(1-t_{\tau(n)})g(||x_{\tau(n)} - T_{\alpha_i x_{\tau(n)}}||).\end{aligned}$$

Applying conditions (i) and (ii) of (8) and using (27), we have that

$$\lim_{\tau(n)\to\infty} |||Uz_{\tau(n)} - z_{\tau(n)}|| = 0. \tag{29}$$

Following the same approach as in Case 1, we obtain from (26) that

$$\limsup_{\tau(n)\to\infty}(\langle g(x^*) - x^*, Uz_{\tau(n)} - x^*\rangle) \leq 0. \tag{30}$$

Together we have from (23) that

$$\begin{aligned}||x_{\tau(n)+1} - x^*||^2 &\leq (1-\gamma_{\tau(n)}(1-(1-\gamma_{\tau(n)})\rho^2))||x_{\tau(n)} - x^*||^2\\ &+ \gamma_{\tau(n)}(1-(1-\gamma_{\tau(n)})\rho^2)\Big[2\gamma_{\tau(n)}(1-\gamma_{\tau(n)})\\ &\big[\langle g(x^*) - x^*, Uz_{\tau(n)} - x^*\rangle\big]\\ &+ \gamma_{\tau(n)}^2[||g(x_{\tau(n)}) - x^*||^2]\Big],\end{aligned}$$

which implies that

$$\begin{aligned}||x_{\tau(n)} - x^*||^2 &\leq ||x_{\tau(n)} - x^*||^2 - ||x_{\tau(n)+1} - x^*||^2\\ &+ \frac{2(1-\gamma_{\tau(n)})\big[\langle g(x^*) - x^*, Uz_{\tau(n)} - x^*\rangle}{1-(1-\gamma_{\tau(n)})\rho^2}\big]\\ &+ \frac{\gamma_{\tau(n)}[||g(x_{\tau(n)}) - x^*||^2]}{1-(1-\gamma_{\tau(n)})\rho^2},\end{aligned}$$

using condition (i) of (8) and (30), we have that

$$\lim_{\tau(n)\to\infty} ||x_{\tau(n)} - x^*|| = 0.$$

This implies that $x_{\tau(n)} \to x^*$ as $\tau(n) \to \infty$. This completes the proof. □

THEOREM 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : H \to H$, for $i = 1, 2, ...,$ be an infinite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$, $\bigcap_{i=1}^{\infty} F(T_i)$ and T is demiclosed at the origin. Let $U : H \to H$ be a directed operator. Suppose $g \in \Pi_C$ with $\rho \in (0, 1)$ and assume that $\{\beta_{n,0}\}$, $\{\beta_{n,i}\}$, $\{t_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ with $\sum_{i=1}^{\infty} \beta_{n,i} = 1$. Suppose that $\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \cap F(U) \neq \emptyset$; then for any $x_1 \in C$, let $\{x_n\}$ be a sequence generated iteratively by*

$$\begin{cases} \hat{x}_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^{\infty} \beta_{n,i}K_i x_n; \\ z_n = t_n x_n + (1 - t_n)\hat{x}_{n+1}; \\ x_{n+1} = \gamma_n g(x_n) + (1 - \gamma_n)U z_n, \ \forall \ n \geq 1; \end{cases} \tag{31}$$

where K_i is defines as stated in Lemma 2.5 and the sequences $\{\beta_{n,0}\}$, $\{\beta_{n,i}\}$, $\{t_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i}(1 - t_n)$;

(iii) $0 < a < \xi_n < \eta_n < b < \frac{1}{1+\sqrt{1+L^2}}$, $\forall \ n \geq 1$.

Then, $\{x_n\}$ generated iteratively by (31) converges strongly to a point $x^* \in \Gamma$ which solves some variational inequality problem

$$\langle g(x^*) - x^*, p - x^*\rangle \leq 0, \ \forall \ p \in \Gamma. \tag{32}$$

Proof. The proof follows from the proof of Lemma 2.5, Lemma 3.1 and Theorem 3.2. □

REMARK 3.4. We observed that authors working on both the implicit and explicit viscosity iterative method considered a nonlinear mapping, mostly a nonexpansive mapping. Marino [15] extended this to a quasi-nonexpansive mapping. Based on these, we consider two different mappings in which one is an infinite family of k-demimetric mappings and the other a directed operator. We also show that our result holds if the demimetric mapping is alternate to an L-Lipschitzian and quasi-pseudocontractive mapping. Furthermore, Chang et. al. [11] proved a strong convergence result by imposing a compactness condition on their mapping. During the course of proving a strong convergence result in this article, we were able to dispense the compactness condition which makes our work extend the works of Chang et. al. [11] and other related works in literature. Another observation is that our iterative algorithm does not require prior knowledge of operator norm as this gives difficulties in real life computation.

