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NONLINEAR FOURTH-ORDER DYNAMIC EQUATIONS ON UNBOUNDED TIME SCALES

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Abstract. In this paper, we investigate nonlinear fourth-order dynamic equations on unbounded time scales. The existence and uniqueness of the solutions for these problems are obtained.

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1. INTRODUCTION

It is well known that the time scale calculus is a unification of continuous and discrete calculus. It was introduced in [10]. The dynamic equations on time scales have received in recent years a considerable attention see for instance [4–7, 11, 12, 15–17].

At present, there have been some studies on fourth-order dynamic equations [6,7,11,12,15–17]. To the authors' knowledge, there is no work on the existence of solutions for singular fourth-order dynamic problems in the lim-4 case so our problem is very different from the papers in the literature. A similar way was employed earlier in the differential and difference operator cases in [1–3, 9, 18].

Let $\mathbb{T}$ be a time scale which is unbounded from above such that $\sup \mathbb{T} = \infty$. We will denote $\mathbb{T}$ also as $[0, \infty)_{\mathbb{T}}$. Some preliminary definitions and theorems on time scales can be found in [5].

The space $L^2_\Delta[a, \infty)_{\mathbb{T}}$ is a Hilbert space consisting of all real-valued functions y such that

$$\int_0^\infty |y(x)|^2 \Delta\xi < \infty$$

with the inner product

$$\langle f, g\rangle := \int_0^\infty f(\xi)\, g(\xi)\, \Delta\xi,\ \ f, g \in L^2_\Delta[0, \infty)_{\mathbb{T}}.$$

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We consider the nonlinear fourth-order dynamic equation

$$(1)\qquad \begin{aligned}(\Upsilon x)(\xi) &:= (p_0 x^{\triangle\nabla})^{\nabla\triangle}(\xi) - (p_1 x^{\nabla})^{\triangle}(\xi) + p_2(\xi)\, x(\xi) \\ &= f(\xi, x(\xi)),\ \xi \in [0,\infty)_{\mathbb{T}},\end{aligned}$$

and assume that p_0, p_1 and p_2 are real-valued; p_0^{-1}, p_1 and p_2 are locally Δ-integrable functions on $[0,\infty)_{\mathbb{T}}$, and $p_0 > 0$ on $[0,\infty)_{\mathbb{T}}$.

Yardımcı and Ugurlu [18] have studied equation (1) for when $\mathbb{T} = [0,\infty)$.

For simplicity of notations, we write

$$\begin{aligned} x^{[0]} &= x \\ x^{[1]} &= x^{\Delta} \\ x^{[2]} &= p_0 x^{\Delta\nabla} \\ x^{[3]} &= p_1 x^{\nabla} - \left(x^{[2]}\right)^{\nabla} \\ x^{[4]} &= p_2 x - \left(x^{[3]}\right)^{\Delta} \end{aligned}$$

Then, Green's formula for solutions $x(.)$ and $z(.)$ is given by

$$\int_0^{\infty} (\Upsilon x)(\xi)\, z(\xi)\, \Delta\xi - \int_0^{\infty} x(\xi)\, (\Upsilon z)(\xi)\, \Delta\xi = [x,z]_{\infty} - [x,z]_0,$$

where

$$[x,z]_{\xi} := x^{[0]}(\xi) z^{[3]}(\xi) - x^{[3]}(\xi) z^{[0]}(\xi) + x^{[1]}(\xi) z^{[2]}(\xi) - x^{[2]}(\xi) z^{[1]}(\xi)$$

and

$$[x,z]_{\infty} := \lim_{\xi\to\infty} [x,z]_{\xi}$$

(see [4]). It is clear that $[x,z]_{\infty}$ exists and is finite.

Let

$$D_{\max} = \left\{ x \in L^2_{\Delta}[0,\infty)_{\mathbb{T}} : \begin{array}{c} \text{the first three } \Delta \text{ derivatives are} \\ \text{locally } \Delta\text{-absolutely continuous in} \\ [0,\infty)_{\mathbb{T}},\ \text{and } \Upsilon(x) \in L^2_{\Delta}[0,\infty)_{\mathbb{T}}. \end{array} \right\},$$

and

$$D_{\min} = \left\{ x \in D_{\max} : \begin{array}{c} x^{[0]}(0) = x^{[1]}(0) = x^{[2]}(0) = x^{[3]}(0) = 0, \\ [x,z]_{\infty} = 0,\ \forall z \in D_{\max}. \end{array} \right\}.$$

Then, the maximal operator $\Gamma_{\max}$ is defined on $D_{\max}$ by the formula

$$\Gamma_{\max} x = \Upsilon x.$$

If we restrict the operator $\Gamma_{\max}$ to the set $D_{\min}$, then we obtain the minimal operator $\Gamma_{\min}$. It is clear that $\Gamma^*_{\min} = \Gamma_{\max}$, and $\Gamma_{\min}$ is a closed symmetric operator with deficiency indices (2,2), (3,3), (4,4) (see [8, 14]).

We will assume that the following conditions are satisfied.

(A1) The lim-4 Case holds for $\Upsilon x = 0$ (see [8]).

(A2) $f(\xi,\zeta)$ is real-valued and continuous in $(\xi,\zeta) \in [0,\infty)_{\mathbb{T}} \times \mathbb{R}$, and, for all (ξ,ζ) in $[0,\infty)_{\mathbb{T}} \times \mathbb{R}$, $f(\xi,\zeta)$ satisfies the following condition:

$$|f(\xi,\zeta)| \leq g(\xi) + \varrho |\zeta|, \tag{2}$$

where $g(\xi) \geq 0$, $g \in L^2_\Delta[0,\infty)_{\mathbb{T}}$, and $\varrho > 0$.

Let $y_i, 1 \leq i \leq 4$, be the solutions of equation (1) subject to the following normalization conditions:

$$p_0^2(t)\, W_\Delta(y_1,y_2,y_3,y_4) = 1$$

and

$$\begin{aligned}
&y_1^{[0]}(0) = \cos\alpha,\ y_1^{[1]}(0) = y_1^{[2]}(0) = 0,\ y_1^{[3]}(0) = -\sin\alpha,\\
&y_2^{[0]}(0) = 0,\ y_2^{[1]}(0) = \sin\beta,\ y_2^{[2]}(0) = -\cos\beta,\ y_2^{[3]}(0) = 0,\\
&y_3^{[0]}(0) = \sin\alpha,\ y_3^{[1]}(0) = y_3^{[2]}(0) = 0,\ y_3^{[3]}(0) = \cos\alpha,\\
&y_4^{[0]}(0) = 0,\ y_4^{[1]}(0) = \cos\beta,\ y_4^{[2]}(0) = \sin\beta\ y_4^{[3]}(0) = 0,
\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$,

$$\begin{bmatrix} [y_1,y_1] & [y_2,y_1] & [y_3,y_1] & [y_4,y_1] \\ [y_1,y_2] & [y_2,y_2] & [y_3,y_2] & [y_4,y_2] \\ [y_1,y_3] & [y_2,y_3] & [y_3,y_3] & [y_4,y_3] \\ [y_1,y_4] & [y_2,y_4] & [y_3,y_4] & [y_4,y_4] \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and

$$W_\Delta(y_1,y_2,y_3,y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1^{[1]} & y_2^{[1]} & y_3^{[1]} & y_4^{[1]} \\ y_1^{[2]} & y_2^{[2]} & y_3^{[2]} & y_4^{[2]} \\ y_1^{[3]} & y_2^{[3]} & y_3^{[3]} & y_4^{[3]} \end{vmatrix}.$$

(see [4]).