COROLLARY 3.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $U : H \to H$ be a nonexpansive mapping. Suppose $g \in \Pi_C$ with $\rho \in (0,1)$ and assume that $\{\beta_n\}$, $\{t_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\Gamma := F(U) \neq \emptyset$; then for any $x_1 \in C$, let $\{x_n\}$ be a sequence generated iteratively by*

$$\begin{cases} \hat{x}_{n+1} = \beta_n x_n + (1-\beta_n)Ux_n; \\ z_n = t_n x_n + (1-t_n)\hat{x}_{n+1}; \\ x_{n+1} = \gamma_n g(x_n) + (1-\gamma_n)Uz_n, \ \forall \ n \geq 1; \end{cases} \tag{33}$$

where the sequences $\{\beta_n\}$, $\{t_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\lim\limits_{n\to\infty} \gamma_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $0 < \liminf\limits_{n\to\infty} \beta_n(1-\beta_n)(1-t_n)$.

Then, $\{x_n\}$ generated iteratively by (33) converges strongly to a point $x^* \in \Gamma$ which solves some variational inequality problem

$$\langle g(x^*) - x^*, p - x^* \rangle \leq 0, \ \forall \ p \in \Gamma. \tag{34}$$

4. APPLICATIONS

1. Variational problems via resolvents mappings.

Given a maximal monotone operator $M : H \to 2^H$, where H is a real Hilbert space, it is well known that its resolvent $J_\lambda^M(x) = (I+M)^{-1}$ is quasi-nonexpansive and $0 \in M(x) \Leftrightarrow x = J_\lambda^M(x)$. More so, the zeroes of M are exactly the fixed points of its resolvent mapping. By replacing T_{α_i} by J_λ^M, the problem under consideration is nothing but find $x^* \in F(U) \cap M^{-1}(0)$, and our new algorithm is defined as follows:

$$\begin{cases} \hat{x}_{n+1} = \beta_n x_n + (1-\beta_n)J_\lambda^M x_n; \\ z_n = t_n x_n + (1-t_n)\hat{x}_{n+1}; \\ x_{n+1} = \gamma_n g(x_n) + (1-\gamma_n)Uz_n, \ \forall \ n \geq 1. \end{cases} \tag{35}$$

2. Equilibrium Problem

The Equilibrium Problem (EP) which was first introduced by Blum and Oettli [8] is to find x^*C such that

$$F(x^*, u) \geq 0, \ \forall \ u \in C; \tag{36}$$

where C is a nonempty, closed and convex subset of a real Hilbert spaces H and $F : C \times C \to \mathbb{R}$ is a bifunction satisfying the following assumptions:

(i) $F(x,x) = 0, \ \forall \ x \in C$;

(ii) F is monotone, that is $F(x,y) + F(y,x) \leq 0 \ \forall \ x, y \in C$;

(iii) For each $x, y, z \in C, \limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(iv) For each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous.

It is well known that operation $T_r^F(x) : H \to C$ defined by (see [1]):

$$T_r^F(x) := \{z \in C, F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\ y \in C\}.$$

is quasi-nonexpansive and its fixed points are exactly the equilibria of F. Setting $T_\alpha = T_r^F$ in (8), then the problem under consideration is nothing but to find a point $x^* \in F(U) \cap F(T_r^F)$. We present our new iterative algorithm as follows:

$$\begin{cases} \hat{x}_{n+1} = \beta_n x_n + (1 - \beta_n)T_r^F; \\ z_n = t_n x_n + (1 - t_n)\hat{x}_{n+1}; \\ x_{n+1} = \gamma_n g(x_n) + (1 - \gamma_n)U z_n,\ \forall\ n \geq 1. \end{cases} \tag{37}$$

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