Now, we will impose the following conditions

$$\begin{gathered}
x^{[0]}(0)\sin\alpha + x^{[3]}(0)\cos\alpha = 0,\\
x^{[1]}(0)\cos\beta + x^{[2]}(0)\sin\beta = 0,\\
[x,y_3]_\infty - d_1[x,y_1]_\infty = 0,\\
[x,y_4]_\infty - d_2[x,y_2]_\infty = 0,
\end{gathered} \tag{3}$$

where $\alpha, \beta, d_1, d_2 \in \mathbb{R}$.

2. MAIN RESULTS

Let us consider the following problem

$$(\Upsilon x)(\xi) = h(\xi), \tag{4}$$

where $\xi \in (0,\infty)$ and $h \in L^2_\Delta[0,\infty)_{\mathbb{T}}$.

Let

$$\varphi(\xi) = \begin{pmatrix} y_1(\xi) \\ y_2(\xi) \end{pmatrix} \text{ and } \psi(\xi) = \begin{pmatrix} y_3(\xi) - d_1 y_1(\xi) \\ y_4(\xi) - d_2 y_2(\xi) \end{pmatrix}.$$

Then, the solution of the boundary-value problem (3), (4) is defined by the formula

$$x(\xi) = \int_0^\infty G(\xi,t)\, h(t)\, \Delta t,$$

where $\xi \in (0,\infty)$ and

$$G(\xi,t) = \begin{cases} \varphi^T(\xi)\,\psi(t), & \text{if } t \le \xi, \\ \varphi^T(t)\,\psi(\xi), & \text{if } t > \xi, \end{cases}$$

where x^T denotes the transpose of the vector x.

Thus, the problem (1), (3) is equivalent to the following equation

$$x(\xi) = \int_0^\infty G(\xi,t)\, f(t, x(t))\, \Delta t. \tag{5}$$

From **(A1)**, we infer that

$$\int_0^\infty \int_0^\infty |G(\xi,t)|^2\, \Delta\xi \Delta t < \infty. \tag{6}$$

Now, we can define an operator

$$T : L^2_\Delta[0,\infty)_{\mathbb{T}} \to L^2_\Delta[0,\infty)_{\mathbb{T}}$$

as follows:

$$(Tx)(\xi) = \int_0^\infty G(\xi,t)\, f(t, x(t))\, \Delta t,\ \xi \in (0,\infty), \tag{7}$$

where $x \in L^2_\Delta[0,\infty)_{\mathbb{T}}$. Hence (5) can be written as $x = Tx$.

THEOREM 2.1. *Suppose that conditions* **(A1)** *and* **(A2)** *are satisfied. Further, let $f(\xi,y)$ satisfy the following Lipschitz condition: there exists a constant $K > 0$ such that*

$$\int_0^\infty |f(\xi, x(\xi)) - f(\xi, z(\xi))|^2\, \Delta\xi \le K^2 \int_0^\infty |x(\xi) - z(\xi)|^2\, \Delta\xi$$

for all $x, z \in L^2_\Delta[0,\infty)_{\mathbb{T}}$. If

$$K \left(\int_0^\infty \int_0^\infty |G(\xi,t)|^2\, \Delta\xi \Delta t \right)^{1/2} < 1,$$

then the problem (1), (3) has a unique solution in $L^2_\Delta[0,\infty)_{\mathbb{T}}$.

Proof. For $x, z \in L^2_\Delta[0,\infty)_\mathbb{T}$, we see that

$$\begin{aligned}
&\left|(Tx)(\xi) - (Tz)(\xi)\right|^2 \\
&= \left|\int_0^\infty G(\xi,t)\left[f(t,x(t)) - f(t,z(t))\right]\Delta t\right|^2 \\
&\leq \int_0^\infty |G(\xi,t)|^2\,\Delta t \int_0^\infty |f(t,x(t)) - f(t,z(t))|^2\,\Delta t \\
&\leq K^2\,\|x-z\|^2 \int_0^\infty |G(\xi,t)|^2\,\Delta t,\ \ \xi \in (a,\infty).
\end{aligned}$$

Thus, we get

$$\|Tx - Tz\| \leq \alpha\,\|x - z\|,$$

where

$$\alpha = K\left(\int_0^\infty \int_0^\infty |G(\xi,t)|^2\,\Delta\xi\Delta t\right)^{1/2} < 1. \qquad \square$$

THEOREM 2.2. *Suppose that conditions* **(A1)** *and* **(A2)** *are satisfied. Further, let us assume that the following condition holds: there exist constants* $M,\ K > 0$ *such that*

$$\int_0^\infty |f(\xi,x(\xi)) - f(\xi,z(\xi))|^2\,\Delta\xi \leq K^2 \int_0^\infty |x(\xi) - z(\xi)|^2\,\Delta\xi$$

for all x *and* z *in*

$$S_M = \left\{x \in L^2_\Delta[0,\infty)_\mathbb{T} : \|x\| \leq M\right\},$$

where K *may depend on* M. *If*

$$\left(\int_0^\infty \int_0^\infty |G(\xi,t)|^2\,\Delta\xi\Delta t\right)^{1/2} \sup_{x\in S_M}\left(\int_0^\infty |f(t,x(t))|^2\,\Delta t\right)^{1/2} \leq M$$

and

$$K\left(\int_0^\infty \int_0^\infty |G(\xi,t)|^2\,\Delta\xi\Delta t\right)^{1/2} < 1,$$

then the problem (1), (3) *has a unique solution satisfying*

$$\int_0^\infty |x(\xi)|^2\,\Delta\xi \leq M^2.$$

Proof. Since S_M is a closed set of $L^2_\Delta[0,\infty)_{\mathbb{T}}$, we will show that T maps S_M into itself. For $x \in S_M$ we get

$$\begin{aligned}\|Tx\| &= \left\| \int_0^\infty G(.,t)\, f(t, x(t))\, \Delta t \right\| \\ &\leq \left\| \int_0^\infty G(.,t)\, f(t, x(t))\, \Delta t \right\| \\ &\leq \left(\int_0^\infty \int_0^\infty |G(\xi,t)|^2 \Delta\xi\Delta t \right)^{1/2} \sup_{x\in S_M} \left\{ \int_0^\infty |f(t,x(t))|^2 \Delta t \right\}^{1/2} \\ &\leq M.\end{aligned}$$

An analysis similar to that in the proof of Theorem 2.1 shows that

$$\|Tx - Tz\| \leq \alpha \|x - z\|, \text{ where } x, z \in S_M.$$

From the Banach fixed point theorem, we get the desired result. □

Now, we show that nonlinear problems may have solutions without uniqueness. In order to get this result, we will use the following Schauder fixed point theorem.

DEFINITION 2.3 ([9]). An operator acting in a Banach space is said to be completely continuous if it is continuous and if it maps bounded sets into relatively compact sets.

THEOREM 2.4 ([9]). *Let* $\mathbf{B}$ *be a Banach space and* $\mathbf{S}$ *a non-empty bounded, convex, and closed subset of* $\mathbf{B}$. *Assume* $A : \mathbf{B} \to \mathbf{B}$ *is a completely continuous operator. If the operator* A *leaves the set* $\mathbf{S}$ *invariant, i.e., if* $A(\mathbf{S}) \subset \mathbf{S}$, *then* A *has at least one fixed point in* $\mathbf{S}$.

THEOREM 2.5. *Suppose that conditions* **(A1)** *and* **(A2)** *are satisfied. Then* T *defined by* (7) *is a completely continuous operator.*

Proof. Let $x_0 \in L^2_\Delta[0,\infty)_{\mathbb{T}}$. Then, we obtain

$$\begin{aligned}&|(Tx)(\xi) - (Tx_0)(\xi)|^2 \\ &= \left| \int_0^\infty G(\xi,t)\,[f(t,x(t)) - f(t,x_0(t))]\,\Delta t \right|^2 \\ &\leq \int_0^\infty |G(\xi,t)|^2 \Delta t \int_0^\infty |f(t,x(t)) - f(t,x_0(t))|^2 \Delta t.\end{aligned}$$

Thus

$$\|Tx - Tx_0\|^2 \leq K \int_a^\infty |f(t,x(t)) - f(t,x_0(t))|^2 \Delta t, \tag{8}$$

where

$$K = \left(\int_0^\infty \int_0^\infty |G(\xi,t)|^2 \Delta\xi\Delta t \right).$$

It is evident that an operator T defined by $Tx(\xi) = f(\xi, x(\xi))$ is continuous in $L^2_\Delta[0,\infty)_\mathbb{T}$ under the condition **(A2)** (see [13]). Hence, for the given $\epsilon > 0$, we can find a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies

$$\int_0^\infty |f(t, x(t)) - f(t, x_0(t))|^2 \Delta t < \frac{\epsilon^2}{K}.$$

From (8), we get

$$\|Tx - Tx_0\| < \epsilon.$$

Let

$$Y = \left\{ x \in L^2_\Delta[0,\infty)_\mathbb{T} : \|x\| \le C \right\}.$$

By (7), we have

$$\|Tx\| \le \left\{ K \int_a^\infty |f(t, x(t))|^2 \Delta t \right\}^{1/2},$$

for all $x \in Y$. Furthermore, using (2), we get

$$\begin{aligned} \int_0^\infty |f(t, x(t))|^2 \Delta t &\le \int_0^\infty [g(t) + \varrho |x(t)|]^2 \Delta t \\ &\le 2 \int_0^\infty \left[g^2(t) + \varrho^2 |x(t)|^2 \right] \Delta t \\ &= 2 \left(\|g\|^2 + \varrho^2 \|x\|^2 \right) \\ &\le 2 \left(\|g\|^2 + \varrho^2 C^2 \right). \end{aligned}$$

Therefore, for all $x \in Y$, we see that

$$\|Tx\| \le \left[2K \left(\|g\|^2 + \varrho^2 C^2 \right) \right]^{1/2}.$$

Further, for all $x \in Y$, we have

$$\int_N^\infty |Tx(\xi)|^2 \Delta\xi \le 2 \left(\|g\|^2 + \varrho^2 C^2 \right) \int_N^\infty \int_0^\infty |G(\xi, t)|^2 \Delta\xi \Delta t.$$

So, from (6), we conclude that for given $\epsilon > 0$ there exists a positive number N, depending only on ϵ such that

$$\int_N^\infty |Tx(\xi)|^2 \Delta\xi < \epsilon^2,$$

for all $x \in Y$.

Hence $T(Y)$ is relatively compact in the space $L^2_\Delta[0,\infty)_\mathbb{T}$. □

THEOREM 2.6. *Suppose that conditions* **(A1)** *and* **(A2)** *are satisfied. Further, we assume that there exists constant $M > 0$ such that*

$$\left(\int_0^\infty \int_0^\infty |G(\xi,t)|^2 \Delta\xi\Delta t\right)^{1/2} \sup_{x\in S_M} \left\{\int_0^\infty |f(t,x(t))|^2 \Delta t\right\}^{1/2} \leq M, \tag{9}$$

where $S_M = \left\{x \in L^2_\Delta[a,\infty)_\mathbb{T} : \|x\| \leq M\right\}$. *Then the problem* (1), (3) *has at least one solution with*

$$\int_0^\infty |x(\xi)|^2 \Delta\xi \leq M^2.$$

Proof. Let $T : L^2_\Delta[0,\infty)_\mathbb{T} \to L^2_\Delta[0,\infty)_\mathbb{T}$ be the operator defined in (7). It follows from Theorems 2.2, 2.5, and (9) that T maps the set S_M into itself. Moreover, the set S_M is bounded, convex and closed. From the Schauder fixed point theorem, we get the desired result. □

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